BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

INCORPORATION OF THE AMERICAN MATHEMATICAL SOCIETY

One of the milestones in the progress of the Society was passed on October 27, 1923, when the unincorporated body turned over the conduct of its affairs to the corporation known as the American Mathematical Society. An announcement was made in this BULLETIN for June, 1923, that a certificate of incorporation had been filed in the District of Columbia and that a meeting of the corporation had been held; but it was later ascertained that this meeting on May 4, 1923, was legally irregular. In consequence, O. S. Adams, Harry English, J. T. Erwin and H. L. Hodgkins met in the office of Dean Hodgkins, 2033 G Street, N. W., Washington, D. C., on October 22, 1923, at 4.00 p. m., for the purpose of arranging all preliminaries necessary to a final transfer of authority to the corporation at the October meeting of the Society. President Veblen was present by invitation.

Dean Hodgkins was elected President of the incorporators and Professor Erwin, Secretary. A waiver of the notice of meeting, signed by all five incorporators, was read.

A set of By-Laws drawn up by a committee of the Society was adopted. According to the revised By-Laws, Ex-Presidents of the Society will after 1923 no longer be ex-officio members of the Council. The designation "Chicago Section" has been dropped, since the meetings in that region have for many years been meetings of the Society. With the addition of a third Vice-President, it is expected that in the future such meetings will be presided over by the President or a Vice-President. To provide secretaries for both regions in which regular meetings of the Society are held, the office of Assistant Secretary has been created in addition to that of Secretary. With these exceptions, the revised By-Laws in general conform as closely to the By-Laws of the unincorporated body as the legal requirements permit. The only essential change *necessitated by incorporation* is the addition of a Board of Trustees to take charge of the financial affairs of the Society. The Council will as heretofore have entire charge of its scientific activities. The By-Laws will be published in full in the List of Members to be issued in the autumn of 1924.

All members of the unincorporated body were elected to membership in the corporation. To fulfil the requirements of the Certificate of Incorporation, the following Board of Trustees was appointed by the incorporators, the list of thirty-one being made up with a view to ensuring the necessary quorum of a majority at the session on October 27 and at the Annual Meeting for 1923: J. W. Alexander, R. C. Archibald, B. A. Bernstein, G. D. Birkhoff, E. W. Brown, F. N. Cole, L. P. Eisenhart, H. B. Fine, W. B. Fite, T. C. Fry, H. E. Hawkes, R. Henderson, H. L. Hodgkins, E. V. Huntington, S. A. Joffe, O. D. Kellogg, E. H. Moore, W. F. Osgood, Anna J. Pell, M. I. Pupin, R. G. D. Richardson, J. F. Ritt, L. P. Siceloff, Clara E. Smith, D. E. Smith, W. M. Strong, H. W. Tyler, Oswald Veblen, H. S. White, J. K. Whittemore, J. W. Young.

The incorporators authorized the Trustees to take over the assets of the unincorporated body and to assume its liabilities. So far as was consistent with the By-Laws, they elected as officers of the corporation the officers of the old organization and for the same terms. The members of the Editorial Committee for the Transactions, formerly members of the Council but not officers, were elected officers with the same terms; as their terms expire, their successors will be elected by the Society, instead of being appointed by the Council. The name of the Committee of Publication was changed to Editorial Committee for the Bulletin, the term of office being adjusted so that instead of all members retiring in 1923, E. R. Hedrick continues until December, 1924, and J. W. Young until December, 1925. The appointed committees of the new organization correspond precisely to those of the old.

As a further necessary step in the proceedings, a meeting of the Board of Trustees was held in the Trustees Room of Columbia University on October 27, 1923, at 9.30 a. m. The call and the waiver, bearing the signatures of all the trustees, were read. Twenty-eight of the trustees responded to the roll-call. President Veblen was elected Chairman of the Board and R. G. D. Richardson, Secretary. An executive committee was appointed, consisting of Oswald Veblen, H. E. Hawkes, Robert Henderson, R. G. D. Richardson, and H. W. Tyler. Resolutions were passed authorizing the taking over of the assets and liabilities of the unincorporated Society and the general dispatch of necessary business.

The final stage in the proceedings was reached at the morning sessions with the adoption of resolutions by the Council and Society transferring all property and incumbrances to the Corporation.

The By-Laws of the incorporated body specify that after May 3, 1924, the Board of Trustees shall consist of five members, and nominations by the Council for these trustees were ordered printed on the ballot to be voted on at the Annual Meeting of 1923.

R. G. D. Richardson,
Secretary.

THE OCTOBER MEETING OF THE SOCIETY

The two hundred thirty-first regular meeting of the Society was held at Columbia University on Saturday, October 27, 1923, extending through the usual morning and afternoon sessions. This meeting marks the transition of the Society into an incorporated body.

The Society was called to order for the transaction of business at 11.15 a. m., in Room 305 of Schermerhorn Hall, Columbia University, New York City, some fifty-five members being present. The Secretary recalled the action of the Society in previously endorsing incorporation and reported the formal action of the Council recommending the final steps. President Veblen made a general statement concerning the procedure adopted on legal advice, and recounted briefly the stages of the incorporation process, which are reported in another article in this issue.

The following resolution was then adopted unanimously:

WHEREAS, a corporation has been organized under the laws of the District of Columbia, known as the American Mathematical Society, for the purpose of carrying on the work heretofore conducted by this Society; and

WHEREAS, it is the judgment of this Society that for the purpose of more efficiently prosecuting said work it is expedient to transfer to the said new corporation the entire property and assets of this Society;

NOW, THEREFORE, BE IT RESOLVED, that the officers of this Society be and they are hereby authorized and directed to transfer to the American Mathematical Society, a corporation as aforesaid, all the property and assets of this Society, including its bank deposits and choses in action, provided the said corporation shall assume and agree to pay all indebtedness and financial obligations of every kind and description for which this Society may now be liable;

AND BE IT FURTHER RESOLVED, that the proper officers

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of this Society be and they are hereby empowered, to execute, acknowledge, and deliver, all contracts, deeds, and other documents necessary and proper for the carrying into effect this resolution.

The attendance at the scientific Sessions included the following sixty-eight members of the Society:

Alexander, Archibald, Ballantine, Bernstein, Birkhoff, E. W. Brown, Abraham Cohen, Cole, Cowley, L. D. Cummings, Douglas, Edmondson, Eisenhart, Fine, Fiske, Fry, Ginsburg, Gleason. Glenn, Grove, Hausle, Hawkes, Hazlett, Hebbert, Robert Henderson, Hille, Himwich, Hodgkins, Hotelling, Huntington, Dunham Jackson, Joffe, Kasner, Kellogg, Kircher, Kline, Langman, MacColl, MacNeish, Meder, Mirick, Molina. Northcott, Osgood, Pell, Pfeiffer, Post, Rainich, Ranum, Reddick, R. G. D. Richardson, Ritt, Schub, Seely, Siceloff, Simons, C. E. Smith. D. E. Smith, Sosnow, Strong, Tyler, Veblen, H. E. Webb, Weiss, M. E. Wells, H. S. White, Whittemore, J. W. Young.

At the meeting of the Council, the following forty-nine persons were elected to membership in the Society:

Miss Ethel Louise Anderton, New Haven, Conn.;
Professor Victor-Elezéar Beaupré, University of Montreal;
Dr. Harry Albert Bender, University of Illinois;
Professor Jacob Roy Bender, University of Idaho;
Miss Lucy Elizabeth Berger, College of William and Mary;
Professor Harry Birchenough, State College for Teachers, Albany, N. Y.;
Mr. Harley W. Chandler, University of Florida;
Professor Myrtie Collier, Southern Branch of the University of California,
Mr. Arthur Herbert Copeland, Harvard University;
Professor Charles Edgar Corbin, College of the Pacific;
Mr. Frederic Warren Darling, United States Coast and Geodetic Survey;
Mr. Charles Herbert Davis, Yale Observatory;
Mr. Joseph Oscar Eckersley, Department of Plant and Structures, New York City;
Mr. Jekuthiel Ginsburg, New York City;
Mr. Kut William Halbert, Harvard University;
Dr. Victor August Hoersch, University of Illinois;
Mr. Julian Laurence Holley, Harvard University;
Dean Charles Ellsworth Horne, College of Agriculture and Mechanic Arts, Mayaguez, Porto Rico,
Professor Charles Applewhite Isaacs, State College of Washington;
Dr Hamilton M. Jeffers, University of Iowa;
Professor William H. Kirchner, University of Minnesota;
Professor Alois Francis Kovarik, Yale University;
Mr. Harry Isler Lane, University of South Dakota;
President William Orville Mendenhall, Friends University,
Mr Aristotle Michal. Rice Institute;
Miss Helen Moon, Lincoln National Life Insurance Company, Fort Wayne, Ind.;
Miss Thirza Adaline Mossman, Kansas State Agricultural College;
Professor George Alexander Newton, Trinity University;
Mr. Frank C. Ogg, University of Illinois;

Professor James Robert Overman, State Normal College, Bowling Green, Ohio;
Professor Arthur Pelletier, Ecole Polytechnique of Montreal;
Mr. Frederick William Perkins, Harvard University;
Professor Edward C. Phillips, Woodstock College;
Mr. Joseph Crawford Polley, Colgate University;
Mr. George Yuri Rainich, Johns Hopkins University;
Professor Charles Henry Rawlins, United States Naval Academy;
Mr. Perry Dean Schwartz, New Haven, Conn.;
Professor Jsaac F. Seiverling, State Normal School, Millersville, Pa.;
Mr. Robert Vincent Sinnott, Hartford, Conn.;
Mr. Morris Miller Slotnick, Harvard University;
Mr. Charles Calvin Steck, Tonawanda, N. Y.;
Dr. Tracy Yerkes Thomas, University of Chicago;
Mr. Kamihayashi Tokusaburo, Iwateken, Japan;
Professor Richard Chace Tolman, California Institute of Technology;
Mr. Glen M. V. Tryon, Fenton, Mich.;
Mr. George Peterkin Unseld, Technical High School, Salt Lake City;
Mr. John Russell Vatnsdal, Yale University;
Mr. Charles Weiss, New York City;
Professor Joseph Wilczewski, St. Xavier College, Cincinnati.

Fourteen applications for membership in the Society were received.

The Committee on Printing announced that arrangements were being made to print both the BULLETIN and the TRANSACTIONS in Hamburg, Germany, during the coming year.

Announcement was made of the appointment of Professors H. E. Hawkes (chairman), E. W. Brown, J. L. Coolidge, and H. S. White as the Committee on the first Josiah Willard Gibbs lecture; of Professor E. E. Moots as representative at the inauguration of President Updegraff of Cornell College on October 19; of Professor E. V. Huntington as representative at the inauguration of President Comstock of Radcliffe College on October 20; and of Professor W. H. Roever, as representative at the inauguration of President Hadley of Washington University on November 10.

Votes of thanks were extended to Boston College for a gift toward the Endowment Fund; to Professor W. H. Roever for a gift toward the expenses of the membership campaign; to the Committee on Incorporation and the incorporators; and to Messrs. Clephane, Harriman, Latimer, and Putnam for legal advice, tendered without fee, in connection with the incorporation proceedings.

It was decided to hold a meeting of the Council in

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conjunction with the western meeting of the Society at Chicago in April, 1924.

A list of nominations for trustees, officers, and other members of the Council was adopted unanimously and ordered printed on the ballot for the annual election.

President Veblen presided at the sessions of the Society, relieved in the morning by Vice-President H. W. Tyler and in the afternoon by Ex-President E. W. Brown. The first part of the afternoon session was devoted to a paper read, at the request of the Programme Committee, by Professor Anna J. Pell, on *Bilinear and quadratic forms in infinitely many variables.* Titles and abstracts of the other papers read at this meeting follow below. Professor Pólya was introduced by Professor Birkhoff. Dr. Douglas's second paper, and the papers of Professor Lipka, Professor Pólya, Dr. Murray, and Professors Alexander and Richardson were read by title.

1. Professor L. P. Eisenhart: *Spaces of continuous matter in general relativity.*

If the field equations for a perfect fluid are taken as

$$R_{ij} - \tfrac{1}{2} g_{ij} R = -k(\sigma u_i u_j - p g_{ij})$$

where R_{ij} is the contracted Riemann tensor with respect to the fundamental tensor g_{ij}, R is the scalar curvature, and σ and p are scalars, the determinant equation $|R_{ij} + \varrho g_{ij}| = 0$ has a simple root and a triple root with simple elementary divisors. Conversely, when the roots of this equation for a space satisfy these conditions and at each point the fundamental quadratic form is reducible to $-dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2$ at each point, the space may be interpreted as the space-time continuum of a perfect fluid. If ϱ' is the simple root and ϱ'' the triple root, then $\sigma = (\varrho' - \varrho'')/k$ and $p = (\varrho' + \varrho'')/2k$. The contravariant components u^i of the line of flow are given by

$$(R_{ij} + \varrho' g_{ij}) u^i = 0.$$

2. Professor J. K. Whittemore: *The deformation of ruled surfaces.*

In this paper it is shown that the equations of any ruled surface, rulings u constant, whose linear element is given by $ds^2 = [(v-\alpha)^2 + \beta^2]\, du^2 + dv^2$, where α and β are arbitrary functions of u, are $x = Av + \int (P\beta - L\alpha)\, du$ with similar equations for y and z, where A, L and P are the cosines of the angles of the x-axis and the tangent, principal normal, and binormal, respectively, of a curve of unit curvature and arbitrary torsion, and u is the arc of this curve. A property of the surface depending on the torsion of this curve is given. The method consists in determining from the Codazzi equations the single unknown coefficient of the second fundamental form of the required surface, then integrating the partial differential equations for the direction cosines of the normal and the tangents to the parametric curves.

3. Mr. G. Y. Rainich: *Analytic vector functions.*

This theory is analogous to the theory of analytic functions of a complex variable; the place of the complex number is occupied by the three-dimensional vector and the part of the product is played by a bilinear vector function, the basis. An analytic function is defined with the aid of a power series; the region of convergence is a parallelepiped or an elliptic cylinder; the condition of analyticity is given by a set of two differential vector equations (analogy with the Cauchy-Riemann equations) or by the vanishing of a contour integral (analogy with the Cauchy-Morera theorem). These functions have no independent importance, because all functions analytic on the same basis have one or three directions along which they have particular properties. To get rid of this peculiarity we consider analytic functions of two variables; substituting for each of these variables a linear vector function of the same variable z we get a function of z which we call semi-analytic; the functions of a semi-analytic class are characterized by one differential vector equation or by the vanishing of a surface integral; they can be expanded into double series. Analogous functions can be formed also starting from ordinary analytic functions.

4. Dr. Jesse Douglas: *Systems of ∞^{2n-2} curves in a Riemann space in which the sum of the angles of every triangle formed by three of the curves is two right angles.*

This paper proves the following theorem. If in a Riemann space of three or more dimensions, $ds^2 = \Sigma a_{ij}\, dx^i dx^j$, there exists a system of ∞^{2n-2} curves, defined by a set of differential equations of the form $d^2x^i/ds^2 = F_i(x, dx/ds)$, which has the

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property that in every triangle formed by three of the curves the sum of the angles is equal to π, then the Riemann space must be conformally equivalent to a euclidean space, and in such a way that the ∞^{2n-2} curves correspond to the straight lines of the euclidean space.

5. Dr. Jesse Douglas: *Necessary and sufficient conditions that a system of ∞^4 curves in space consist of the mutual intersections of ∞^3 surfaces.*

The ∞^4 straight lines of space are the mutual intersections of the ∞^3 planes. Not every system of ∞^4 curves in space has a so related system of ∞^3 surfaces. Assuming the curve system to be defined by the differential equations $y'' = F(x,y,z,y',z'), z'' = G(x,y,z,y',z')$ the present paper derives necessary and sufficient conditions in the form of three partial differential equations of the second order bearing on F and G.

6. Professor Joseph Lipka: *On Ricci's coefficients of rotation.*

Ricci's coefficients of rotation, γ_{ijk}, play an important rôle in the absolute calculus of any space whatsoever. He first defines the γ's analytically and then shows their geometric meaning be means of the tangent spaces. Following a suggestion made to the writer by Professor Levi-Civita, the γ's are here defined geometrically by means of the notion of parallelism. As a consequence, the expressions for the geodesic curvature and several other invariants in terms of the γ's are all found and interpreted intrinsically. Applying the notion of conformal parallelism, the γ's are generalized so that they involve an arbitrary function of the coordinates, and geometric interpretations of certain invariants in the generalized coefficients are given.

7. Professor Joseph Lipka: *Types of alignment charts in three variables.*

This paper contains a study of a systematic classification of alignment or nomographic charts, based upon parallel coordinates. In particular, it gives the specific form which an equation in three variables, $f(u, v, w) = 0$, must have in order that it may be represented by certain combinations of curved and straight line scales. It also gives the equations of the scales in each case, and their method of construction.

8. Professor Georg Pólya: *On the mean-value theorem corresponding to a given linear homogeneous differential equation.*

This paper, which will appear in the TRANSACTIONS OF THIS SOCIETY, deals with the conditions which must be imposed on a linear differential form $L(y)$ of the nth order in order that whenever y vanishes $n+1$ times on the given interval, $L(y)$ shall necessarily vanish on that interval at least once.

9. Dr. F. H. Murray: *Note on stability à la Poisson.*

This paper appears in full in the present issue of this BULLETIN.

10. Professor J. W. Alexander: *On infinitely connected plane regions.*

Some theorems are proved on the topology of infinitely connected plane regions, and a class of regions is determined such that each region is characterized by a pair of invariants, one finite, the other transfinite.

11. Professor J. W. Alexander: *On the deformation of an n-cell.*

Let $r' = R(r, \theta)$, $\theta' = \Theta(r, \theta)$ be the equations of a one-to-one continuous transformation T carrying the interior and boundary of the unit circle into itself and leaving all points on the boundary of the unit circle invariant. The transformation may be extended to the entire plane by saying that it leaves all points without the unit circle invariant. Construct, now, a transformation T_λ such that $r' = \lambda R(r/\lambda, \theta)$, $\theta' = \Theta(r/\lambda, \theta)$ for $\lambda > 0$ and such that T_0 is the identity. Then, as λ varies from 0 to 1, T_λ determines a one-to-one continuous deformation generating the transformation T and leaving the boundary of the unit circle pointwise invariant. The theorem generalizes to n dimensions.

12. Professor R. G. D. Richardson: *On the reality of the zeros of a λ determinant.*

This paper appeared in full in the December number of this BULLETIN.

13. Professor E. V. Huntington: *Sets of completely independent postulates for cyclic order.*

The author gives two sets of five postulates for the theory of cyclic order, each set being shown to be "completely independent" in the sense of E. H. Moore. An abstract of some of the results appeared in the PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 2 (1916), pp. 630–631. The paper is closely related to the exhaustive treatment of *Sets of independent postulates for betweenness*, published by E. V. Huntington and J. R. Kline in the TRANSACTIONS OF THIS SOCIETY, vol. 18 (1917), pp. 301–325, though the results in the present case are much more easily obtained. The paper will be offered to the TRANSACTIONS.

14. Professor Dunham Jackson: *Some corollaries of Bernstein's theorem.*

The theorem of Bernstein referred to in the title states that if $T_n(x)$ is a trigonometric sum of order n, and if $|T_n(x)| \leqq L$ for all values of x, then $|T_n'(x)|$ can never exceed nL. The hypothesis requires that the condition on $|T_n(x)|$ be satisfied for *all* (real) values of x, or, what amounts to the same thing, that it be satisfied throughout an interval of length 2π. The present paper obtains results of similar character, though less simple in form, on the assumption merely that the condition is satisfied throughout an interval of length less than 2π, the conclusions then being valid for the restricted interval also. These results are deduced indirectly from Bernstein's theorem itself.

15. Dr. E. L. Post: *Theory of generalized differentiation.*

The theory of derivatives of non-integral orders, begun by Liouville and Riemann, may be said to be complete. On the other hand, the theory of more general operators considered as functions of D, the derivative, is fragmentary. In the present paper, the problem is attacked by an entirely new method, and results are obtained that generalize those of Riemann. Although the present investigation was inspired purely by an interest in the bizarre, it has been found to connect with many fields in the most recent branches of analysis. In particular, it has applications in the theory of the Laplace transformation, Volterra's permutable functions and functions of composition (of the closed cycle group), and the Heaviside operational classes.

R. G. D. RICHARDSON,
Secretary.

THE SEPTEMBER MEETING OF THE SAN FRANCISCO SECTION

The forty-second regular meeting of the San Francisco Section of the American Mathematical Society was held at the University of Southern California, Los Angeles, on September 18, 1923. Professor Cajori, chairman of the section, opened the meeting and presided until the election of the new chairman. The attendance included the following seventeen members of the Society:

Bateman, Cajori, A. F. Carpenter, Daus, Glazier, Hoskins, Glenn James, F. R. Morris, Russell, Sherwood, Showman, Steed, Touton, Wear, Willett, Winger, Wolfe.

The election of officers for the ensuing year resulted as follows: Chairman, Professor A. F. Carpenter; Secretary-Treasurer, Professor B. A. Bernstein; Program Committee, Professors B. A. Bernstein, D. N. Lehmer, E. T. Bell.

Titles and abstracts of the papers read at this meeting follow below. The papers of Professor Bell, Griffin, and Porter, and Mr. Romig were read by title. Mr. Romig was introduced by Professor Cajori.

1. Professor E. T. Bell: *Notes on recurring series of the third order.*

Important new historical material regarding the u, v of Lucas is given in the first part of this paper. The remainder is devoted to the algebra of the three linearly independent functions defined by a recurrence relation of the third order. The most general possible addition theorems for each of the functions are obtained. By slight changes in notation all results of the paper can be extended at once to series of order $n > 3$. As was to be expected, the results are much more complicated than those for Lucas' functions. There are a few applications to Diophantine analysis.

2. Professor E. T. Bell: *Analogies between elliptic functions and the u, v of Lucas.*

This paper appeared in full in the November, 1923, number of this BULLETIN.

3. Professor Florian Cajori: *Did Pitiscus use the decimal point?*

While Pitiscus is his *Trigonometry* of 1612 used fractions extensively, he did not use the point as separatrix between units and tenths. Most modern historians of mathematics state incorrectly that he used the decimal point in his trigonometric tables of 1612. The point does occur hundreds of times, but simply as a mark to divide the numeral figures into groups easier to read and copy.

4. Professor Florian Cajori: *Empirical generalizations on the growth of mathematical notations.*

Several empirical observations on the nature and growth of notation are given, which point to haphazard developments usually failing of general adoption, and which set forth the difficulties encountered in securing universal notations. The efforts of international committees towards uniformity of notation are recounted.

5. Professor A. F. Carpenter: *Linear complexes associated with the general ruled surface.*

In a paper read before the San Francisco Section of the Society in April, 1923, a number of theorems were proved concerning the primary and secondary flecnodal and complex cubics associated with each element of a ruled surface. Each of these cubics determines a linear complex. In the present paper, the author discusses the properties of these complexes and certain new cubics defined by means of them.

6. Professor A. F. Carpenter: *Note on a net of curves on a general surface.*

By a method analogous to that which enables us to define lines of curvature of a surface, and the Dupin indicatrix at a point, it is possible to obtain a quadratic differential form whose vanishing determines the extreme values of the geodesic torsion, and which may therefore be taken as defining a net of curves on the surface. These curves, which may properly be called lines of torsion, constitute an orthogonal system, the two curves of this system through any surface point bisecting the angles between the lines of curvature. The maximum and minimum values of the geodesic torsion are equal but opposite in sign for all real surfaces, and the indicatrix for the geodesic torsion is a rectangular hyperbola whose asymptotes coincide with the tangents to the lines of curvature.

7. Dr. P. H. Daus: *Numerical illustrations of a theorem in ternary continued fractions.*

Connected with the equation $x^3+qx-(p_1^3+p_1q+1)=0$, there exists a normal ternary continued fraction expansion $(p_1, p_1^2+q;\ 2p_1, 3p_1^2+q;\ \overline{3p_1, 3p_1^2+q;})$. This note considers certain numerical cases in the light of a theorem by Lehmer (PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 4 (1918), p. 360) which determines whether the ratios $A_n:B_n:C_n$ formed from the nth convergent set do or do not approach limits.

8. Mr. H. P. Robertson: *The absolute differential calculus of a non-Pythagorean, non-Riemannian space.*

Fundamental in the theory of relativity is the assumption that the line element of four-dimensional space-time is represented by the square root of an invariant quadratic differential form, and the subsequent generation of its related tensors by means of the absolute differential calculus. If, instead, it is assumed (a possibility suggested by Riemann) that the line element is represented by the $2m$th root of a differential form of degree $2m$ (m an integer) a new calculus is needed for the generation of tensors connected therewith. The present paper develops such an instrument, which applies to differential forms of degree $2m$ in n variables, and in which the Ricci-Levi-Civita calculus is included as a special case.

9. Dr. Victor Steed: *On a system of equations connected with the lines on the cubic surface.*

To show that there are lines on the cubic surface, Salmon suggested eliminating two of the cartesian coordinates from the equations of a line and the equation of the surface, and then equating to zero the coefficients of the resulting equation in the third coordinate. The four equations in the four coordinates of the line thus obtained are here interpreted as four cubic hypersurfaces in space of four dimensions. If these were not special, they would have, by the theorem of Bezout, 81 distinct points of intersection. The question of how to account for the fact that these hypersurfaces have only 27 points of intersection in finite space is considered, it being shown that all four hypersurfaces have in common three skew lines lying entirely in the hyperplane at infinity.

10. Professor Harry Bateman: *Derivation of three-dimensional potentials from four-dimensional potentials.*

When a solution of Laplace's equation in four variables is integrated with respect to one of these variables from $-\infty$ to $+\infty$ the result is generally a solution of Laplace's equation in three variables. Since potential functions in four variables are often easier to calculate than analogous potential functions in three variables, the well known device just mentioned may be of real value. At any rate, it leads to some interesting identities, as is shown in this paper.

11. Professor F. L. Griffin: *Curves functionally conjugate with respect to a given curve.*

The author defines a general category of relationships among coplanar curves as follows: Let the radii vectores of three coplanar curves be $\varrho_1, \varrho_2, \varrho_3$, and let there exist among the radii corresponding to any common value of θ a functional relation $F(\varrho_1, \varrho_2, \varrho_3) = 0$. Regarding one of the curves as a basic reference curve, the other two will be called *functionally conjugate* with respect to the first. Ordinary inverse curves, cissoids of curves, etc., constitute special classes of such conjugates. For various simple selections of the function F and the basic curve, the conjugates of conics are found to be well known curves. This type of relationship admits of generalization to any number of curves and of extension to surfaces and loci in any number of dimensions.

12. Professor M. B. Porter: *Second mean-value theorem for integrals of summable functions.*

This paper appeared in full in the November, 1923, number of this BULLETIN.

13. Mr. H. G. Romig: *Division by zero (a historical study).*

The author describes the first use of zero as a divisor among the Hindus in 628 A. D., and traces historically the observance of the difference between the infinitesimal and absolute zero, the rigorous exclusion of division by absolute zero by Martin Ohm in 1828 and Axel Harnack in 1881, and the more recent treatment of division by absolute zero in algebra, modern geometry, and function theory.

14. Professor R. M. Winger: *Singular points of self-projective rational septimics.*

In an earlier paper, the author obtained the types of self-projective rational septimics. In the present paper he analyzes the singularities which occur in great variety and considerable complexity. The standard methods of expansion in the neighborhood of the singularity and birational transformation are not suitable for rational curves which are normally given in parametric form. Moreover, the double-point equation is tedious to expand since it leads to an equation of degree 30 in the case of the septimic. Dr. Scott's theory of excess is, however, of great assistance. Two general theorems of interest are proved: (1) If a rational curve of order n is invariant under a cyclic group of order n (and no higher group) this group cannot contain an homology. (2) A rational curve of order n which is invariant under a cyclic G_n (and no higher group) has a multiple point of order greater than 2, having, however, but two distinct parameters. The parameters of the multiple point are the fixed parameters of the binary group, while the multiple point is the only fixed point of the ternary group which lies on the curve.

L. M. Hoskins,
Acting Secretary of the Section.

NOTE ON STABILITY À LA POISSON*

BY F. H. MURRAY

In this note will be developed certain consequences of a theorem due to Poincaré concerning "Stabilité à la Poisson".†

Suppose given a differential system

$$\frac{dx_i}{dt} = X_i(x_1, x_2 \cdots x_n), \qquad (i = 1, 2, \ldots, n), \tag{1}$$

such that the functions X_i are analytic in their arguments within an ordinary‡ closed region D, and suppose the equations (1) admit a positive multiplier M within this region. Then

$$\frac{\partial}{\partial x_1}(MX_1) + \frac{\partial}{\partial x_2}(MX_2) + \cdots + \frac{\partial}{\partial x_n}(MX_n) = 0. \tag{2}$$

In addition one can adopt either of the following hypotheses: (α) every solution of (1) which takes on a system of values $(x_1^0, x_2^0, \ldots, x_n^0)$ belonging to D at the time $t = t_0$ remains within or on the boundary of D for all values of t; (β) every solution of (1) which takes on a system of values $(x_1^0, x_2^0, \ldots, x_n^0)$ at $t = t_0$, belonging to an ordinary closed region D' interior to D remains within or on the boundary of D for all values of t.

In the discussion given by Poincaré, hypothesis (α) is made explicitly; but an examination of the argument shows that the same method of reasoning can be applied under hypothesis (β), to the solutions of (1) which, at $t = t_0$, take on values belonging to D'. The result may be stated as follows:

If $P_0(x_1^0, \ldots, x_n^0)$ is any interior point of D', Δ any ordinary closed region containing P_0 and interior to D', then there exists at least one point $P'(x_1', x_2', \ldots, x_n')$ belonging to $\varDelta$ such that, given t_0 and $T > t_0$, there exists a value $\tau > T$ such that the solution of (1) taking on the values $(x_1', \ldots, x_n')$ at $t = t_0$ will take on values $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ belonging to $\varDelta$ at the time $t = \tau$.

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* Presented to the Society, October 27, 1923.

† *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 3, chapter 26.

‡ By "ordinary closed region" we mean an n-dimensional region bounded by an ordinary $(n-1)$-dimensional surface, and possessing a certain volume different from zero.

An application will be made to the case in which $X_i(0, 0, \ldots, 0) = 0$, $(i = 1, 2, \ldots, n)$, and in which the system (1) is known to possess *ordinary stability* in a certain neighborhood $|x_i| < A$. Ordinary stability may be said to exist in the given region under the following conditions: given any region (b): $|x_i| \leqq b$, $b \leqq A$, there exists a region (c): $|x_i| \leqq c$, $c \leqq b$, such that any solution of (1) which takes on a system of values belonging to (c) at $t = t_0$ lies within (b) for all values of t.

Suppose $c = C$ the value corresponding to $b = A$; the assumption of ordinary stability implies hypothesis (β) in which $D' = (C)$, $D = (A)$. Consequently there is stability à la Poisson within the region (C) if equations (1) admit a positive multiplier M within the region (A). This condition will be satisfied if equations (1) are in the canonical form

$$(3) \qquad \frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}, \quad (i = 1, 2, \ldots, m),$$

since equations (2) are satisfied by $M = 1$.

From an examination of the chapter of *Les Méthodes Nouvelles* mentioned above it can easily be seen that the assumptions concerning the functions X_i can be generalized. It is sufficient that the solutions of (1) be continuous functions of the initial values for $t = t_0$, and that they possess continuous partial derivatives of the first order with respect to them. These conditions will be satisfied if the partial derivatives $\partial X_i/\partial x_k$ exist and are continuous within and on the boundary of the region D considered,* $(i, k = 1, 2, \ldots, m)$.

DALHOUSIE UNIVERSITY

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* Goursat, *Cours d'Analyse*, vol. 3, 2d edition, p. 13.

APPLICABILITY WITH PRESERVATION OF BOTH CURVATURES*

BY W. C. GRAUSTEIN

1. *Introduction.* In his celebrated memoir of 1867, Bonnet† proved that there is in general no surface applicable to a given surface with preservation of both the total and mean curvatures, but that in certain exceptional cases there exists a unique surface or a one-parameter family. The primary purpose of this note is to exhibit conditions of simple form characterizing the second of these exceptional cases. The method of treatment applies equally well to the general case.

2. *Necessary and Sufficient Condition.* It is well known that a necessary condition that a surface S admit ∞^1 surfaces applicable to it with preservation of both curvatures is that the surface S be isometric.‡ Accordingly, we can assume that S is an isometric surface, referred to its lines of curvature, and that the parameters are isometric. The linear element of S is then of the form

$$ds^2 = \lambda(du^2 + dv^2), \tag{1}$$

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* Presented to the American Mathematical Society, September 7, 1923.

† *Mémoire sur la théorie des surfaces applicables sur une surface donnée,* Journal de l'École Polytechnique, vol. 42 (1867), pp. 72 et seq.

‡ This fact appears to have been discovered first by Raffy, *Sur une classe nouvelle de surfaces isothermiques et sur les surfaces déformables sans alteration des courbures principales,* Bulletin de la Société de France, vol. 21 (1893), pp. 70-72. His proof, however, is faulty, in that he bases part of it on the incorrect statement, made by Caronnet, that every W-surface applicable to a surface of revolution is isometric. If that statement were correct, every helecoidal surface would be isometric, which is not the case. A valid proof is given by Hazzidakis, *Biegung mit Erhaltung der Hauptkrümmungsradien,* Journal fur Mathematik vol. 117 (1897), p. 46.

and the Codazzi equations become

$$(2)\qquad \frac{\partial e}{\partial v}=\frac{e+g}{2\lambda}\frac{\partial \lambda}{\partial v},\quad \frac{\partial g}{\partial u}=\frac{e+g}{2\lambda}\frac{\partial \lambda}{\partial u},$$

where $e, f\,(=0)$, g are the differential coefficients of the second order. Recalling that $1/r_1=e/\lambda$, $1/r_2=g/\lambda$, and setting

$$\frac{1}{r_1}+\frac{1}{r_2}=2M,\quad \frac{1}{r_1}-\frac{1}{r_2}=2N,$$

we find that equations (2) can be rewritten in the forms*

$$(3)\qquad \frac{\partial M}{\partial u}=N\frac{\partial \log \lambda N}{\partial u},\quad \frac{\partial M}{\partial v}=-N\frac{\partial \log \lambda N}{\partial v}.$$

If there exists a surface S' applicable to S with preservation of both curvatures, then

$$e'g'-f'^2=eg,\quad e'+g'=e+g.$$

Hence it can be shown that

$$(4)\quad e'=e-\frac{2\lambda N}{1+z^2},\quad f'=\frac{2\lambda N z}{1+z^2},\quad g'=g+\frac{2\lambda N}{1+z^2},$$

where

$$z=\operatorname{ctn}\alpha,$$

and where α is the angle under which the curves corresponding to the lines of curvature on S' cut the lines of curvature on S.

The Codazzi equations for S', by virtue of the corresponding equations (2) for S, can be written in the forms

$$\frac{\partial(e'-e)}{\partial v}=\frac{\partial f'}{\partial u},\quad \frac{\partial(g'-g)}{\partial u}=\frac{\partial f'}{\partial v}.$$

Substituting for $e'-e$, f', and $g'-g$ their values as given by (4), and solving for $\partial z/\partial u$, $\partial z/\partial v$, we obtain two equations which reduce to

$$(5)\qquad \frac{\partial}{\partial u}\frac{z}{\lambda N}=\frac{\partial}{\partial v}\frac{1}{\lambda N},\quad \frac{\partial}{\partial v}\frac{z}{\lambda N}=-\frac{\partial}{\partial u}\frac{1}{\lambda N}.$$

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* We have hereby assumed that $N\neq 0$. But a sphere obviously admits no surface applicable to it in the manner prescribed.

But these equations are compatible if and only if the reciprocal of λN is a harmonic function.

THEOREM 1. *There exists a one-parameter family of surfaces applicable to a given surface S with preservation of both curvatures if and only if the lines of curvature on S form an isometric system, and if, when isometric parameters for this system are introduced, the reciprocal of $\lambda(1/r_1 - 1/r_2)$ is a harmonic function.*

If one function, ψ, to which the reciprocal of λN is conjugate, is obtained by a quadrature, the general solution of equations (5) is

$$z = \lambda N(\psi + k),$$

where k is an arbitrary constant. Substituting this value of z in (4), we obtain e', f', and g' for the surfaces S'. Thus these surfaces are completely determined.

That a surface S of constant mean curvature admits ∞^1 surfaces S' applicable to it with preservation of both curvatures is well known.*

It is in this case only that the angle α is constant and that the lines of curvature on S correspond to the lines of curvature on one of the surfaces S'. For, by (3), the mean curvature is constant only if λN is constant, and, by (5), λN is constant only when z is constant.

3. *Applicability in an Infinity of Ways.* Consider now an arbitrary one-parameter family of surfaces, every pair of which are applicable with preservation of both curvatures. An obvious necessary condition that this applicability be possible in a continuous infinity of ways is that the surfaces be W-surfaces, i. e., that the curvatures be functionally related. It is surprising to find that, if we exclude the surfaces of constant mean curvature, this condition is also sufficient.

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THEOREM 2. *Every pair of a one-parameter family of (isometric) W-surfaces satisfying the conditions of Theorem 1, but of variable mean curvature, are applicable in a continuous infinity of ways with preservation of both curvatures.*

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[*] Cf. Bonnet, loc. cit.

For, if we except surfaces of constant mean curvature, an isometric W-surface is either a surface of revolution or a surface of one of several special types. The theorem is obvious in the first case. In the second case, it follows directly from certain investigations by the author, which are to be published elsewhere.

A surface of constant mean curvature admits ∞^1 surfaces each applicable to it in a continuous infinity of ways with preservation of both curvatures if and only if it is applicable to a surface of revolution. The surfaces of constant mean curvature with this property have been determined by Raffy.*

4. *Conclusion.* In conclusion, we sketch the application of the method of § 2 to the general problem. In this case we assume merely that the lines of curvature on S are parametric. If S' is applicable to S with preservation of both curvatures,

$$e'g'-f'^2 = eg, \qquad Eg'+Ge' = Eg+Ge.$$

Then

$$e' = e-\frac{2EN}{1+z^2}, \qquad f' = \frac{2\sqrt{EG}\,Nz}{1+z^2}, \qquad g' = g+\frac{2GN}{1+z^2},$$

where $z = \operatorname{ctn}\alpha$, and α has the same significance as before.

The Codazzi equations for S', after application of those for S, become

$$\frac{\partial(e'-e)}{\partial v} = f'\frac{\partial}{\partial u}\log\left(f'\sqrt{\frac{G}{E}}\right),$$

$$\frac{\partial(g'-g)}{\partial u} = f'\frac{\partial}{\partial v}\log\left(f'\sqrt{\frac{E}{G}}\right).$$

Proceeding as in § 2, we obtain finally the equations

$$(6) \qquad \frac{\partial z}{\partial u} = Az-B\sqrt{\frac{E}{G}}, \qquad \frac{\partial z}{\partial v} = Bz+A\sqrt{\frac{G}{E}},$$

---

* *Sur certaines surfaces dont les rayons de courbure sont liés par une relation,* BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 89 (1890-91), pp. 158-169.

where

$$
\begin{aligned}
A &= \frac{1}{N}\frac{\partial M}{\partial u} = \frac{\partial \log GN}{\partial u}, \\
B &= -\frac{1}{N}\frac{\partial M}{\partial v} = \frac{\partial \log EN}{\partial v}.
\end{aligned} \tag{7}
$$

But equations (6) are compatible if and only if

$$Pz = Q,$$

where

$$
\left\{
\begin{aligned}
P &= \frac{\partial A}{\partial v} - \frac{\partial B}{\partial u} = \frac{\partial^2}{\partial u \partial v}\log\frac{G}{E}, \\
Q &= \frac{\partial}{\partial u}\left(A\sqrt{\frac{G}{E}}\right) + \frac{\partial}{\partial v}\left(B\sqrt{\frac{E}{G}}\right) - A^2\sqrt{\frac{G}{E}} - B^2\sqrt{\frac{E}{G}}.
\end{aligned}
\right. \tag{8}
$$

THEOREM 3. *There are* ∞^1 *surfaces* S' *applicable to a given surface* S *with preservation of both curvatures if* $P = 0$, $Q = 0$; *there is a unique surface* S' *if* $P \neq 0$ *and* $z = Q/P$ *satisfies equations* (6). *Otherwise no surface* S' *exists.*

According to (8), the vanishing of P is the condition that S be an isometric surface. If, then, isometric parameters are introduced, so that $E = G = \lambda$, $Q = 0$ becomes the condition that $1/\lambda N$ be a harmonic function, provided that for A and B the second values given in (7) are used. If the first values are substituted instead, $Q = 0$ reduces to a condition more complicated in form, but involving merely the principal radii of curvature.

HARVARD UNIVERSITY

COMPLETE SETS OF REPRESENTATIONS OF TWO-ELEMENT ALGEBRAS*

BY B. A BERNSTEIN

1. *Introduction.* Any algebra consists of one or more classes of elements and one or more operations or relations among the elements. An operation $\oplus$ of a class K of two elements a, b is given by a $\oplus$-table of the form

$\oplus$	a	b
a	c_1	c_2
b	c_3	c_4

where the c's all belong to K, or do not all belong to K, according as the operation is K-closing or not. A relation R in K is given by an R-table of the form

R	a	b
a	$\pm$	$\pm$
b	$\pm$	$\pm$

where the sign $+$ indicates that aRb holds, the sign $-$ that aRb does not hold. The object of this paper is to give convenient representations of these $\oplus$-tables and R-tables, hence of all two-element algebras, and to point out the usefulness of these representations in connection with fundamental questions relating to postulate-sets.

2. *Representation of Class-closing Operations.* There are 2^4 operations $\oplus$ possible in a class of two elements when the condition of closure is satisfied. The following theorem gives us two sets of representations of these operations, one arithmetic and one boolean. In the boolean representation the symbols 0, 1, a', $a+b$, ab denote respectively

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* Read before the San Francisco Section of the Society April 7, 1923. The paper includes the substance of the following papers, presented before the San Francisco Section October 21, 1922:

(1) *An arithmetic representation of boolean logic,* (2) *Arithmetic independence systems for the Whitehead-Huntington postulates for boolean algebras,* (3) *A boolean representation of a number field.*

the logical "zero," the "whole," the negative of a, the logical sum of a and b, the logical product of a and b.

THEOREM A. *The* 2^4 *systems* $(K, \oplus)$ $(K\colon 0, 1;\ a \oplus b = 0, 1)$ *are equivalent to the arithmetic functions*

$$\text{(I)} \qquad A_1ab + A_2a + A_3b + A_4 \ \textit{modulo}\ 2, \qquad (A_i = 0, 1)$$

and also to the boolean functions

$$\text{(II)} \qquad Aab + Bab' + Ca'b + Da'b', \quad (A, B, C, D = 0, 1).$$

The theorem is true for the following reasons:

(1) Every system $(K, \oplus)$ of the theorem is equivalent to a $\oplus$-table of the form

$$\text{(III)} \qquad \begin{array}{c|cc} \oplus & 0 & 1 \\ \hline 0 & e_1 & e_2 \\ 1 & e_3 & e_4 \end{array} \qquad (e_i = 0, 1).$$

(2) Every function of form (I) or of form (II) determines a $\oplus$-table (III).

(3) Every $\oplus$-table (III) determines a function $f(a, b)$ of form (I) from the equations (modulo 2)

$$f(0, 0) = A_4, \qquad f(0, 1) = A_3 + A_4,$$
$$f(1, 0) = A_2 + A_4, \quad f(1, 1) = A_1 + A_2 + A_3 + A_4.$$

(4) Every $\oplus$-table (III) determines a function $F(a, b)$ of form (II) from the fact that

$$F(0, 0) = D, \quad F(0, 1) = C, \quad F(1, 0) = B, \quad F(1, 1) = A.$$

Table A gives the representations determined by Theorem A.

3. *Representation of Relations.* Since functions (I), as well as functions (II), of Theorem A are all distinct, we have the following theorem.

THEOREM B. *The* 2^4 *systems* (K, R) $(K : 0, 1;$ *R a dyadic relation*) *are equivalent to the arithmetic equations* (*modulo* 2)

$$\text{(I}'\text{)} \qquad A_1ab + A_2a + A_3b + A_4 = 0, \qquad (A_i = 0, 1),$$

and also to the boolean equations

$$\text{(II}'\text{)} \quad Aab + Bab' + Ca'b + Da'b' = 0, \quad (A, B, C, D = 0, 1).$$

Table B gives the representations determined by Theorem B.

TABLE A

REPRESENTATIONS OF SYSTEMS $(K, \oplus)$

$(K: 0, 1;\ a \oplus b = 0, 1)$

No.	$a \oplus b$	Arithmetic Representation	Boolean Representation
1	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}$	0	0
2	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$	$ab \quad (\mathrm{mod}\ 2)$	ab
3	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}$	$ab + b \quad (\mathrm{mod}\ 2)$	$a'b$
4	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 1 \end{array}$	b	$ab + a'b = b$
5	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}$	$ab + a \quad (\mathrm{mod}\ 2)$	ab'
6	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 1 \end{array}$	a	$ab + ab' = a$
7	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$	$a + b \quad (\mathrm{mod}\ 2)$	$ab' + a'b$
8	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}$	$ab + a + b \quad (\mathrm{mod}\ 2)$	$ab + ab' + a'b = a + b$
9	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}$	$ab + a + b + 1 \quad (\mathrm{mod}\ 2)$	$a'b'$
10	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$	$a + b + 1 \quad (\mathrm{mod}\ 2)$	$ab + a'b'$
11	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 0 \end{array}$	$a + 1 \quad (\mathrm{mod}\ 2)$	$a'b + a'b' = a'$
12	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}$	$ab + a + 1 \quad (\mathrm{mod}\ 2)$	$ab + a'b + a'b' = (ab')'$
13	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 1 & 0 \end{array}$	$b + 1 \quad (\mathrm{mod}\ 2)$	$ab' + a'b' = b'$
14	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 1 & 1 \end{array}$	$ab + b + 1 \quad (\mathrm{mod}\ 2)$	$ab + ab' + a'b' = (a'b)'$
15	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}$	$ab + 1 \quad (\mathrm{mod}\ 2)$	$ab' + a'b + a'b' = (ab)' = a' + b'$
16	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}$	1	1

TABLE B

REPRESENTATIONS OF SYSTEMS (K, R)

(K: 0, 1; R A DYADIC RELATION)

<table>
<tr><th>No.</th><th>$a\ R\ b$</th><th>Arithmetic Representation</th><th>Boolean Representation</th></tr>
<tr><td>1</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & - & - \end{array}$</td><td>$1=0$</td><td>$1=0$</td></tr>
<tr><td>2</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & - & + \end{array}$</td><td>$ab+1=0 \pmod 2$</td><td>$ab'+a'b+a'b'=(ab)'$
$=a'+b'=0$</td></tr>
<tr><td>3</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & - & - \end{array}$</td><td>$ab+b+1=0 \pmod 2$</td><td>$ab+ab'+a'b'=(a'b)'=0$</td></tr>
<tr><td>4</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & - & + \end{array}$</td><td>$b+1=0 \pmod 2$</td><td>$ab'+a'b'=b'=0$</td></tr>
<tr><td>5</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & + & - \end{array}$</td><td>$ab+a+1=0 \pmod 2$</td><td>$ab+a'b+a'b'=(ab')'=0$</td></tr>
<tr><td>6</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & + & + \end{array}$</td><td>$a+1=0 \pmod 2$</td><td>$a'b+a'b'=a'=0$</td></tr>
<tr><td>7</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & + & - \end{array}$</td><td>$a+b+1=0 \pmod 2$</td><td>$ab+a'b'=0$</td></tr>
<tr><td>8</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & + & + \end{array}$</td><td>$ab+a+b+1=0 \pmod 2$</td><td>$a'b'=0$</td></tr>
<tr><td>9</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & + & - \\ 1 & - & - \end{array}$</td><td>$ab+a+b=0 \pmod 2$</td><td>$ab+ab'+a'b=a+b=0$</td></tr>
<tr><td>10</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & + & - \\ 1 & - & + \end{array}$</td><td>$a-b=0 \pmod 2$</td><td>$ab'+a'b=0$</td></tr>
<tr><td>11</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & - & - \end{array}$</td><td>$a=0$</td><td>$ab+ab'=a=0$</td></tr>
<tr><td>12</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & - & + \end{array}$</td><td>$ab+a=0 \pmod 2$</td><td>$ab'=0$</td></tr>
<tr><td>13</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & + & - \\ 1 & + & - \end{array}$</td><td>$b=0$</td><td>$ab+a'b=b=0$</td></tr>
<tr><td>14</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & + & - \\ 1 & + & + \end{array}$</td><td>$ab+b=0 \pmod 2$</td><td>$a'b=0$</td></tr>
<tr><td>15</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & + & - \end{array}$</td><td>$ab=0 \pmod 2$</td><td>$ab=0$</td></tr>
<tr><td>16</td><td>$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & + & + \end{array}$</td><td>$0=0$</td><td>$0=0$</td></tr>
</table>

4. *Representation of Operations that are not Class-closing.* Representations of operations $\oplus$ that do not satisfy the condition of closure are given by the following theorem.

THEOREM C. *Let* $\sum \equiv (K, \oplus)$ *be a system of two elements* 0, 1, *with* $\oplus$ *such that* $a \oplus b = x$[*] *not in* K *for some values of* a, b; *let* $f(a, b)$ *be an arithmetic function equivalent* (*Theorem* A) *to a system obtained from* $\sum$ *by replacing the* x*'s by* 0 *or* 1; *and let* $\Phi(a, b)$ *be an arithmetic function equivalent to the system obtained from* $\sum$ *by replacing* (1) *the* x*'s by* 0 *and* (2) *the non-*x*'s by* 1; *then the function*

$$F(a, b) \equiv f(a, b) + \frac{0}{\Phi(a, b)}$$

is an arithmetic equivalent of $\sum$.

For $F(a, b) = f(a, b) + (0/1)$ when a, b are such that $a \oplus b = 0$ or 1 in $\sum$; and $F(a, b) = f(a, b) + (0/0)$ when a, b are such that $a \oplus b = x$ in $\sum$.

By defining a/b as the (unique) element y which satisfies the boolean equation $by = a$, we can obtain boolean representations of systems $\sum$ of Theorem C. Since 0/0 is not an element of a boolean algebra, we have

THEOREM C'. (The same as Theorem C with the word *arithmetic* everywhere replaced by the word *boolean.*)

Thus, to find a representation of the system $\sum \equiv (K, \oplus)$ determined by the table

$$\begin{array}{c|cc} \oplus & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & x \end{array} \qquad (x \text{ not in } K),$$

take from Table A system 7[†] for $f(a, b)$ and system 15 for $\Phi(a, b)$. An arithmetic representation of $\sum$ is then

$$a + b \pmod 2 + \frac{0}{ab + 1 \pmod 2},$$

---

[*] I am using the expression "$a \oplus b = x$ not in K" for some values a, b, to mean that for the values a, b in question $a \oplus b$ is meaningless.

[†] Or also system 8.

and a boolean representation is

$$ab' + a'b + \frac{0}{a' + b'}.$$

5. *Applications.* The representations of tables A, B obviously have a bearing on postulational questions. I point out here a few facts obtained with the help of the tables.

(1) A two-element boolean algebra $(K, \oplus, \odot)$ may be represented by an arithmetic system; namely, by

$$K{:}\,0, 1;\quad a \oplus b = ab + a + b \pmod 2;\quad a \odot b = ab \pmod 2,$$

or, dually, by

$$K{:}\,0, 1;\quad a \oplus b = ab \pmod 2;\quad a \odot b = ab + a + b \pmod 2.$$

(2) The two-element boolean algebra $(K, <)$, expressed in terms of the relation $<$ of inclusion, may be represented by the arithmetic system

$$K{:}\,0, 1;\quad a < b = ab + a = 0 \pmod 2.$$

(3) Each of the $\oplus$-tables 7 and 10 of Table A satisfies the conditions for an abelian group.* Hence, the logical elements 0, 1 form an abelian group with respect to each of the operations

$$ab' + a'b, \qquad ab + a'b'.\dagger$$

(4) In Table A, $\oplus$-tables 7 and 2 together define a field $(K, \oplus, \odot)$.‡ Hence the logical elements 0, 1 form a field with respect to the pair of operations $ab' + a'b$, ab, and also with respect to the dual pair $ab + a'b'$, $a + b$.

(5) All boolean operations can be defined in terms of each of the operations $a'b'$, $a' + b'$.§ Hence all the arith-

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* For postulates for an abelian group see TRANSACTIONS OF THIS SOCIETY, vol. 4 (1903), p. 27.

† It can be shown that the totality of the elements of *any* boolean algebra, finite or infinite, form an abelian group with respect to each of these operations.

‡ For postulates for fields see TRANSACTIONS OF THIS SOCIETY, vol. 4 (1903), p. 31.

§ See Sheffer's postulates for boolean algebras, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), p. 481.

metic operations of Table A can be defined in terms of each of the operations $ab+a+b+1 \cdot (\mathrm{mod}\ 2)$, $ab+1\ (\mathrm{mod}\ 2)$. And any two-element algebra can be defined by means of an appropriate postulate-set involving the single operation 9 or the single operation 15.

(6) Huntington's abstract systems (K, $\oplus$, $\odot$) proving the independence of his first set of postulates for boolean algebras* may be replaced by the arithmetic systems† of the following table (Systems I'_a, I'_b, ..., VI$'$ are independence systems respectively for postulates I_a, I_b, ..., VI).

TABLE C

	K	$a \oplus b$	$a \odot b$
I'_a	0,1	$a+b\,(\mathrm{mod}\,2)+\dfrac{0}{ab+1\,(\mathrm{mod}\,2)}$	ab
I'_b	0,1	ab	$a+b\,(\mathrm{mod}\,2)+\dfrac{0}{ab+1\,(\mathrm{mod}\,2)}$
II'_a	0,1	0	ab
II'_b	0,1	ab	0
III'_a	0,1	a	ab
III'_b	0,1	ab	a
IV'_a	0,1	$a+b\ (\mathrm{mod}\ 2)$	ab
IV'_b	0,1	ab	$a+b\ (\mathrm{mod}\ 2)$
V$'$	0,1	$ab+a+b\ (\mathrm{mod}\ 2)$	$ab+a+b\ (\mathrm{mod}\ 2)$
VI$'$	0	$0+0$	$0 \cdot 0$

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* See TRANSACTIONS OF THIS SOCIETY, vol. 5 (1904), p. 288.

† With the help of Table A we can, of course, write down boolean systems for Huntington's abstract systems.

BROUWER'S CONTRIBUTIONS TO THE FOUNDATIONS OF MATHEMATICS*

BY ARNOLD DRESDEN

1. *Introduction.* In a number of papers, published from 1907 on, Professor L. E. J. Brouwer, of the University of Amsterdam, has developed ideas which affect the foundations of mathematics in a fundamental way. While some of his papers are readily available to American mathematicians,† there are several others which are less accessible. On account of its critique of some of our most fundamental concepts and methods, the position of Brouwer may have a far reaching effect upon the future development of mathematics. In his *Begründung der Mengenlehre,* he has made a beginning with a revision of a basic field of modern mathematics in accordance with his point of view. But, whatever their ultimate significance may be, the conclusions which Brouwer reaches are certainly interesting. Moreover, they are indispensable as a background for an appreciation of his *Begründung der Mengenlehre,* as well as for understanding the controversial discussions on the foundations of mathematics of Weyl and Hilbert.‡

For this reason, it has seemed worth while to present Brouwer's most important ideas concerning the foundations of mathematics to American readers. For this purpose, we have used, besides the material referred to above, his book on the foundations of mathematics, (*Over de Grondslagen der Wiskunde,* Amsterdam, Maas & van Suchtelen, 1907) and his article on the unreliability of logical principles (*De onbetrouwbaarheid der logische principes,* TIJDSCHRIFT VOOR WIJSBEGEERTE, vol. 1 (1908)). Our discussion falls into three parts, viz., mathematics and experience, mathematics and mathematical language, mathematics and logic.

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* Presented to the Society, December 29, 1923.

† See *Intuitionism and formalism,* this BULLETIN, vol. 20 (1913), p. 81; Review of Schoenflies-Hahn, *Die Entwickelung der Mengenlehre,* JAHRESBERICHT DER VEREINIGUNG, vol. 23 (1914), p 78; *Intuitionistische Mengenlehre,* JAHRESBERICHT DER VEREINIGUNG, vol. 28 (1920), p. 203; *Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten,* Amsterdam, 1918–19.

‡ See Weyl, *Über die neue Grundlagenkrise der Mathematik,* MATHEMATISCHE ZEITSCHRIFT, vol. 10 (1921), p. 39; Hilbert, *Neubegründung der Mathematik,* ABHANDLUNGEN DER HAMBURGISCHEN UNIVERSITÄT, vol. 1 (1922), p. 157, and *Die logischen Grundlagen der Mathematik,* MATHEMATISCHE ANNALEN, vol. 88 (1922), p. 151.

2. *Mathematics and Experience.* Brouwer conceives of mathematical thinking as a process of construction, which builds its own universe, independent of the universe of our experience, somewhat as a free design, under the control of nothing but arbitrary choice, restricted only in so far as it is based upon the fundamental mathematical intuition. This intuition, upon which not only mathematical thinking, but all intellectual activity is held to be based, is found in the abstract substratum of all observation of change, "a fusion of continuous and discrete, a possibility of conceiving simultaneously several units, connected by a '*between*' that cannot be exhausted by the interpolation of new units."

It is not to be expected that all mathematicians will agree with this point of view. It is in this conception of the source and of the character of mathematical thinking that the ideas of Brouwer have their root. Its relation to the thought of Plato* and of other Greek philosophers, interesting as it is, must be left untouched here except for the observation that the acceptance of this union of discrete and continuous as the rock bottom of mathematical thinking disposes of the paradoxes of Zeno and of the conflicts of Parmenides somewhat as the theory of relativity disposed of the drag of the ether.†

In a similar way it disposes of many questions in point set theory, which have occupied the attention of mathematicians. For, by combining continuous and discrete in one fundamental concept, it renders futile all attempts at building up one of these by means of the other, conceived as independent of the first. But these matters have been dealt with extensively in *Intuitionism and Formalism.*

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* See, e. g., Brunschvicg, *Les Étapes de la Philosophie Mathématique,* p. 49 et seq.

† Compare Brunschvicg, loc. cit., p. 155.

The fundamental intuitive concept of mathematics must not be thought of as in the nature of an undefined idea, such as occur in postulational theories, but rather as something in terms of which all undefined ideas which occur in the various mathematical systems are to be intuitively conceived, if they are indeed to serve in mathematical thinking. It manifests itself in the intuition of time, which makes possible "repetition, as being object in time and again object."

The position of Brouwer on this point is directly opposed to those of Kant and of Russell, who hold that mathematical thinking cannot be based on the one-dimensional time continuum alone, but that it requires also three-dimensional euclidean space (Kant, *Transcendental Ethics*), or projective space (Russell, *Foundations of Geometry*).

In the first chapter of *Over de Grondslagen der Wiskunde,* Brouwer constructs on the basis of this fundamental intuition the order types ω and η, and the elementary propositions of algebra and geometry. In these building processes, experience plays no part, and Brouwer holds that "in this constructive process, bound by the obligation to notice with care which theses are acceptable to the intuition and which are not, the only possible foundation for mathematics is to be looked for."*

It must of course be remembered that these statements concerning the role of experience are to be taken in the philosophical sense, not in the historical sense. For no one could deny that in the historical development of mathematics, experience played a permanent part. On the other hand, one will have to admit that, while "experimental *science* is linked up with mathematics, experience can never force the choice of a *particular* mathematical system." The question as to the role of experience in the development of mathematics seems to the present author still to have a good many aspects that call for further study. Enough has been said however to indicate Brouwer's fundamental thesis and to discuss some of its important consequences.

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* It may be of interest to compare with this statement, the one found on p. 50 of Boutroux, *L'Idéal Scientifique des Mathématiciens,* in a discussion of the Hellenic conception: "And if it frequently happens that we make mistakes, it is because we have obscured our vision by insufficient exercise of our intuitive faculty."

Two other points need however still to be made clear. It must already have become evident to the reader that Brouwer is not seeking to build up a system of postulates for mathematics, either in whole or in part. The freedom of choice in the construction of mathematics, once the fundamental intuition is recognized, leaves the way open for setting up various postulate systems for any part of mathematics or for the whole science. This seems indeed to be the most reasonable attitude towards postulational theory. Each system of postulates for a particular field of knowledge is to be looked upon as a set of pronouncements in terms of undefined ideas, which are verifiable in that particular field, if the undefined ideas are suitably particularized. In the measure in which these postulates are independent they enlarge our knowledge of the structure of the field; in the measure in which they are non-categorical, they establish relations with other fields.

In the second place, even though constructed without any direct interplay of experienced reality, mathematics is not without value for practical life. Because through the agency of mathematical building, phenomena are linked together in causal sequences, which enable man to control the external world. "The conduct of man aims to observe as many as possible of these mathematical sequences, in order that, whenever an earlier element in such a sequence offers in actuality a better opportunity for taking hold of the situation than a later element in the same sequence, even though only the later one appeals to his instincts, he may choose the earlier one as the object of his acts." The mathematical universe thus becomes an accompaniment of the phenomenal universe, which assists man in his control of the latter.

3. *Mathematics and Mathematical Language.* The relation of Brouwer's thought to the Platonic point of view, hinted at above, is brought out once more in his insistent separation of mathematics from the language of mathematics. In Book VII of *The Republic*, Socrates is made to say, that "on one point at any rate we shall encounter no opposition from those who are even slightly acquainted with geometry, when we assert that this science holds a position which flatly contradicts the language employed by those who handle it." In quoting this passage and in commenting upon it, Boutroux gives more emphatic utterance to its thought: "It is known indeed, that the Platonists established a profound distinction between 'discourse' and 'intelligence', between written science, which is a didactic exposition of truths already known, and the *conception* of scientific truths, which is the direct product of our faculty of intuition in its dealing with the world of ideas."* For Brouwer, mathematics is a process of construction, and "of the mathematical building and reasoning, and in particular of the logical reasoning which men do within themselves, they try to evoke copies in other men by means of sounds and symbols, which also serve to aid their own memory." It seems that creative mathematicians cannot but receive with approval Brouwer's remark, that "in arguments concerning experiential realities, fitted into mathematical systems, logical principles are not the guide, but rather a regularity observed a posteriori in the accompanying language; and if one speaks in accordance with this regularity, but detached from mathematical systems, there is always a danger of paradoxes, like that of Epimenides." Mathematical proof without the use of words consists in establishing relations between different parts of the mathematical edifice, i. e. "when mathematical objects are given by means of their relations to elements or fragments of a mathematical edifice, one transforms these relations by a series of tautologies and thus one progresses step by step to the relations of the objects with other parts of the edifice."

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* P. Boutroux, loc. cit., p. 35.

It is only in the language which accompanies this process for purposes of communication and memory, that logical forms arise. "The words of your mathematical demonstration are but the accompaniment of wordless mathematical building", Brouwer says to the logician, "and when you establish a contradiction, I simply observe that the construction cannot go on, that in the given edifice no room can be found for the posited structure. And when I make this observation I do not think of the Law of Contradiction." It is clear that the role of logic, as here conceived, is very different from the one usually attributed to it.

Before taking up more fully Brouwer's views of the relation of mathematics to logic, it will be of interest to insert the following passage: "The mathematical fact is independent of the logical or algebraic dress in which we seek to represent it; indeed, the idea which we have of it is richer and fuller than all the definitions which we can give of it, than all the forms or combinations of signs or of propositions by means of which we can express it. The expression of a mathematical fact is arbitrary, conventional. But the fact itself, that is to say, the truth which it contains, forces itself upon our mind apart from all conventions. Thus, one could not account for the development of mathematical theories, if one tried to consider the algebraic formulas and the logical combinations as the objects whose study the mathematician pursues. However, all the characteristics of these theories are easily explained, once one admits that the algebra and the logical propositions are but the language into which one *translates* a set of ideas and of objective facts."*

4. *Mathematics and Logic.* Indeed, Aristotelian logical reasoning is but a special kind of mathematical reasoning, namely that kind which is "concerned exclusively with relations of 'whole and part'." And the language which accompanies such logical reasoning is the *language of logical reasoning,* just as *mathematical language* is that which

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* P. Boutroux, loc. cit., p. 203.

accompanies mathematical reasoning. Furthermore, these languages, themselves, like other parts of the phenomenal world can become the object of mathematical observation and study; thus arise *theoretical logic* as the mathematics of the language of logical reasoning, and *logistics* as the mathematics of the language of mathematical reasoning.

In view of these characterizations, it is not surprising to find little sympathy with the attempts to lay down logical foundations for mathematics. "A logical building up of mathematics, independent of the mathematical intuition is impossible — because in this way we obtain but a verbal edifice irrevocably apart from mathematics proper — and moreover a contradiction in terms, because a logical system, as well as mathematics itself, requires the fundamental intuition of mathematics."

The fundamental difference between the point of view of Brouwer concerning the nature of mathematics and that of Hilbert, as expressed in the latter's *Neubegründung*, referred to above, comes out clearly in the following sentences: "Suppose we have proved by some method or other, without having a mathematical interpretation in mind, that a logical system built up on the basis of some verbal axioms, is non-contradictory, i. e., that at no point of the development of the system two contradictory propositions will arise; and suppose that we then find a mathematical interpretation of the axioms, (which consists of requiring a construction of a mathematical edifice from elements which satisfy given mathematical relations). Does it then follow from the non-contradictoriness of the logical system that such a mathematical structure exists? No such thing has ever been proved by the postulationists" . . . "so, e. g., it has nowhere been proved, that if a finite number must satisfy a set of conditions which can be shown to be non-contradictory, that then this number actually exists."* If we compare this paragraph with Hilbert's system of undefined ideas and of postulates for mathematics, one is reminded of the phrase of Poincaré, quoted elsewhere by Brouwer,* "Les hommes ne s'entendent pas, parce qu'ils ne parlent pas la même langue et qu'il y a des langues qui ne s'apprennent pas."

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* *Grondslagen*, p. 141.

Concerning the logical paradoxes which have disturbed some mathematicians during the last twenty-five years and the attempts to resolve them by means of more refined logical methods, Brouwer holds that "they arise whenever regularity in the language which accompanies mathematics is extended so as to apply also in a language of mathematical words, which do not accompany mathematics." Moreover, "logistics concerns itself with mathematical language, instead of with mathematics, and consequently does not clarify mathematics; finally all paradoxes disappear if one restricts oneself to dealing with systems that are explicitly constructible on the basis of the fundamental intuition," i. e., if one gives priority to mathematics instead of to logic. This aspect of Brouwer's position again finds support elsewhere: "In other words, the most important advances which mathematicians make, are obtained not in perfecting the form, but in modifying the basis of the theory. These advances cannot be regarded as being of logical character."† . . . "In order to give mathematical theories a firm structure, we have decided to give them the form of logical systems; but, observing that these systems are artificial and can moreover be infinitely diversified, we realize that they neither constitute the whole of mathematics, nor its principal part. Behind the logical form there is something else."‡

But Brouwer does not merely indulge in a general criticism of the role of logic in mathematics; he proceeds to a discussion of the profoundly important question: "Is it allowable, in dealing with purely mathematical constructions and transformations temporarily to neglect the idea of the constructed mathematical system and to work with the accompanying verbal structure, guided by the principles of the *syllogism*, of *contradiction,* and of the *excluded middle,* confident that by evoking temporarily the idea of the reasoned mathematical constructions, every part of the argument could be validated in turn?"*

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* See this Bulletin, vol. 20 (1913), p. 96.

† P. Boutroux, loc. cit., p. 168.

‡ Ibid. p. 170.

It should be clear that for the actual work of mathematical research this question, once one adopts Brouwer's conception of the character of mathematical thinking, is of primary importance. And his answer is *Yes,* as concerns the principles of the syllogism and of contradiction, but *No* for the law of the excluded middle. While the law of contradiction asserts that it is impossible for a proposition to be both true and false, the law of the excluded middle (L. E. M.) says that every proposition is either true or false. Its acceptance leads therefore to a belief in the solvability of every mathematical problem.† For, from Brouwer's point of view, this principle asserts that "for every hypostatized fitting into each other of systems in a definite way, either the actual construction can be made, or an insurmountable obstruction can be erected." If the proposition deals with fragments of a finite, definite, discrete system, this possibility will readily be granted, so that the L. E. M. may be considered as valid in dealing with such cases. For instance, of two positive integers, it can be affirmed that either they are relatively prime, or they possess a common divisor different from unity.

But the situation becomes different when we are dealing with infinite systems. Propositions concerning infinite systems can be dealt with systematically only when the use of complete induction is possible; in such a case the infinite system can be fitted in by the use of properties of an *arbitrary element.* On the other hand, the totality of the mathematical properties and contradictions derivable by means of complete induction forms what Brouwer calls a "denumerably unfinished set," i. e., a set of which nothing but a denumerable subset can ever be explicitly exhibited, and such that whenever a denumerable subset is given, a new element of the set can always be derived from it by means of a previously defined process.* The possibility of systematically establishing the truth or falsity of a proposition concerning an arbitrarily proposed infinite system depends therefore upon finding among the denumerably unfinished set of mathematical properties and contradictions one which (eventually by means of complete induction, i. e. "by means of an element invariant over a denumerably infinite sequence") enables us to put the proposition as one which can be dealt with and decided, one way or the other, by the use of complete induction.

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* A statement of the laws of thought will be found in any text on formal logic; see, e. g. Jevons, *Elementary Lessons in Logic,* p. 117.

† Compare Hilbert, *Mathematische Probleme,* GOTTINGER NACHRICHTEN, 1900.

But the search for such a structure of property or contradiction cannot be carried out systematically; hence its success depends more or less upon good luck and cannot be assured a priori. Hence it is uncertain whether for an arbitrary proposition concerning a given infinite system either the construction or the obstruction can be established, and hence it is equally uncertain whether the L. E. M. is valid in such a case. But, still further, unjustified assumption that one or the other must be possible can never be detected; for that would mean that both the hypothesis of construction and that of obstruction would lead to an obstruction in the further process of construction, which conflicts with the law of contradiction.

It is on the basis of these considerations that Brouwer denies unlimited validity to the L. E. M., and that he reaches the following conclusion: "In mathematics, it is not certain whether or not all logic is permissible, and it is not certain whether it can be decided, whether or not all logic is permissible."

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* This notion, of which the set of well-ordered ordinals is an example, plays an important part in much of Brouwer's work.

A QUALITATIVE DEFINITION OF THE POTENTIAL FUNCTIONS*

BY PHILIP FRANKLIN

1. *Introduction.* In this paper we aim to set up postulates completely characterizing the potential functions, which do not involve derivatives or integrals, and are thus of a more qualitative nature than the definitions previously given. Incidentally, we may interpret all of our postulates as statements of properties of such physical quantities as give rise to potential functions, and when so interpreted, we see from physical grounds that they hold for the quantities in question. They thus furnish a means of going directly from certain physical problems to the potential functions, without the use of Laplace's equation. Our results will be stated in full only for potential functions of two and three variables, although they may evidently be extended to the n-dimensional case.

2. *Postulates for Two Dimensions.* Consider a class of functions of two variables, x and y, thought of for convenience as Cartesian coordinates, and let each function have associated with it a region R of the plane. Our assumptions are:

(1) Each function is continuous in both variables at all interior points of its region R.

(2) If R_1 and R_2, the regions for two functions of the class $f_1(x, y)$ and $f_2(x, y)$, have a region R_3 in common, then any linear combination of these functions, such as $Af_1(x, y) + Bf_2(x, y)$, is a function of the class, whose region contains all the points of R_3.

(3) If an orthogonal transformation of the variables (i. e., a change of Cartesian axes) converts the function $f(x, y)$ with region R into $F(x', y')$, this latter function is a member of the class, its region being the region R expressed in the new variables.

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* Presented to the Society, April 28, 1923.

(4) The function $f(x, y) = 1$ is a function of the class, its region R being the entire plane.

(5) If a given circle lies entirely inside each region R_i, belonging to $f_i(x, y)$, these $f_i(x, y)$ constituting an infinite sequence of functions of the class, and if, for points x_c, y_c on the circumference of the circle $\lim_{i \to \infty} f_i(x_c, y_c) = 0$, the limit being approached uniformly over the circumference, then the sequence of values of the functions $f_i(x, y)$ at the center of the circle cannot approach a limit different from zero.

We shall show that any family of functions satisfying these five postulates is necessarily a class of potential functions, each function being harmonic at any interior point of its region R. Since every linear class of harmonic functions satisfies the above postulates, the widest class of functions satisfying them is the totality of harmonic functions.

By way of motivating our choice of postulates, we note that postulates 1, 2, and 4 are natural requirements since the functions we are trying to define satisfy a linear homogeneous partial differential equation. Postulate 3 is allied to the fact that the Laplacian operator is the only linear differential operator of the second order invariant under orthogonal transformations of coordinates. The last postulate is added as a fairly weak condition which completes the characterization, since the preceding ones are not by themselves sufficient for this.

It is illuminating to consider the significance of the postulates for a physical example, say the functions giving the temperature of points of a set of thin plates, when various boundary temperatures are assigned. The postulates state, essentially, that these functions are continuous point functions, as a class independent of the choice of coordinates, which possess the property of combination by superposition; that if the boundary temperatures are constant, the temperature inside will be this same constant, and finally that if the temperatures at the circumference of a circular plate are altered so that they approach zero uniformly, the temperature at the center of the circular plate will approach zero.

3. *Deduction of the Mean-Value Property.* We shall now prove that the class of functions which satisfy the postulates of § 2 have the mean-value property, that is, if x_0, y_0 is the center of a circle C lying entirely inside of R, the region for $f(x, y)$, any function of the class, then $f(x_0, y_0)$, its value at the center of C is equal to the average value of $f(x, y)$ taken over the circumference of C.

To show this, we first form a series of positive constants ε_i such that $\lim_{i\to\infty} \varepsilon_i = 0$. We select a particular ε_i, and divide the circumference of the circle C into N_i equal parts, taking N_i so large that the oscillation of $f(x, y)$ on any one part is less than ε_i. This is possible since postulate 1 makes $f(x,y)$ continuous, and therefore uniformly continuous on the circumference of C. We next form $N_i - 1$ new functions, existing in and on C, obtained from $f(x, y)$ by rotating the axes about the center of C through an angle $2k\pi/N_i$ $(0 < k < N_i)$. These functions, by postulate 3, are members of the class, as is also the function $f_i(x, y)$, defined as the sum of these $N_i - 1$ functions and the original function, divided by N_i, by postulate 2. Finally, from postulates 2 and 4, we see that the function $F_i(x, y) = f_i(x, y) - f_c$, where f_c denotes the average value of the original function on the boundary of C, is also in the class.

The function $F_i(x, y)$ satisfies the inequality

$$(1) \qquad |F_i(x_c, y_c)| < \varepsilon_i,$$

where x_c, y_c denote any point on C. For, if p_k is the average value of $f(x, y)$ in the kth subdivision of C, we have

$$(2) \qquad f_c = \sum_{k=1}^{N_i} p_k/N_i,$$

and from the choice of N_i, for x_c, y_c in the kth subdivision

$$(3) \qquad |f(x_c, y_c) - p_k| < \varepsilon_i.$$

Recalling the method of forming $f_i(x. y)$, we deduce from (2) and (3) that

$$(4) \qquad |f_i(x_c, y_c) - f_c| < \varepsilon_i,$$

for all points on the circumference of C, and this is equivalent to (1) above.

Moreover, at x_0, y_0, the center of the circle C, we have

(5) $$F_i(x_0, y_0) = f_i(x_0, y_0) - f_c = f(x_0, y_0) - f_c.$$

Thus the values of the functions $F_i(x, y)$ at x_0, y_0 are the same for all values of i, and consequently approach a limit when i becomes infinite. But from (1) and postulate 5, this limit must be zero, and (5) gives

(6) $$f(x_0, y_0) = f_c,$$

which is the mean-value property.

4. *The Harmonic Character of the Functions.* We may now readily prove that the functions of our class are harmonic at all interior points of their regions R. For, with any interior point of a region R as center, let us draw a circle lying wholly inside R, the region for the function $f(x, y)$. We also form, by Poisson's integral or otherwise, a harmonic function $h(x, y)$ having the same values along the circumference of the circle as the given function $f(x, y)$. Since both $h(x, y)$ and $f(x, y)$ possess the mean-value property, their difference $f(x, y) - h(x, y)$ will also possess it. Consequently this function takes on its maximum and minimum values on the circumference of the circle; and since it is zero on the circle, it must be zero identically. Hence $f(x, y)$ agrees with $h(x, y)$ in a neighborhood of the center of the circle and is harmonic at this point.

5. *Postulates for Three Dimensions.* It is fairly obvious what revisions we must make in the postulates of § 2 to make them applicable to potential functions in three-dimensional space. Here we deal with a class of functions of three variables, or Cartesian coordinates, x, y and z, where each function has associated with it a three-dimensional region R, of space. Our assumptions now are:

(1) Each function is continuous in all three variables at all interior points of its region R.

(2) If R_1 and R_2, the regions for two functions of the class $f_1(x, y, z)$ and $f_2(x, y, z)$, have a region R_3 in common, then any linear combination of these functions, such as $Af_1(x, y, z) + Bf_2(x, y, z)$, is a function of the class, whose region contains all the points of R_3.

(3) If an orthogonal transformation of the variables (i. e., a change of Cartesian axes) converts the function $f(x, y, z)$ with region R into $F(x', y', z')$, this latter function is a member of the class, its region being the region R expressed in the new variables.

(4) The function $f(x, y) = 1$ is a function of the class, its region R being the whole of space.

(5) If a given sphere lies entirely inside each region R_i, belonging to $f_i(x, y, z)$, these $f_i(x, y, z)$ constituting an infinite sequence of functions of the class, and if, for points x_s, y_s, z_s on the surface of the sphere $\lim_{i \to \infty} f_i(x_s, y_s, z_s) = 0$, the limit being approached uniformly over the surface; then the sequence of values of the functions $f_i(x, y, z)$ at the center of the sphere cannot approach a limit different from zero.

Any class of functions satisfying these five postulates is, as we shall show presently, a class of Newtonian potential functions. Consequently the widest class of functions satisfying them is the totality of Newtonian potential functions. The remarks made about our earlier set of postulates at the end of § 2 apply here, mutatis mutandis.

Our reason for writing out the postulates at length and carrying out the proof of their sufficiency for the three-dimensional case, is that the transition to the three-dimensional case requires an essential modification in the proof. For, in the earlier proof of § 3, use was made of a series of regular polygons approximating a circle. As no analogous configuration exists in space, we must resort to a property of functions on a sphere, to which we next proceed.

6. *Functions on a Sphere.* If we are given a function on a sphere, we may form a new function from it by the following process. We select a particular diameter of the sphere as an *axis*, and so determine the second function that its value is constant on every circle on the sphere whose plane is perpendicular to the axis, and is equal to the average of the original function on this same circle. We define this process as that of *averaging the original*

function about an axis. We may now state the following lemma.

LEMMA. *If a continuous function on a sphere is averaged about one axis, the resulting function about a second axis perpendicular to the first one, this third function about the first axis, and so on, indefinitely, using the two axes alternately, the sequence of functions so obtained will approach a limit. Furthermore, this limit will be constant over the whole sphere, and equal to the average of the original function over the sphere.*

To prove this, we first observe from the continuity of the function that it possesses a maximum and a minimum value, which are actually reached, since the surface of the sphere is closed. Also, from the nature of the averaging process, the successive functions are all continuous, and the maximum values M_i form a never increasing sequence, while the minimum values m_i form a never decreasing one. Hence these quantities approach limits, which we denote by M and m respectively.

Another consequence of the character of the averaging process is that if we determine a positive δ such that the oscillation of our original function is less than a preassigned ε in every circle on the sphere of radius δ, which we may do since this function is continuous and therefore uniformly continuous on the sphere, the same δ will retain its relation to ε after the averaging, and hence for all subsequent functions in the series. This follows from the fact that the oscillation in a circle of radius δ after averaging cannot exceed the maximum oscillation in the set of circles of equal radius having their centers in the plane perpendicular to the axis containing the center of the original circle.

Let us now select an ε and determine a δ in the way described above, and let us set

$$\eta = \frac{2\varepsilon\delta}{\pi r}, \tag{7}$$

where r is the radius of the sphere. We next consider the result of averaging the original function i times, taking i so great that F_i, the resulting function, has its maximum and minimum values each within η of their respective limits, i. e.,

$$(8) \qquad \begin{cases} M \leqq M_i \leqq M + \eta, \\ m - \eta \leqq m_i \leqq m. \end{cases}$$

Let E be the constant value of the function F_i on the equator, i. e., on the great circle perpendicular to the axis about which we have just averaged, and consequently having as one diameter the other axis which we must use to get F_{i+1}. From the way we selected δ, F_i cannot differ from E by more than ε in the zone of width 2δ bounded by small circles parallel to and at distances δ above and below the equator. Hence, if we construct two lunes lying entirely in this zone, each of angular width $2\delta/r$, bounded by planes through the new axis, the value of F_i will be at most $E+\varepsilon$ inside these lunes. The remaining portion of the sphere consists of two lunes each of angular width $\pi - 2\delta/r$, and in these lunes, the function is at most M_i or $M+\eta$ by (8). Consequently, if we average once more to get F_{i+1}, since this function cannot exceed the average of the upper bounds in the lunes in question, we must have

$$(9) \qquad F_{i+1} \leqq \frac{\frac{4\delta}{r}(E+\varepsilon) + \left(2\pi - \frac{4\delta}{r}\right)(M+\eta)}{2\pi}.$$

Since F_{i+1} is at least as great as M at some point, this implies that

$$(10) \qquad M \leqq M + \eta + \frac{2\delta}{\pi r}(E + \varepsilon - M - \eta),$$

from which, using (7), we get

$$(11) \qquad E \geqq M + \eta - 2\varepsilon \geqq M - 2\varepsilon.$$

By an entirely analogous argument, using the minimum values of F_i and F_{i+1}, we show that

$$(12) \qquad E \leqq m + 2\varepsilon.$$

These last two equations give

$$(13) \qquad E + 2\varepsilon \geqq M \geqq m \geqq E - 2\varepsilon,$$

which shows that $M = m$, since ε can be made arbitrarily small.

Hence the value of the function at all points approaches this constant value, and since the averaging process leaves the average of the function taken over the entire sphere unchanged, the constant must be equal to this average, or the average of the original function over the sphere.

7. *Deduction of the Mean-value Property.* We are now in a position to prove that the class of functions which satisfy the postulates of § 5 have the mean-value property. That is, if x_0, y_0, z_0 is the center of a sphere S lying entirely inside of R, the region for $f(x, y, z)$ any function of the class, then $f(x_0, y_0, z_0)$, its value at the center of S, is equal to the average value of $f(x, y, z)$ taken over the surface of S.

Since the process of averaging about an axis discussed in the preceding section involves limiting processes not covered by our postulates, we cannot directly apply it; but we can obtain a function of our class from $f(x, y, z)$ having the same value at the center of S, and on S approximating the result of averaging $f(x, y, z)$ about any axis as closely as we please. For we have merely to select an N so large that the oscillation of the original function in any circle of radius $\pi r/N$ is less than ε, form functions obtained from the original function $f(x, y, z)$ by rotating the coordinates about the chosen axis through angles $2k\pi/N\,(0 < k < N)$, sum these functions and divide by N. The proof that this is the desired function is entirely parallel to the argument given in § 3.

Let us now form a series of positive constants ε_i such that $\lim_{i\to\infty} \varepsilon_i = 0$. For each ε_i we shall form a function $f_i(x, y, z)$ as follows. First form a function $f_{i1}(x, y, z)$ which has the same value at the center of S as $f(x, y, z)$ and which, on the surface of S, differs from the result of averaging $f(x, y, z)$ about a fixed axis by less than $\varepsilon_i/4$, which can be done by the method just described. Next select a second axis perpendicular to the first, and form $f_{i2}(x, y, z)$ which, on the surface of S, differs from the result of averaging $f_{i1}(x, y, z)$ about this second axis by less than $\varepsilon_i/8$. Then form $f_{i3}(x, y, z)$ differing on the surface of S from the result of averaging $f_{i2}(x, y, z)$ about the first axis by less than $\varepsilon_i/16$ and so on; $f_{it}(x, y, z)$ differing on the surface of S from the result of averaging $f_{i,t-1}(x, y, z)$ about the first or second axis, according as t is odd or even, by less than $\varepsilon_i/2^{t+1}$. The functions on the surface of S, $F_1(x, y, z), \ldots, F_t(x, y, z), \ldots$ obtained from $f(x, y, z)$ by actually averaging about the two axes in succession, will be related to the set just constructed, in that, on the surface of S,

$$(14)\qquad |F_1 - f_{i1}| < \varepsilon_i/4, \qquad |F_2 - f_{i2}| < \varepsilon_i/2^2 + \varepsilon_i/2^3,$$

and for all values of t,

$$(15)\qquad |F_t - f_{it}| < \varepsilon_i(1/2^2 + 1/2^3 + \cdots + 1/2^{t+1}) < \varepsilon_i/2.$$

But, by the lemma of § 6, the limit of $F_t(x, y, z)$ as t becomes infinite is f_s, the average of $f(x, y, z)$ over the surface of S. Hence, by taking t large enough, we may make

$$(16)\qquad |F_t - f_s| < \varepsilon_i/2; \qquad i \geqq T.$$

We put

$$(17)\qquad f_i(x, y, z) = f_{iT}(x, y, z).$$

Then, from (15) and (16), we have

$$(18)\qquad |f_i(x, y, z) - f_s| < \varepsilon_i.$$

Furthermore, $f_i(x_0, y_0, z_0)$ is equal to $f(x_0, y_0, z_0)$, since all the processes used to obtain $f_i(x, y, z)$ from $f(x, y, z)$ left its value at the center of S unchanged. Thus the sequence of functions

$$(19)\qquad G_i(x, y, z) = f_i(x, y, z) - f_s$$

satisfies all the conditions of our fifth postulate. Since its value at the center of S is constant, this constant must be zero, and we have

$$(20)\qquad f(x_0, y_0, z_0) = f_i(x_0, y_0, z_0) = f_s$$

which is the mean-value property.

8. *The Harmonic Character of the Functions.* We may now apply the reasoning of § 4 to show that inside any sphere S entirely within its region R, any function of our

class $f(x, y, z)$ is identical with the harmonic function set up by Poisson's integral which has the same values on the boundary of S as $f(x, y, z)$. Thus all the functions of our class are harmonic at all points interior to their regions. Also since every linear class of harmonic functions in three variables satisfies them, the widest such class of functions is the totality of Newtonian potential functions.

9. *Concluding Remarks.* We might extend our postulates to the case of n dimensions, the proof being accomplished by a proper generalization of the lemma of § 6.

It should also be noticed that the postulates for two dimensions apply to functions on a sphere* (and for three dimensions to functions on a hypersphere) provided we replace the orthogonal transformations which correspond to a rotation of the plane (or 3-space) by the transformations on the sphere (hypersphere) corresponding to rotations about a diameter, as the proof given still applies with but slight modifications.

Moreover, since postulate 5 is a consequence of the maximum-minimum property, which is, of course, much more restrictive, we may obtain a set of postulates, by postulating the maximum-minimum property, which, though more stringent than ours, is stated in terms of more familiar properties. Since this substitution enables us to dispense with postulate 4, the new assumptions may be stated in the following form:

A linear family of functions of two (three) variables, invariant as a family under orthogonal transformations of the variables, each function being continuous at all interior points of some region R, and such that for any subregion of R it takes its maximum and minimum values on the boundary of this region, is necessarily a family of harmonic functions.

This statement is of interest, because it shows one set of conditions which, on being added to the maximum-minimum property, characterize the class as harmonic.

HARVARD UNIVERSITY

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* This extension was called to my attention by Dr. Norbert Wiener.

ON THE LOCATION OF THE ROOTS OF POLYNOMIALS*

BY J. L WALSH

It is the object of this note to prove a number of results (particularly Theorems I, II, V, VI, below) concerning the roots of polynomials, generalizations of former results established by the writer. Our main new result is the following theorem.

THEOREM I. *Let (the interiors and boundaries of) the circles $C_1, C_2, \ldots, C_k$ whose centers are the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the respective loci of $n_1, n_2, \ldots, n_k$ roots of a variable polynomial $f(z)$ which has no other roots, where the circles C_i are all equal and their centers α_i all lie on a line parallel to the axis of reals. If the polynomial*

$$(1)\quad a_0 z^n + n a_1 z^{n-1} + n(n-1) a_2 z^{n-2} + \cdots + n(n-1)\ldots 2\cdot 1\, a_n, \quad (n = n_1 + n_2 + \cdots + n_k),$$

has only real roots, and if the circles C_i are sufficiently small, then the locus of the roots of the polynomial

$$F(z) = a_0 f(z) + a_1 f'(z) + a_2 f''(z) + \cdots + a_n f^{(n)}(z)$$

consists of the circles C_j' which are equal to the circles C_i and whose centers are the roots of $F(z)$ when the roots of $f(z)$ are the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ of multiplicities $n_1, n_2, \ldots, n_k$ respectively.† *Any circle C_j' which has no point in common with any of the other circles C_m' contains a number of roots of $F(z)$ equal to the multiplicity of its center as a root of $F(z)$ when the roots of $f(z)$ are the points α_i.*

We shall later make clear the exact meaning of the words, "if the circles C_i are sufficiently small". Theorem I is to be proved by iteration of the following theorem.

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† When the roots of $f(z)$ are the points $\alpha_1, \alpha_2, \ldots, \alpha_k$, the roots of $F(z)$ are all collinear with the roots of $f(z)$. This is a well known theorem due to Hermite, which can easily be established by proving the result in succession for the polynomials $F_1(z), F_2(z), \ldots, F_n(z)$ used below.

THEOREM II. *Let the circles* $C_1, C_2, \ldots C_k$ *whose centers are the points* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *be the loci respectively of* $n_1, n_2, \ldots, n_k$ *roots of a variable polynomial* $f(z)$ *which has no other roots, where the circles* C_i *are all equal and their centers* α_i *all lie on a line parallel to the axis of reals. If* c_1 *is real, then the locus of the roots of the polynomial*

$$F_1(z) = f(z) + c_1 f'(z) \tag{2}$$

consists of the circles C'_j *which are equal to the circles* C_i *and whose centers are the roots of* $F_1(z)$ *when the roots of* $f(z)$ *are the points* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *of multiplicities* $n_1, n_2, \ldots, n_k$ *respectively. Any circle* C'_j *which has no point in common with any other of the circles* C''_m *contains a number of the roots of* $F_1(z)$ *equal to the multiplicity of its center as a root of* $F_1(z)$ *when the roots of* $f(z)$ *are the points* α_i.

Theorem I has already been established for the case

$$a_0 = a_1 = \cdots = a_{e-1} = a_{e+1} = a_{e+2} = \cdots = a_n = 0, \quad a_e \neq 0.^*$$

Theorem I has also been proved with no restriction on (1) nor on the size of the circles C_i for the case $k = 1$.† The limiting case for $c_1 = \infty$ of Theorem II is also true, and is proved as a limiting case of I, Theorem VIII. For the case that the circles C_i of Theorem II are sufficiently small, Theorem II is contained in Theorem I.

The limiting case for $c_1 = \infty$ of Theorem II gives a theorem which is essentially a well known theorem due to Lucas:

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* TRANSACTIONS OF THIS SOCIETY, vol. 24 (1922), pp. 31–69, Theorem X, p. 53. We shall refer to this paper as I.

† TRANSACTIONS OF THIS SOCIETY, vol. 24 (1922), pp. 163–180, Theorem VI. We shall refer to this paper as II.

The note in the COMPTES RENDUS of which II is the development was published and II itself was written before the publication of a paper by S. Kakeya, PROCEEDINGS OF THE PHYSICO-MATHEMATICAL SOCIETY OF JAPAN, (3), vol. 3 (1921), pp. 94–100. Kakeya treats the main theorem of II, and by essentially the methods of II, although he makes the restriction that the circular region C involved shall be bounded by a circle whose center is the origin. See also T. Takagi, PROCEEDINGS OF THE PHYSICO-MATHEMATICAL SOCIETY OF JAPAN, (3), vol. 3 (1921), pp. 175-179, who gives a proof of Theorem V of the present paper.

If a circle contains all the roots of a polynomial, that circle contains all the roots of its derivative.

Theorem I contains the more general theorem of Lucas which deals with any derivative of the original polynomial, for it turns out that in this case the circles C_i are "sufficiently small" in the sense which we shall give to those words.

Let us proceed to the proof of Theorem II. In the case $c_1 = 0$, the hypothesis and conclusion are the same; we turn to the case $c_1 \neq 0$. The roots of $F_1(z)$ are the roots of

$$(3) \quad -\frac{1}{c_1} = \frac{f'(z)}{f(z)} = \frac{1}{z-z_1} + \frac{1}{z-z_2} + \cdots + \frac{1}{z-z_n},$$

where $z_1, z_2, \ldots, z_n$ are the roots of $f(z)$. It follows from the form of (3) that we can interpret the conjugate imaginary complex quantity of the right-hand member of (3) as the force due to particles situated at the respective points $z_1, z_2, \ldots, z_n$, each of which repels with a force equal to the reciprocal of the distance. The roots of (3) are the positions of equilibrium in the field of force due to these particles and an additional constant force of magnitude $1/c_1$ at every point of the plane. It is to be noted that a multiple root of $f(z)$ is always a root of $F_1(z)$ although not a root of (3).

The proof of Theorem II can be given so as to be almost identical with the proof of the limiting case for $c_1 = \infty$ of Theorem II.* That is, we consider one particular root z of $F_1(z)$, and for that point z we replace the n repelling particles of (3) by n equivalent particles which coincide at a point ζ. When the n particles have the circles C_i as their respective loci, the n-fold particle ζ has a circular region as its locus. Study of the locus of ζ will give us the theorem. Let us proceed to the general outline of the proof. We shall omit details if they are similar to the details given in I.

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* This remark applies not to the proof of I, Theorem X for $k = 1$ as actually given in I, but to the proof there given in detail for I, Theorem VI. That proof can of course be given so as to apply directly to I, Theorem X, for $k = 1$.

Theorem II is surely true when there is only one circle C_i. In this case, if a point z is on the boundary of the locus of the roots of $F_1(z)$ and is not on or within the circle C_1 which contains more than one particle, then it is necessary that

$$z_1 = z_2 = \cdots = z_n,$$

and these points lie on C_1.* Thus the line zz_1 is parallel to the axis of reals.

If the locus of all the roots of $f(z)$ is the *exterior* of a circle C_1, the locus of the roots of $F_1(z)$ consists of the exterior of C_1 (if $n > 1$), and the exterior of another circle C_1' defined as the circle traced by that root of $F_1(z)$ other than z_1 when the point $z_1 = z_2 = \cdots = z_n$ traces the circle C_1. If z is a root of $F_1(z)$ interior to the circle C_1 and on the boundary of its locus, z must lie on C_1', the point $z_1 = z_2 = \cdots = z_n$ lies on C_1, and the line zz_1 is parallel to the axis of reals.

We return to Theorem II for the polynomial $F_1(z)$, and suppose that there are precisely two circles C_i, namely C_1 and C_2. It is clear from the field of force that no point z not lying on or between the common external tangents to C_1 and C_2 can be a root of (3). Moreover it follows from the special case of Theorem I mentioned above that not every point z lying between these tangents can be a point of the locus of the roots of (3). In the general case $n_1 > 1$, $n_2 > 1$, there are four circles C_j' of which two are the circles C_1 and C_2, and one of which has its center interior to the interval α_1, α_2.

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* Here and below we make use of the lemma: *The force at a point P due to k particles situated in a circular region not containing P is equivalent to the force at P due to k coincident particles also in C.* See Walsh, Transactions of this Society, vol. 22 (1921), pp. 101–116, Lemma I, p. 102.

Let z be a fixed point on the boundary of the locus of the roots of $F_1(z)$. Then z is not interior to a circle C_1 or C_2 which is the locus of more than one root of $f(z)$, and is not interior to both circles, for in either of these cases z as well as any neighboring point can be made a multiple root of $f(z)$ and hence a root of $F_1(z)$. Suppose also that z is not on C_1 if $n_1 > 1$ nor on C_2 if $n_2 > 1$. Then (as in I, p. 47) we may replace the n_1 particles of C_1 by n_1 equivalent but coincident particles ζ_1 in C_1, the n_2 particles of C_2 by n_2 equivalent but coincident particles ζ_2 in C_2; and finally we may replace these particles ζ_1 and ζ_2 by an equivalent n-fold particle ζ. When ζ_1 and ζ_2 have the circles C_1 and C_2 as their loci, the locus of ζ will be either the interior of a circle C or the exterior of a circle C, and C will cut the line zz' through z parallel to the axis of reals either at the same angle as do the circles C_1 and C_2 or at the supplementary angle according as the locus of ζ is interior or exterior to C.*

Corresponding to the particular point z that we are considering on the boundary of its locus, we may choose definite points $z_1, z_2, \ldots, z_n, \zeta_1, \zeta_2, \zeta$, so that z is a position of equilibrium in the field of force. The point ζ must lie on the boundary of its locus C, and z must lie on the boundary of the locus of the roots of $F_1(z)$ when the locus of ζ $(= z_1 = z_2 = \cdots = z_n)$ is considered merely to be the locus C. For a small change in z makes only a small change in C. If z, ζ, and C have not the relation stated, a sufficiently small but arbitrary change of z can be made, and ζ can be chosen in the new locus C so that z is still a root of $F_1(z)$. That is, the original point z is not on the boundary of its locus.

The point ζ must therefore lie at an intersection of the line zz' and the circle C; ζ is on the boundary of its locus. By the properties of the circles C_1, C_2, C, and the angles in which they are cut by the line zz' (as in I, p. 49), the points ζ_1 and ζ_2 must lie on C_1 and C_2 respectively and on the line zz'. In fact, ζ_1 and ζ_2 must be those intersections of zz' with C_1 and C_2 which cause z, the root of the corresponding polynomial $F_1(z)$, to lie on one of the circles C_j'. Thus we have proved that whenever z lies on the boundary of its locus, it lies on one of the circles C_j'.

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* It is conceivable that this locus of ζ should consist of the entire plane. This slight difficulty can be overcome as in I, p. 50. It is possible that the locus of ζ should be a half-plane. The reader will make the necessary modifications in the present treatment, which are indeed modifications of phraseology rather than of method.

The locus of the roots of $F_1(z)$ contains all the points interior to the circles C_j'; indeed, it is obvious by a simultaneous translation of the points α_i as the roots of $f(z)$ and of the roots of the corresponding polynomial $F_1(z)$ that every point on or within a circle C_j' is a point of the locus. Every point of the boundary of the locus is a point of one of the circles C_j', no point not between the two common external tangents of the circles C_i, C_j' is a point of the locus, so the locus consists of precisely the points stated in Theorem II.

The remark in Theorem II concerning the number of roots of $F_1(z)$ in the regions C_j' follows (as in I, p. 50) from a consideration of the roots of $F_1(z)$ when the roots of $f(z)$ are the points α_i, and from the continuity of the roots of $F_1(z)$ considered as functions of the roots of $f(z)$. Theorem II is now completely proved for the case $k = 2$.

The same method is used in every case. We replace the particles $z_1, z_2, \ldots, z_n$ by a single equivalent particle ζ, whose locus is a circular region. This particle ζ is on the boundary of its locus if z is on the boundary of its locus, and the properties of the boundary of the locus of ζ in connection with the properties of the circles C_i enable us to prove that the point z is on one of the circles C_j', and to prove the theorem in its generality. Further details are left to the reader.

Theorem II is a result dealing with the interiors of circles as the loci of roots of polynomials. We may use the same method to prove a result similarly dealing with the *exteriors* of equal circles whose centers lie on a line parallel to the axis of reals, or dealing with half-planes bounded by lines perpendicular to the axis of reals. Rotation of the plane in any of these cases or in Theorem I gives a new result.

We proceed to prove Theorem I by successive application of Theorem II. Suppose $a_0 \neq 0$ and denote by $c_1, c_2, \ldots, c_n$ the negatives of the various roots of (1); these have been assumed real. Theorem I has been proved for the polynomial

$$F_1(z) = f(z) + c_1 f'(z). \tag{4}$$

From this fact follows Theorem I for the polynomial

$$F_2(z) = F_1(z) + c_2 F_1'(z) = f(z) + (c_1 + c_2) f'(z) + c_1 c_2 f''(z).$$

For, by the part of the theorem already proved and applied to $F_1(z)$, the locus of the roots of $F_1(z)$ consists of certain circles. Since the roots of $F_1(z)$ lie in these circles, it follows that the roots of $F_2(z)$ lie in certain other circles, which are precisely the circles C_j' of Theorem I that pertain to $F_2(z)$ as a linear combination of $f(z)$ and its derivatives. It follows from a simultaneous translation of the α_i and the roots of $f(z)$ that every point of the last-mentioned circles is a point of the locus of the roots of $F_2(z)$. Thus Theorem I is proved for $F_2(z)$ and by induction can be proved for

$$\tag{5} \begin{cases} F_3(z) = F_2(z) + c_3 F_2'(z), \\ F_4(z) = F_3(z) + c_4 F_3'(z), \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad , \\ F_n(z) = F_{n-1}(z) + c_n F_{n-1}'(z) = F(z). \end{cases}$$

It should be noted, however, (as in I, p. 53), that the reasoning just used is not of universal validity. For if the circles C_j' which contain the roots of $F_1(z)$ are not mutually external, we cannot say that they contain respectively the proper number of roots of $F_1(z)$ for a new application of Theorem II.* Our reasoning is valid only in the case that the circles C_j' which are the loci of the roots of $F_1(z)$ have no point in common one with another, and similarly for the sets of circles C_j' which are the loci of the roots of

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* This is, in fact, not merely a fault of our particular statement or method of proof of Theorem II. See I, pp. 36, 37.

$F_2(z), F_3(z), \ldots, F_{n-1}(z)$. This condition will always be satisfied in Theorem I if, when the polynomial (1) and the points α_i are given, the common radius of the circles C_i is sufficiently small, and this is the meaning we give to the restriction in the enunciation of the theorem, "if the circles C_i are sufficiently small". In order to prove the theorem up to but not including the last sentence, we need not require that no two of the circles C_j' which are the loci of the roots of $F_n(z) = F(z)$ should have a point in common, but if this condition is satisfied the number of roots of $F(z)$ in those circles C_j' is as indicated. This fact is proved (as in I, p. 50) from the continuity of the roots of $F(z)$ considered as functions of the roots of $f(z)$.

It still remains, in the proof of Theorem I, to remove the restriction $a_0 \neq 0$. This can be accomplished by remembering that Theorem I has already been proved* when $F(z)$ is simply a derivative of $f(z)$. Hence application of Theorem I for the case $a_0 \neq 0$ to a derivative of $f(z)$ shows that the roots of $F(z)$ must lie in the circles C_j'. Every point on or within the circles C_j' is a point of the locus, for this follows by a translation. We naturally require as before that the circles C_i be sufficiently small. The number of roots of $F(z)$ in a circle C_j' can be determined by continuity, so Theorem I is completely established.

We add the statement of a theorem which is almost trivial when the preceding development is considered, but which seems nowhere to have been mentioned in the literature.†

THEOREM III. *If the roots of a polynomial $f(z)$ of degree n lie in a region R bounded by two parallels to the axis of reals, and if the roots of* (1) *are all real, then the roots of*

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* In I, Theorem X. The proof of the present note is also valid for this case; we merely choose the left-hand member of (3) to be zero to prove the result for the first derivative, and apply repeatedly for the other derivatives.

† In connection with these theorems for the polynomial $F_1(z)$, see Fujiwara, TÔHOKU MATHEMATICAL JOURNAL, vol. 9 (1916), pp. 102–108. See also Uchida, TÔHOKU MATHEMATICAL JOURNAL, vol. 10 (1916), pp. 139–141.

$$F(z) = a_0 f(z) + a_1 f'(z) + a_2 f''(z) + \cdots + a_n f^{(n)}(z)$$

also lie in R.

The generalization of Theorem I for the case $k = 1$ but which makes no restriction on (1) nor on the size of the circles C_i (i. e., II, Theorem VI) gives us the following theorem.

THEOREM IV. *If all the roots of a polynomial* $f(z)$ *lie in the strip*

$$a_1 \leqq y \leqq a_2$$

of the complex ($z = x + iy$) *plane, and if all the roots of the polynomial* (1) *lie in the strip*

$$b_1 \leqq y \leqq b_2,$$

then all the roots of

$$F(z) = a_0 f(z) + a_1 f'(z) + a_2 f''(z) + \cdots + a_n f^{(n)}(z)$$

lie in the strip

$$a_1 + b_1 \leqq y \leqq a_2 + b_2.$$

Innumerable theorems of the same nature as Theorem IV can be proved by II, Theorem VI. We give one further example, after a preliminary remark.

If two convex regions R_1 and R_2 are the respective loci of points z_1 and z_2, then the region R which is the locus of the point $z = z_1 + z_2$ is also a convex region. We shall speak of R as the *sum* of the regions R_1 and R_2. Then we have

THEOREM V. *If two convex regions are the respective loci of the roots of* $f(z)$ *and of* (1), *then their sum is the locus of the roots of* $F(z)$.

Theorem V obviously includes Lucas's theorem for convex regions on the roots of the derivative of a polynomial. Theorem V includes, and can be proved from Theorem IV. Theorem IV includes Theorem III, and Theorem III includes the theorem due to Hermite which we have been using in Theorem I, to the effect that if the roots of $f(z)$ and of (1) are real, then the roots of $F(z)$ are also real.*

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* Reference has been made Takagi's work in this connection. Theorem V does not directly extend to the case of infinite convex

Let us study the case of real polynomials in more detail. We shall prove for them a theorem analogous to Theorem I.

THEOREM VI. *Let intervals* I_i $(i = 1, 2, \ldots, k)$ *of the axis of reals, whose extremities are* α_i, β_i $(\alpha_i \leq \beta_i)$ *be the respective loci of* n_i *roots of a variable polynomial* $f(z)$ *which has no other roots. Then if the roots of the polynomial* (1) *are all real, the locus of the roots of* $F(z)$ *is composed of a number of intervals* I_j' *of the axis of reals. The left-hand extremities of the intervals* I_j' *are the roots of* $F(z)$ *when the roots of* $f(z)$ *are concentrated at the points* α_i; *the right-hand extremities are the corresponding roots of* $F(z)$ *when the roots of* $f(z)$ *are concentrated at the points* β_i. *Any interval* I_j' *which has no point in common with any other interval* I_j' *contains a number of roots of* $F(z)$ *equal to the multiplicity of its left-hand extremity as a root of* $F(z)$ *when the roots of* $f(z)$ *are the points* α_i. *If the intervals* I_i *are all of the same length, the intervals* I_j' *are also all of this same length.*

Theorem VI has been proved (II, footnote, p. 180) for the case $k = 1$, and has been proved (I, Theorem XV) for general k for the case that $F(z)$ is a derivative of $f(z)$. The proof of this latter theorem is to be followed closely in the proof of Theorem VI. We shall prove Theorem VI under the assumption that none of the intervals I_i is a point; to include this more general case requires merely a slight change in phraseology. In the theorem as stated the intervals are assumed to be finite, but the theorem can be extended to include infinite intervals. We prove the theorem first for the case of the polynomial

$$F_1(z) = f(z) + c_1 f'(z).$$

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regions, using the ordinary extension of the definition of convexity, although the following is an immediate result of II, Theorem VI:

If the roots of (1) *have as their locus any region* R *and if the roots of* $f(z)$ *have as their locus the region* C,

$$|z - a| \leq r, \text{ where } a \text{ and } r \text{ are fixed,}$$

then the locus of the roots of $F(z)$ *is the region* S *which is the locus of points*

$$\zeta = \rho + z,$$

where ρ *and* z *have the respective loci* R *and* C. *If* R *is the interior or exterior of a circle, so also is* S.

Let us denote by $\alpha_i^{(m)}$ the roots of $F_m(z)$ when the roots of $f(z)$ are concentrated at the points α_i, $\alpha_i^{(m)} \leqq \alpha_j^{(m)}$ when $i<j$, and similarly by $\beta_i^{(m)}$ the roots of $F_m(z)$ when the roots of $f(z)$ are concentrated at the points β_i, $\beta_i^{(m)} \leqq \beta_j^{(m)}$ when $i<j$. We shall show that the intervals $(\alpha_i^{(m)}, \beta_i^{(m)})$ form the locus of the roots of $F_m(z)$.

We start with the roots of $f(z)$ concentrated at the points α_i and move these roots continuously toward the right until they reach the points β_i. The roots of $F_1(z)$ also vary continuously; they start at the points α_i' and reach the points β_i'. We can even say that the rth root z_r' of $F_1(z)$ (roots numbered in order from the left) varies continuously. Let us now prove that z_r' always moves, if at all, toward the right.

The equation determining z_r' is of the form

$$\Phi = \frac{1}{c_1} + \frac{f'(z)}{f(z)} = \frac{1}{c_1} + \frac{m_1}{z_r' - \gamma_1} + \frac{m_2}{z_r' - \gamma_2} + \cdots + \frac{m_s}{z_r' - \gamma_s}, \tag{6}$$

where the γ_i are the roots of $f(z)$, coinciding in any multiplicities desired. We compute the values

$$\frac{\partial \Phi}{\partial z_r'} = -\frac{m_1}{(z_r'-\gamma_1)^2} - \frac{m_2}{(z_r'-\gamma_2)^2} - \cdots - \frac{m_s}{(z_r'-\gamma_s)^2},$$

$$\frac{\partial \Phi}{\partial \gamma_i} = \frac{m_i}{(z_n'-\gamma_i)^2}.$$

It is always true that $\partial z_r'/\partial \gamma_i$ is positive, so z_r' always increases with γ_i.*

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* This result is the chief tool in the proof of Theorem VI, and as it is true that $\partial z_r'/\partial \gamma_i \neq 0$ for a large class of polynomials, Theorem VI can be suitably modified to apply to much more general polynomials, in particular to $f'(z)$, where $f(z)$ is real but has not necessarily all real roots. Theorem VI holds without change for polynomials and the iterates of polynomials of the type

$$(a - b^2 z) f(z) + c f'(z),$$

where a, b, c are real and the roots of $f(z)$ are real. This last class of polynomials has recently been considered by Nagy, JAHRESBERICHT DER VEREINIGUNG, vol. 31 (1922), pp. 238–251.

Equation (6) is no longer valid to determine z_r' if z_r' is located at a multiple root of $f(z)$. Under these circumstances, if γ_i does not coincide with z_r', the motion of γ_i does not change the position of z_r'. If γ_i does coincide with z_r' and if γ_i is moved to the right, z_r' is either unchanged or moved to the right; this follows immediately from the fact that a t-fold root of $f(z)$ is a $(t-1)$-fold root of $F_1(z)$, and from the fact that every interval of the axis of reals bounded by roots of $f(z)$ contains at least one root of $F_1(z)$.

From the general fact, then, that the rth root z_r' of $F_1(z)$ varies continuously and in one sense under the indicated variation of the roots of $f(z)$, it follows that z_r' traces the entire interval from α_r' to β_r' and can never lie outside that interval. The determination of the locus of Theorem VI is now complete for the polynomial $F_1(z)$. The number of roots of $F_1(z)$ in an interval (α_r', β_r') having no point in common with any other interval (α_s', β_s') is easily found by continuity.

When the roots of $f(z)$ move from the points α_i continuously to the right, the roots of $F_1(z)$ move from the points α_i' continuously to the right, and from the formulas (5), the roots of $F_2(z)$ move from the points α_i'' continuously to the right, and the roots of $F_m(z)$ move from the points $\alpha_i^{(m)}$ continuously and to the right. This gives us Theorem VI for the case $a_0 \neq 0$; the restriction $a_0 \neq 0$ can be removed with no difficulty as in the proof Theorem I of the present paper.

HARVARD UNIVERSITY

NATIONAL RESEARCH COUNCIL REPORT ON ALGEBRAIC NUMBERS

Algebraic Numbers. Report of the Committee on Algebraic Numbers, National Research Council. By L. E. Dickson, H. H. Mitchell, H. S. Vandiver, G. E. Wahlin. BULLETIN OF THE NATIONAL RESEARCH COUNCIL, vol. 5, part 3, number 28, February, 1923. Washington, D. C. 96 pp.

Nowhere is the craft and subtlety of the mathematician more ingeniously developed than in the theory of algebraic numbers. The subject is dotted over with gins and snares, not to say boobytraps, and more than one reputable arithmetician has fallen foul of its pitfalls, thereby exhibiting himself as a warning to the initiated who have sense enough to heed his misfortunes. The unwary, with all the courage of their kind, rush in and are quickly slain. But for those who are gifted with caution and perseverance the theory has rich rewards. As an emancipator of the intellect the theory of ideals is at least the peer of non-euclidean geometry, yet it is but little cultivated by professional mathematicians and is almost wholly unappreciated by mathematical philosophers. The reason is not far to seek: even schoolboys are easily if superficially impressed by the spectacular heterodoxies of geometry, while hard labor and a mature judgment are prerequisites for the mere apperception of the quieter and perhaps more profound beauties of modern arithmetic. Should some mathematician with the necessary talents popularize this subject as non-euclidean geometry has been popularized, he would do an important service to both mathematics and philosophy.

The Report by Dickson, Mitchell, Vandiver, and Wahlin has the twofold object of bringing up to date Hilbert's classic *Bericht* of 1894–95 on the theory of algebraic number fields, and of supplementing the Bericht by accounts of the literature on fields of functions and related topics which Hilbert omitted. It is stated in the preface that Hilbert's report and the present one exhaust the literature. So far as the reviewer can judge this claim is substantiated. There are a few printer's errors, but none that will cause any inconvenience.

This Report differs in one essential respect from both Hilbert's Bericht and from the Bulletins on mathematical physics issued by the National Research Council. It would not be possible to gain a working knowledge of the topics discussed from the extremely condensed statements in the 96 pages of the Report. Nevertheless this Report is admirably fitted to the use of the experts for whom it doubtless is intended.

The responsibility for the several main divisions of the Report is as follows: quadratic, galois and abelian fields, units in a general field, (MITCHELL); cubic fields, norms and congruences in a general algebraic field, classes of ideals, Klassenkorper, complex multiplication of elliptic functions, higher reciprocity laws and Kummer fields, (VANDIVER); distribution of prime ideals, miscellaneous topics in algebraic numbers, cyclotomy, fields of functions, (DICKSON); Hensel's p-adic numbers (WAHLIN). Remarkable uniformity of treatment has been maintained throughout, the style following closely that of Dickson's *History of the Theory of Numbers.* In practically all instances the chief results of the memoir abstracted are concisely stated; there are also occasional indications of the methods followed or short summaries of the proofs. A particularly valuable feature is the correction of such errors in the literature as have been noted by the authors, each of whom is an expert in the topics which he discusses.

Although it would be absurd to attempt a précis of this work which is itself a concise summary and which carries condensation to the limit, it may not be out of place here to note one or two points of general interest, particularly for the encouragement of beginners in this extensive and difficult province of modern arithmetic. First, the relatively large amount of space devoted to cyclotomy is proof that this venerable subject—it was born with Gauss' discovery of 1796—is still lusty. Indeed several of the reports hint at unexhausted lodes where a novice might yet dig out nuggets of value. Quite recently, for example, Burnside has considered octosection. These smaller finds are valuable chiefly as whetters of the appetite for discovery; the mother lode lies elsewhere. Again, the fascinating highway of Hensel's p-adic numbers, but lately opened up to general traffic and bearing as it does toward analysis, offers an attractive and less hazardous entrance to the entire theory than do the older, strictly arithmetic approaches. The literature of this subject has a refreshing newness; it has not yet grown dishearteningly vast, nor is it overloaded with minute and harrassing technicalities. To reach quickly the heart of the classical theory by well travelled roads one cannot do better than follow Landau's exposition (Teubner, 1918), as yet unsurpassed for brevity and clearness.

Although, as can be seen from an inspection of this Report and of current literature, the theory of algebraic numbers owes most of its extraordinarily rapid development to the German school, evidence is not lacking that at last the mathematical public in America, England, and France is becoming aware of the splendid creations of Kummer, Dedekind, Kronecker, Hensel and their followers. The contributions of American mathematicians have been few but choice. To cite only three, all mentioned in the Report, we may recall the fundamental and far reaching researches of E. H. Moore and Dickson on the classes of residues of a prime ideal modulus, Mitchell's correction of errors in the work of Kummer which escaped the notice of so penetrating a mind as H. J. S. Smith, and Vandiver's numerous contributions to the same end. Last there is the recent work of Dickson, which bids fair to be epoch making, on *Algebras and their Arithmetics*,* where the classic theory of algebraic numbers finds a simple and profound generalization.

With the books of Landau and Dickson, the report of Hilbert and that of the present authors now available, it is to be hoped that algebraic numbers, one of the major divisions of modern mathematics, will not much longer remain in learned obscurity, but will take its rightful place as one of the chief glories of any liberal mathematical education.

E. T. BELL

VOLUMES II AND III OF DICKSON'S HISTORY

History of the Theory of Numbers. By L. E. Dickson. Vol. II, *Diophantine Analysis,* xxv + 803 pp. Vol. III, *Quadratic and Higher Forms,* (with a chapter on the class number by G. H. Cresse). iv + 313 pp. The Carnegie Institution, Washington, 1923.

Since the time of Gauss, the theory of numbers has developed in a number of different directions. Let us examine this development prior to the year 1890. Dirichlet and Riemann founded the analytic prime number theory; Kummer, Kronecker, and Dedekind created the theory of algebraic numbers; Eisenstein, Hermite, Smith, and Minkowski developed the arithmetic theory of forms; Jacobi, Eisenstein, Kronecker, Smith, and Hermite applied the theory of elliptic functions to various problems. It will be noted that, in the main, this progress centered about a few great names. The discoveries of these men did not excite the attention of other mathematicians in many cases because the contents of the original papers were often complicated and difficult to read, and few suitable texts were provided to meet the needs of the beginner.

In considering the period between 1890 and 1900, however, a decided change is noted. In this interval appeared the *Lehrbuch der Algebra* of Weber and Hilbert's *Bericht über die Theorie der Algebraischen Zahlen.* These works and the original papers of the same authors appear to have exercised a profound influence on a number of able young mathematicians. In another line, Hadamand and de la Vallée Poussin obtained epoch making results in the theory of the Riemann zeta function, with applications to the asymptotic distribution of prime numbers. Minkowski founded a geometry of numbers which has bearing on many parts of the number theory. Dickson initiated his extensive contributions to the subject by developing the theory of finite fields, having direct connection with the theory of congruences, and attacked arithmetic problems arising in the theory of linear groups.

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* This sentence was written by the reviewer before the award to Professor Dickson of the Cincinnati Prize. See page 90 of this issue. THE EDITORS.

Since 1900 the theory of numbers has been investigated not only by a large proportion of mathematicians of the first rank, but by a great number of less gifted individuals, until at the present time it would perhaps be unwise to assert that the subject has not received sufficient attention from men of research.

The progress has been continuous, and, it seems to the writer, comparable with that during any similar interval of time in the past. In particular, it does not appear unlikely that the year 1909 will be referred to in the future as a landmark in the theory of numbers. In that year A. Thue published his proof of the theorem that the equation

$$f(x, y) = c,$$

where $f(x, y)$ is a homogeneous binary form of degree n with integral coefficients, irreducible in the rational field, and c is an integer, has only a finite number of solutions if $n > 2$; Hilbert's proof of Waring's theorem that any integer can be expressed as the sum of not more than k nth powers, k being an integer depending on n only, appeared; Wieferich showed that if

$$x^p + y^p + z^p = 0$$

has a solution in integers, x, y, and z prime to the odd prime p, then

$$2^{p-1} \equiv 1 \pmod{p^2};$$

and Furtwängler announced and proved a general theorem of reciprocity between two integers in a relative cyclotomic field of which the law of quadratic reciprocity between two odd rational integers is a special case. Each of these results attracted much attention and was the starting point of investigations of wide scope.

In recent years the subject has developed rapidly and in various ways. Based on important discoveries by Hecke, Landau and others regarding the Dedekind function, the methods employed in treating the theory of the distribution of rational primes have been extended in such a way as to yield corresponding theorems concerning the distribution of ideal primes in an algebraic field. Furtwangler and Fueter developed the theory of the class field (Klassenkörper) of an algebraic field, which has close connection with the higher laws of reciprocity and the complex multiplication of elliptic functions. Dickson has been the leader in applications of modern algebraic methods to various problems; has founded theories of modular invariants, and of arithmetics of linear associative algebras, the latter having important uses in ordinary Diophantine analysis. Hardy and Littlewood have discovered new analytic methods in additive number theory, obtaining therefrom remarkable results concerning partitions, representations of a number as sums of squares, Waring's theorem (referred to above) and the representation of an integer as the sum of primes. Humbert, Mordell, and others have contributed extensively in recent years to the arithmetic theory of forms.

It is fitting, during such an interesting stage in the development of number theory, that the monumental work under review should appear. Volume I of the *History* was reviewed by D. N. Lehmer in this BULLETIN (vol. 26 (1919), p. 125).

The plan of the history is to cover all parts of the subject with the exception of algebraic numbers. Volume IV, which has not as yet appeared, is to comprehend, in particular, the extensive literature on the laws of quadratic reciprocity.

For the second volume Dickson has given a resumé of the contents of each chapter in the preface; in the third volume a resumé is at the beginning of each chapter. In view of this we shall not take up in detail the material covered, aside from a few points which expecially interest the reviewer.

The chapter headings of volume II are: Polygonal, pyramidal and figurate numbers; Linear Diophantine equations and congruences; Partitions; Rational right triangles; Triangles, Quadrilaterals and tetrahedra; Sum of two squares; Sum of three squares; Sum of four squares; Sum of n squares; Number of solutions of quadratic congruences in n unknowns; Liouville's series of eighteen articles; Pell equation; Further single equations of the second degree; Squares in arithmetic or geometric progression; Two or more linear functions made squares; Two quadratic functions of one or two unknowns made squares; Systems of two equations of degree two; Three or more quadratic functions of one or two unknowns made squares; Systems of three or more equations of degree two in three or more unknowns, Quadratic form made an nth power; Equations of degree three; Equations of degree four; Equations of degree n; Sets of integers with equal sums of like powers; Waring's problem and related results; Fermat's last theorem,

$$ax^r + by^s = cz^t,$$

and the congruence

$$x^n + y^n \equiv z^n \pmod{p}.$$

Volume III contains: Reduction and equivalence of binary quadratic forms; Representation of integers; Explicit values of x, y in

$$x^2 + \Delta y^2 = g;$$

Composition of binary quadratic forms; Orders and genera and their composition; Irregular determinants; Number of classes of binary quadratic forms with integral coefficients; Binary quadratic forms whose coefficients are complex integers or integers of a field; Number of classes of binary quadratic forms with complex integral coefficients; Ternary quadratic forms; Quaternary quadratic forms; Quadratic forms in n variables; Binary cubic forms; Cubic forms in 3 or more variables; Forms of degree $n \geqq 4$; Binary Hermitian forms; Hermitian forms in n variables and their conjugates; Bilinear forms, matrices, linear substitutions; Representation by polynomials modulo p; Congruencial theory of forms.

On reading the chapter entitled "Pell's Equation," which cites 314 references, one is impressed by the number of writers who independently obtained special methods of solution approximating in character the well known continued fraction algorithm. This seems particularly noteworthy as an example of the evolution of a mathematical idea.

Those chapters in volume II beginning with "Squares in arithmetic or geometric progression" and ending with "Equations of degree n," seem to the writer to be a very unusual contribution to mathematical literature. They deal with subjects which particularly interest amateur investigators and a large proportion of the references are to be found in the problem columns of lesser known publications. Nothing has ever appeared which compares to these chapters in furnishing information about special Diophantine problems.

It will be seen from the above list that the material treated in the third volume is much more technical, in the main, than that taken up in the other volumes and the form in which some of the chapters have been written indicates that more space has been devoted to explanations of various general theories. The chapter on the binary quadratic form class number contains 375 references. indicating the fascination which this topic has had for numerous writers. Legendre, Gauss, Jacobi, Cauchy, Dirichlet, Eisenstein, Smith, Dedekind, Liouville, Hermite, Kronecker, Poincaré, Weber and Klein are some of the well known names associated with it.

Perhaps less known than the material in the other chapters is that contained in the chapters on cubic and Hermitian forms. Following Eisenstein, consider the cubic form

$$f=(a, b, c, d)=ax^3+3bx^2y+3cxy^2+dy^3$$

with rational integral coefficients, and set

$$F=Ax^2+Bxy+Cy^2,\ A=b^2-ac,\ B=bc-ad,\ C=c^2-bd.$$

The *determinant* of the quadratic form for $2F$ is called the determinant D of the cubic form f.

Using linear transformation, the terms proper, improper, equivalent, and class are defined for f in a manner analogous to that in quadratic form theory. Also, correspondences are obtained between the classes of F and certain classes of f. The representation of an integer by f depends on finding integers representable by F, (if $(A, B, C)=1$), whose cubes can be represented in the form

$$V^2 - DV^2.$$

(Cf. the *History*, Chap. XII, Eisenstein[2], Pepin[13]). The topic has been investigated extensively by Eisenstein, Arndt, and Pepin.

After Hermite we write

$$v = x + iy,\ v_0 = x - iy,$$

and u_0 for the conjugate imaginary of u, and we have the binary Hermitian form

$$f(v, u;\ v_0, u_0) = Avv_0 + Bvu_0 + B_0 v_0 u + Cuu_0$$

where A and C are real, while B, B_0 are conjugate imaginaries. Then f takes only real values under a linear transformation and $BB_0 - AC$ is an invariant, called the determinant of f. Definite, indefinite, and reduced forms are defined and a theory has been built up which is analogous to that of binary quadratic forms. The subject has applications to the theory of quaternary quadratic forms and has been developed mainly by Hermite, Picard, Bianchi and Humbert.

We shall now consider the place of the work in mathematical literature. On page xx of the preface of volume II the author says:

"Conventional histories take for granted that each fact has been discovered by a natural series of deductions from earlier facts and devote considerable space in the attempt to trace the sequence. But men experienced in research know that at least the germs of many important results are discovered by a sudden and mysterious intuition, perhaps the result of subconscious mental effort, even though such intuitions have to be subjected later to the sorting processes of the critical faculties. What is generally wanted is a full and correct statement of the facts, not an historian's personal explanation of those facts. The more completely the historian remains in the background or the less conscious the reader is of the historian's personality, the better the history." In the reviewer's opinion, Dickson has come very close to realizing these ideals, and this circumstance gives the work its greatest value. In particular, it appears that he has had always in mind the needs of the investigator. An endeavor is made to supply every scrap of information that might possibly aid in research on a particular problem.

It is characteristic of number theory that from time to time results which are quite new and of an entirely elementary character are obtained by methods also elementary. This being the case, one often finds it of value to study papers as old as those of Euler, Lagrange and Legendre, as *all* of their ideas have not yet been incorporated in general theories, but are, however, reported on in full in the *History*.

Muir's *History of the Theory of Determinants*, which resembles Dickson's *History* closely in giving detailed reports of each paper cited, differs, however, in other respects. Much space is devoted by Muir to criticising the results of articles or in praise of various writers. This type of material does not, in general, particularly interest the man of research, and the published opinions of individuals as to the value of mathematical results rarely have any interest after a term of years. This brings out a quality of Dickson's *Hystory* which it seems well to emphasize, namely, that these volumes may be regarded as a permanent foundation on which to bring the history of number theory up to date at various intervals in the future. It is hardly conceivable that anyone should want to re-examine all the material that Dickson has already gone over and re-write his work. He has put things in such form that no more is perhaps needed in order to cover future progress in number theory than to make additions to his work from time to time.

The material shows in itself how useful the history will be in enabling investigators to avoid duplication in published results. It is to be noted in glancing over the pages, how often the contents of a published article, is, unknown to the writer of it, largely a repetition of previous research of some other author.

After some experience in the use of the first and second volumes, the reviewer has noted a number of instances, where the author has expressed in a nut-shell the main results of a long and involved paper in a much clearer way than the writer of the article did himself. The ability to reduce complicated mathematical arguments to simple and elementary terms is highly developed in Dickson.

It often happens in the history of mathematics that a mathematician becomes a specialist in a particular topic, and, after years of experience with it, he publishes a treatise giving a harmonious and comprehensive development of the subject, the material being all arranged and presented according to his own particular point of view. This treatise may become a classic and its readers are likely to get in the habit of ignoring, to a considerable extent, the literature that preceded its publication. In this way the points of view of the older writers are often lost sight of, as these treatises rarely, if ever, reproduce all the older material on a particular topic. It would seem that there is too great a preponderance of books of this sort in the literature and too few histories or reports of the type of Dickson's work.

It may happen that some readers of this review are not interested particularly in number theory, but are specialists in some other field. Let them consider what a work of similar character to Dickson's *History* on their favorite subject would mean to them. Perhaps they will then be convinced of the supreme importance of producing detailed histories or reports in all branches of mathematics.

H. S. VANDIVER

THREE BOOKS ON RELATIVITY

The Mathematical Theory of Relativity. By A. S. Eddington. Cambridge, University Press, 1923. VI + 247 pp.

The Meaning of Relativity. By Albert Einstein. Translated into English by E. P. Adams. Princeton University Press, 1923. 123 pp.

The Theory of General Relativity and Gravitation. By L. Silberstein. New York, Van Nostrand, 1922. IV + 141 pp.

In attempting a combined review of these three books, I do not wish to give the impression that each does not merit extensive consideration. However, there are necessarily many points of similarity, and by pointing these out and emphasizing the differences I may be able to give an idea of the character of each book. Eddington's book is by far the most comprehensive and contains practically all of the mathematical treatment appearing in the other books, but the latter contain many helpful and stimulating observations and interpretations.

Einstein recalls in his first lecture some of the more fundamental ideas and equations of pre-relativity physics, and converts them into tensor form. In the second lecture the equations for the same physical concepts as interpreted in special relativity are given tensor form. This should be particularly helpful to those, who, not being entirely familiar with mathematical processes, have tried to acquire a knowledge of tensor analysis from the general treatments, such as was given by Einstein in his 1916 paper, and is largely followed in the books under discussion (Eddington, Chapter 2; Einstein, Lecture 3; Silberstein, Chapter 3). After following the first two lectures the reader begins to feel that, as Eddington says (3), "our knowledge of conditions in the external world, as it comes to us through observation and experiment, is precisely of the kind which can be expressed by a tensor and not otherwise." He may not yet be prepared to agree with the last part of this statement, but as he comes to appreciate the effective use which Einstein has made of tensor calculus in his general theory of relativity, he is forced to the conclusion that here is a great contribution to mathematical physics. Scientists may agree or not with Einstein's interpretation of his equations as regards the character of physical space and, in particular, the significance of the well known crucial tests of his theory, but they cannot afford to ignore the guidance of tensor calculus in their attempts to give mathematical formulation to the results of experiment.

The postulates and ideas of special relativity are set forth, more or less briefly, by all three authors in preparation for the transition to general relativity. In their generalized form the postulates may be stated in the explicit form:

1. Everything connected with location which enters into observational knowledge—everything we can know about the configuration of events—is contained in a relation of extension between pairs of events; this relation is called the interval and its measure is denoted by ds; in any system of coordinates $ds^2 = \Sigma g_{ij}\, dx^i\, dx^j$, where g_{ij} are functions of the coordinates which describe the metrical relations of the space-time continuum and also the gravitational field.

2. The path of a freely moving particle, whether in the presence or absence of a gravitational field, but not in an electromagnetic field, is a geodesic of the quadratic form ds^2.

3. The track of a light-wave satisfies the condition $ds = 0$. (Ed. 10, 36; S. 20; E. 71, 87, 102.)

The generalization consists in replacing inertial, or Galilean, systems by general systems of coordinates and in interpreting the g's as potentials of the gravitational field. As a first step in this and other generalizations, we have Einstein's postulate:

4. For infinitely small regions there will be an inertial system (i. e. a local reference system for which the g's have the Galilean values) relative to which the laws of special relativity are valid. (E. 70; S. 12.) However, the assumption underlying the generalization is Einstein's "principle of equivalence". Eddington (40) considers the latter to be "a hypothesis to be tested by experiment as opportunity offers"; ... "as a suggestion rather than a dogma admitting of no exceptions." Under the guidance of this principle Einstein was led to the postulates:

5. Inertial mass and gravitational mass are identical.

6. The law of gravitation for empty space is $R_{ij} = 0$, or $R_{ij} = \lambda g_{ij}$, where λ is a very small constant. It may be that this is equivalent to the "principle of equivalence," in other words, that the principle breaks down for all other curved worlds.

Proceeding from these postulates, Eddington (83–90) and Silberstein (92–100) derive the Schwarzschild solution of the equations $R_{ij} = 0$ for a radially symmetric field and obtain the equations of the geodesics which are identified with the paths of the planets about the sun; from these are derived Einstein's formula for the motion of the perihelion of Mercury; Einstein (105–07) merely states these results. Eddington adds to Postulate 3 the requirement that the tracks of light be geodesics; and, by allowing ds to approach zero in the preceding discussion, he obtains the formula for the deflection of light; Silberstein gives a similar derivation and also states that it is a consequence of Fermat's principle, as shown by Levi-Civita and de Sitter. Einstein does not assume that the paths of light are geodesics. He adheres to Postulate 3 and derives (103) the formula for deviation of light from an approximate solution of the equations

$$R_{ij} - \tfrac{1}{2} g_{ij} R = -kT_{ij}. \tag{1}$$

From this solution he determines the value of the constant k and upon it he bases also his conclusion concerning the shift of spectral lines; the discussion of this question by Eddington and Silberstein is based on the expression $ds^2 = (1 - 2m/r)\,dt^2$, obtained from the Schwarzschild form for an atom at rest or practically so.

Before giving the results for planetary motion as stated above, Einstein (88) considers an approximate solution somewhat after the manner of his 1916 paper. Silberstein (35–8) follows a similar course and obtains the equations given by Einstein with added terms. He thinks that Einstein failed to obtain these "through a too-hasty computation of the Christoffel symbols." It seems to be a question rather of what is meant by "first approximation." As interpreted by Einstein in both places these added terms vanish. In any case the interpretation of these terms as due to an acceleration exerted upon the frame of reference by a velocity field cg_{4i} $(i = 1, \ldots, 4)$ is interesting.

Einstein's derivation (90–94) of equations (1), as a generalization of the Laplace-Poisson equation,

$$\Delta\varphi = 4k\pi\rho, \tag{2}$$

is one of the most interesting parts of the book. The tensor T_{ij}, the energy-momentum tensor of matter, "includes in itself the energy density of the electromagnetic field and ponderable matter", ... "It is only the circumstance that we have not sufficient knowledge of the electromagnetic field of concentrated charges that compels us, provisionally, to leave undetermined in presenting the theory the true form of this tensor." However, in accordance with special relativity, the principle of conservation of momentum and energy is expressed by the vanishing of the divergence, that is

$$\frac{\partial T_{ij}}{\partial x^i} = 0. \tag{3}$$

This and Postulate 4 serve as a guide in the determination of the left-hand member of (1). Einstein does not say that this is equivalent to

$$T^{ij}_i = 0 \tag{4}$$

in general coordinates, where the subscript indicates covariant differentiation, but he explicitly states (91) "we shall have to assume (4) as valid." We wish to emphasize that there is an assumption involved in replacing ordinary derivatives of special relativity by covariant derivatives in general relativity—it is the assumption that in applying Postulate 4 the coordinates at the point are geodesic. Eddington (119) makes the transition from (3) to (4) by an appeal to the "principle of equivalence." Since the components g_{ij} are interpreted as the potentials of the gravitational field—a fundamental assumption of the Einstein theory—the generalization of equation (2) requires of the left-hand member of (1) that it contain no differential coefficients of g_{ij}

higher than the second, that it be linear and homogeneous in the second differential coefficient, and that its divergence be zero. Einstein states without proof that the only tensor of the second order satisfying the first two conditions is $R_{ij} + a g_{ij} R$, in the usual notation, where a is an arbitrary constant.* He then shows that the third condition requires that $a = -1/2$, and in doing so makes use of a particular type of geodesic coordinates at a point. In view of his assumption mentioned above, he is then justified in his statement that the vanishing of the divergence is proved for any system of coordinates. Eddington (119) and Silberstein (81) establish the same result by making use of the four fundamental identities

$$R^{i}_{ij} = \tfrac{1}{2}\frac{\partial R}{\partial x^{j}}.$$

So far as I know, these identities were first established by Levi-Civita (RENDICONTI LINCEI, 1917, p. 388) in a general way by means of the identities of Bianchi; Eddington's proof (115) is based upon the use of geodesic coordinates.

The tensor T_{ij} for an electromagnetic field alone was written by Einstein in his 1916 paper (§ 20) in the form

$$T^{i}_{j} = -F_{j\alpha}F^{i\alpha} + \tfrac{1}{4}\delta^{i}_{j}F_{\alpha\beta}F^{\alpha\beta}, \tag{5}$$

where $F_{j\alpha}$ is the curl $\partial\varphi_{j}/\partial x^{\alpha} - \partial\varphi_{\alpha}/\partial x^{j}$ of the electromagnetic potential, δ^{i}_{j} are the Kronecker deltas, and

$$F^{\alpha\beta} = g^{\alpha i}g^{\beta j}F_{ij},$$

using the conventional notation. In his second lecture (53) he shows that this expresses the principles of energy and momentum as developed by Maxwell, and that the four-dimensional formulation of special relativity serves as the guide to the amalgamation into a tensor. By similar considerations Maxwell's equations are written in the tensor form

$$\left\{\begin{aligned} &\frac{\partial F_{ij}}{\partial x^{k}} + \frac{\partial F_{jk}}{\partial x^{i}} + \frac{\partial F_{ki}}{\partial x^{j}} = 0,\\ &\frac{\partial}{\partial x^{j}}(\sqrt{-g}\,F^{ij}) = \sqrt{-g}\,\rho_0\frac{dx^{i}}{ds}, \quad \text{or} \quad (F^{ij})_{j} = \rho_0\frac{dx^{i}}{ds}, \end{aligned}\right. \tag{6}$$

where ρ_0 is the proper density of electricity and $(F^{ij})_{j}$ is the covariant derivative. (Cf. Ed. 170–5; S. 106–113.) In this generalization from special relativity by means of Postulate 4, there is the added assumption that the coordinates are geodesic. Applying covariant differentiation to (5), we have (Ed. 182; S. 121) for the divergence of T^{i}

$$T^{i}_{j,i} = F_{j\alpha}\rho_0\frac{dx^{\alpha}}{ds}. \tag{7}$$

Thus the divergence vanishes in regions outside of charged particles,

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* For a proof the reader is referred to Weyl, *Space, Time and Matter,* p. 315; also Birkhoff, *Relativity and Modern Physics,* pp. 209–221.

"the only regions in which we can believe that we have the complete expression for the energy tensor", according to Einstein (55). The expression in the right-hand member of (7) is the ponderomotive force four-vector which replaces the electric force vector in the equations of electrostatics, when these equations have been put in tensor form. These equations were derived for special relativity by Einstein in his 1905 paper and again in his second lecture (52). They agree to a high degree of accuracy with the results of experiments on β-particles, and, when written in the form for general relativity, they show that the paths of an electron are not geodesics. Eddington (189) "wants to know what the electron is trying to accomplish by deviating from a geodesic". He considers the field outside the electron and assumes that the field within the electron counterbalances it; he then concludes that an electron in an external field of force having the acceleration given by the equations referred to is "a miracle", whatever that may mean.

As previously remarked, Einstein obtained an approximate solution of equations (1) and therefrom reached the conclusions concerning physical phenomena which have become generally known. These solutions were based on the assumption that the potentials g_{ij} have the Galilean values at infinity and differ little from these values in the neighborhood of matter. In 1917 he presented his views on the so-called *cosmological problem*. Mach held that the inertia of any particle depends upon the whole matter in the universe. If this is true, then (1) "the inertia of a body must increase when ponderable masses are piled up in its neighborhood; (2) a body must experience an accelerating force when neighboring masses are accelerated, and, in fact, the force must be in the same direction as the acceleration; (3) a rotating hollow body must generate inside itself a 'Coriolis field', which deflects moving bodies in the sense of a rotation, and a radial centrifugal field as well" (E. 110). Einstein obtained an approximate solution, involving each of these effects, but of magnitude too small to be tested by experiment, in doing so he was led to make the hypothesis that the physical universe, as distinguished from the space-time continuum, is spherical and closed, and he proposed suitable expressions for the functions g_{ij} and for the tensor T_{ij} which were in keeping with this hypothesis. We cannot here discuss adequately the merits and demerits of this proposal, and of the one suggested by de Sitter; the reader will find this done fully by Eddington (155–168) and by Silberstein (124–137).

Suppose that we had not started with the six postulates previously set down, but had merely taken the first one, which may be interpreted roughly as saying that the physical world (space-time) is a Riemannian manifold of four dimensions with a fundamental tensor g_{ij}. From this tensor others may be derived, such as R_{ij}, B_{ijkl}, and so on. Accepting the fundamental principle of relativity that physical laws are expressible by tensor equations, the next step is to find these equations by a principle of identification with the results of experiment. If, as Eddington (119) says, we take "the view that energy, stress and momentum belong to the world and not to some extraneous substance in the world, we must identify the energy-tensor with some fundamental tensor of the world". Since the divergence of $R_{ij} - \frac{1}{2} g_{ij} R$ vanishes, it is natural to try the equation (1) and see what happens. The left-hand member vanishes only when $R_{ij} = 0$; this, then, is the condition for empty space. From (1) it can be shown that the path of a particle in empty space is a geodesic (Ed. 127). By identifying geodesic coordinates at a point with the Galilean coordinates of special relativity, Eddington (178) shows that the path of a light-pulse is a geodesic; it is a question, however, whether this proof is not equivalent to the assumption involved in the identification. We shall not consider the further consequences of this point of view, but will turn our attention to the more general world-geometry, as developed by Weyl and Eddington.

Weyl and Eddington consider a general four-dimensional continuum in the sense of Analysis Situs and define an affine connection for the manifold by means of forty functions Γ^i_{jk}, symmetric in j and k. By considering the change in a general vector A^i as it undergoes a parallel displacement round a small circuit, they are led to a tensor of the fourth order $*B^i_{jkl}$ which is a generalization of the Riemann tensor. When this tensor is contracted for i and l, the resulting tensor is the sum of a symmetric tensor R_{jk} and the curl of a vector φ_i. Eddington puts $R_{jk} = \lambda g_{jk}$, where λ is a universal constant and takes

$$ds^2 = g_{ij}\, dx^i dx^j$$

as the metric of the space; he leaves the functions Γ^i_{jk} perfectly general, which is equivalent to having an arbitrary tensor K^i_{jk}, symmetric in j and k. Weyl, on the other hand, specializes his geometry by expressing the Γ's as certain functions of the components of a symmetric tensor g_{ij}, their first derivatives, and the components of a vector, which is equivalent to φ_i mentioned above.*

Before we proceed further with the development of this geometry it is advisable to consider its relation to the actual space of physical phenomena. Eddington (197) says "Two possible ways of generalizing our geometric outlook are open. It may be that the Riemannian geometry assigned to actual space is not exact; and that the true geometry is of a broader kind leaving room for the electromagnetic potential vector to play a fundamental part and so receive geometric recognition as one of the determining characters of actual space." ... "The alternative is to give all our variables, including the electromagnetic potential, a suitable graphical representation in some new conceptual space—not actual space." ... "We have then to distinguish between Natural Geometry, which is the single *true* geometry in the sense understood by the physicist, and World Geometry, which is the pure geometry applicable to a conceptual graphical representation of all the quantities concerned in physics." The latter is Eddington's idea of the significance of his geometry and of Weyl's; Weyl at first held that his geometry was the geometry of actual space, but according to Eddington (198, 208) he now holds to the latter view also. Commenting upon the theory of electromagnetism leading to equation (7), Einstein (108) says "This inclusion of the theory of electricity in the scheme of the general theory of relativity has been considered arbitrary and unsatisfactory by many theoreticians. Nor can we in this way conceive of the equilibrium of the electricity which constitutes the elementary electrically charged particles. A theory in which the gravitational field and the electromagnetic field enter as an essential unity would be much preferable." He reiterates this statement in his latest paper (Preussische Akademie der Wissenschaften 1923, pp. 32–38) in which he undertakes to present such a theory. However, he gives no indication that, in adopting Eddington's geometry for the basis of his work, he feels that the geometry is merely a graphical representation of the physical world. On the contrary the idea that he is dealing with the actual world seems to be fundamental in all of his writings concerning the theory of relativity.

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* Eddington's world-geometry is essentially the same as the Geometry of Paths as developed by Veblen and myself in a number of articles in volumes eight and nine of the Proceedings of the National Academy, and by me in the Annals of Mathematics, vol. 24 (1923). This geometry is not equivalent to the more restricted type considered by Weyl, as Eddington states (243). Furthermore, the equations of the paths are fundamental, which has a bearing on Eddington's remark (216): "It may be asked whether there is any other way of obtaining tensors, besides the consideration of parallel displacement round a closed circuit. I think not ..."

Weyl, Eddington, and Einstein identify the vector φ_i of their geometries with the electromagnetic potential. Eddington (208) says it is "the electromagnetic potential because that is the way in which we choose to represent the potential graphically." He also identifies $R_{ij}\,dx^i dx^j = 0$ as the equation of light-pulses, but he does not prove that this is a consequence of the former identification. In other words, it seems to me that if Eddington's view of graphical representation is adopted and one identification is made, the other must be established. It seems reasonable that in some way φ_i is connected with the electromagnetic potential, but if one is dealing with actual space is it in fact the electromagnetic potential itself, or if one is making a graphical representation, is that the best interpretation to place upon φ_i? Recently I gave (Proceedings of the National Academy, June, 1923) another identification of φ_i based upon the equations of a moving electron. In the present state of the theory, a statement that this is the correct interpretation would be dogmatic. I mention it merely for the purpose of pointing out that if it is found that the conclusions of Weyl, Eddington, and Einstein are not in accordance with physical phenomena, as they become more clearly understood, the difficulty may be due to their identification of the vector φ_i.

It would not be proper to bring this review to a close without saying something about the Hamiltonian principle of action. Lorentz and Hilbert seem to have been the first to try to take over this principle from classical mechanics and determine its bearing upon general relativity. Einstein considers it in his 1916 paper (§ 15) and later in the same year made a more extensive study of it. Silberstein (88) feels that "the importance of the Hamiltonian principle seems to be unduly overestimated," and Eddington, who gives (131–144) a full discussion of it, together with what he calls the Hamilton derivative, says (138) "We have thus to remember that when a writer begins to talk about action, he is probably going to consider impossible conditions of the world. (That does not mean that he is talking nonsense—he brings out the important features of the possible conditions by comparing them with the impossible conditions)." There are two distinct methods of approaching this question. One is to build up an integral such that on equating its variation to zero, known results are obtained. This is what Weyl has done (*Space, Time, and Matter*, pp. 209–216; pp. 230–237). The other is to start with the idea that an integral can be found which will be the philosopher's stone to reveal the hidden treasures of the physical world. Thanks to the guiding hand of the tensor calculus the search for such an integral (if it exists) is not a game of blind man's buff. Weyl (295) chooses an integral and discusses the consequences of the choice, although he questions whether his action-principle is realized in nature exactly in the form chosen. Eddington (232) suggests and considers several integrals, one of which Einstein develops in his recent paper in such a manner as to obtain an expression for the Γ's in terms of a tensor s_{ij} and a vector i_j. The results of these theories await verification. It may be that one of them is the hoped-for *omnium gatherum* with or without the correct identification.

L. P. EISENHART

WINGER ON PROJECTIVE GEOMETRY

An Introduction to Projective Geometry. By R. M. Winger. Boston, D. C. Heath, 1923. V + 443 pp.

Among the books on projective geometry in English which have appeared so far, Winger's *Introduction* is a distinct novelty. In fact it justly breaks away from the more or less traditional Cremona-Reye style of a passed period and presents, on the whole, such topics which for a more advanced study of geometry are of essential importance. For this reason, the reviewer is glad to declare from the start that Winger has written an excellent text-book.

The author himself says that "this book is intended as an introductory account for senior-college and beginning graduate students — for the prospective teacher who is seeking proper orientation of elementary mathematics, as well as the university student who lacks the preparation for an intelligent reading of the general treatises on higher geometry and the modern books on higher algebra". As mathematical preparation for a proper understanding of the book collegiate training in algebra, analytic geometry and calculus is all that is required. It is perhaps not necessary to review in detail the thirteen chapters which in order deal with essential constants; duality; the line at infinity; projective properties; double ratio; projective coordinates; the conic; collineations and involutions in one dimension; binary forms; algebraic invariants; analytic treatment of the conic; collineations in the plane; cubic involutions and the rational cubic curve; non-euclidean geometry. The book is very clearly and expressively written throughout, and the propositions are stated concisely and in simple straightforward English. Undoubtedly it will have a very refreshing effect upon the student.

The few criticisms which the reviewer wishes to make concern in some instances the method of presentation rather than the choice of contents. Thus, on page 8, the statement "to show that a condition is necessary and sufficient entails the proof of a proposition and its converse" is, of course, not obvious and needs qualification. In the definition of isotropic lines by $x \pm iy - k = 0$, it would probably be better not to introduce the new term *circular rays* for the special case when $k = 0$. The old term *ray* (*Strahl*) which has an optical meaning when qualified should be abandoned. The word *line* (straight) is sufficient for this purpose. Thus one might speak of $x \pm iy = 0$ as the principal isotropic lines. The paradoxical statements at the top of page 54 might have been omitted without harm; since from a purely geometrical standpoint they are absolutely meaningless.

In Chapter IV on projective properties the statement that "between a figure and its projection there exists the relation of *one-to-one-correspondence*" should be qualified as *algebraic*, in order to be projective, since it is not difficult to establish non-projective one-to-one-correspondences, as for example between the x- and y-axis of a cartesian system by means of $y = \sqrt[3]{x}$ when restricted to real numbers.

Discussing metric and projective properties the author says that "since a circular cone is merely a perspection of a circle from the vertex it follows that any section, not on the vertex; i. e., any proper conic is a projection of the circle". Now there is certainly no obvious conclusion impelled by such an assumption. No matter whether the spaces in which one operates are metrically or projectively defined, such a statement needs proof.

The treatment of projective coordinates is in the opinion of the reviewer the least satisfactory in the whole treatise. The term projective applied to any mathematical doctrine means independence of any particular metrical concept whatsoever. Hence from a scientific standpoint, the establishment of projective coordinates should not be based either on euclidean or cartesian space, even when disguised in the form of a homogeneously made cartesian space. The proper starting point is the famous, indefinitely repeated harmonic quadrangle construction of v. Staudt, plus Dedekind's axiom of continuity, etc., and the establishment of a linear (1,1)-correspondence between the elements thus obtained and the continuum of real numbers. Thus on a line every point A_i is individualised by a parameter $\lambda_i = x_1/x_2$. Projectively, the distance between two points A_1 and A_2 is defined by $A_1 A_2 = \lambda_2 - \lambda_1$. A similar procedure leads to the definition of projective coordinates of spaces of two, three, or, in fact, of any number of dimensions. We easily obtain more general spaces by admitting complex values for the parameters λ. In this manner, a perfectly rigorous projective geometry may be built up without the least reference to euclidean or cartesian metrics. Also from a didactic standpoint this procedure is simple enough and does not offer greater difficulties to the student than other more or less metric methods. In a purely projective geometry (of the plane) the statement that "the projective coordinates of a point are proportional to fixed multiples of the distances of the point from the sides of the triangle of reference" has no place at all.

The chapters on binary forms and algebraic invariants are very well done. In courses on the algebraic theory of invariants the greater part of the time is usually spent in the study and assimilation of the purely formal algebraic processes, so that little time is left for geometric applications. Winger's presentation of the subject, based on the polarizing process, may be called ideal from the standpoint of the geometrician. The treatment of apolarity in the ternary field, however, must seem rather abrupt and disconnected to the student, since it is based merely on a suggested generalization from the binary field.

"Absolute coordinates," defined by

$$x = X + iY, \ \bar{x} = X - iY,$$

might better be called *isotropic coordinates*, as introduced by Laguerre. If one speaks of isotropic lines it would seem natural to speak also of isotropic coordinates. The term *absolute* introduced by Cayley is not expressive of the idea it intends to convey. There certainly is nothing "absolute" about such real or imaginary boundary conics of non-euclidean geometry.

The chapter on collineations brings out the important properties of these transformations. Involutions on the rational cubic curve are treated in a very instructive manner. The idea of order and class of a curve resulting from the principle of duality is illustrated effectively by a number of well chosen examples. The restriction of involutive properties to rational curves reflects, of course, the author's preoccupation with such curves in his researches on self-projective rational curves. On the other hand, the elliptic cubic, as is well known, offers a beautiful example for the application of projective theories.

In the concluding chapter we find a short account of non-euclidean geometry in the plane after the style of Klein, and a historic note thereupon. That a historic account in a book on projective geometry should be limited to non-euclidean geometry seems somewhat peculiar. It would have been perfectly proper to give a general historic sketch on the entire subject of projective geometry.

It is proper to point out an error which is very common in histories of mathematics and which is contained in the following statement on page 420. "Little progress was made until about a hundred years later when Gauss (1777-1855) and his friends and pupils became deeply interested in the subject." Now the fact is that Gauss's deeper interest in the subject was subsequently aroused by the brilliant discoveries of Lobatschewsky and Bolyai. As a matter of fact, Gauss, in the beginning, hoped to be able to prove what is known as the euclidean parallel axiom and assumed a rather skeptical attitude towards the new discoveries. Subsequent deeper meditations, however, led Gauss to his own establishment or verification and acceptance of the new theory.

A very valuable feature of Winger's introduction is the number of well chosen exercises. Altogether the book will serve a very useful purpose. It will enable the student to acquire the basic concepts and theories which are nowadays a necessity for students who wish to enter into the study of more advanced geometric theories. In this sense the book deserves to be very highly recommended.

ARNOLD EMCH

SHORTER NOTICES

The Making of Index Numbers. By Irving Fisher. Boston and New York, Houghton Mifflin Company, 1922. xxxi and 526 pages.

This book is an elaborate inductive study of price index numbers, the prime object being the discovery of the best, or at least the most accurate index number. To begin with, six fundamental types are considered, the arithmetic, geometric, harmonic, median, mode and aggregative. Each of these may be weighted in different ways, thus giving rise to 24 non-identical formulas. According to Fisher an index number should satisfy two chief tests. Since an index number implies two dates, one of which is the base year, the interchange of years yields a second formula called the *time antithesis* of the first. If P_{01} represents the first index number, the time antithesis is $1/P_{10}$, and the first test or *time reversal* test is $P_{01}\cdot P_{10}=1$.

The second test, original with Professor Fisher, is called the *factor reversal* test, and assumes that index formulas should give consistent results if applied to prices and to quantities. That is

$$P_{01}\cdot Q_{01}=V_{01}=\frac{\sum p_1\, q_1}{\sum p_0\, q_0}$$

where p_0, p_1 represent prices and q_1, q_0 quantities in the years 0 and 1 respectively. Q_{01} is obtained from P_{01} by interchanging the p's and q's. V_{01} is called the value index. $V_{01}\div Q_{01}$ is called the *factor antithesis* of P_{01}.

Each time antithesis and each factor antithesis is an index number, and after dropping out identicals, 22 new formulas are added to the list making now a total of 46. But only four of these satisfy the first test and none the second. Taking the geometric mean of a pair of time antitheses or of a pair of factor antitheses gives an index number which conforms to the first or to the second test respectively. Performing both operations gives a number which satisfies both tests. By these means the number of formulas is increased to 96 and forms the main series of index numbers discussed in the book. Certain other supplementary formulas giving slight variations to those in the main series are derived by a process called "crossing the weights" and bring the total to 134.

These 134 index numbers now go through a sifting process to find the best one. First all unweighted formulas and all formulas based upon the median or the mode are ruled out because of freakishness or lack of sensibility. Others are dropped because of more or less bias in one direction, leaving 47 fairly good index numbers. In testing the formulas, the prices and quantities of 36 commodities in the six rather extraordinary years from 1913 to 1918 are used. A surprising thing is the close agreement of this group of index numbers, thus giving the author one of his answers to the argument that the index number to be used should depend upon the purpose for which it is to be employed. Further sifting yields 13 index numbers all satisfying the two tests. Of these the formula

$$\sqrt{\frac{\sum p_1 q_0}{\sum p_0 q_0} \cdot \frac{\sum p_1 q_1}{\sum p_0 q_1}}$$

"is at least equal in accuracy and is probably slightly superior in accuracy to any of the others". This index number is called by Fisher the "ideal" index number, but will probably go into the literature as *Fisher*'s index number, although he insists that the names Walsh and Pigou should be used also.

Eight other numbers are given honorable mention. One in particular, on account of its rapidity of calculation, accuracy and simplicity

$$\frac{\sum (q_0 + q_1) p_1}{\sum (q_0 + q_1) p_0}$$

is really to be preferred to the ideal except when the utmost accuracy is desired, notwithstandig the fact that it is not included in the group of 13 satisfying both tests.

This brief summary gives but a poor idea of the elaborate detail of the book, written as it is for the general reader as well as the specialist. There are worked out examples, tables of comparisons, graphical explanations, notes, appendices for mathematical details until no point is left which is not thoroughly discussed. Chapter 15 on the speed of calculation is a fair example of the painstaking work throughout the book. In this chapter the index numbers for prices and quantities of the above mentioned 36 commodities for the years 1914–1918 were computed, timed and ranked for each of the 134 formulas. The times varied from one hour to 64.5 hours. The ideal formula ranked 29 and took 14.3 hours to calculate. The second of the formulas mentioned above ranked 16 and took 9.6 hours.

We cannot say that Professor Fisher has actually proved that his ideal index number is really the best one, if a best one exists. It seems to be one among several good ones. But the book does prove that some of the index numbers in use are bad ones.

A. R. CRATHORNE

Statistical Method. By Truman L. Kelley. New York, The Macmillan Company, 1923. xi + 390 pp.

There is throughout this book such a close coordination between the theoretical developments and the practical applications as to make it fairly obvious that the mathematical problems were suggested largely by the work of the author on practical problems of statistics. The method of the book is inductive, starting with quantitative data to be described, and developing the appropriate mathematical methods for the suitable description and elucidation of various kinds of data. The book seems remarkable with respect to the large number of topics treated in the given space. This seems to have been made possible in part by using in the earlier parts of the text a good many concepts whose meanings are developed later. In general, this plan involves considerable departure from logical sequence, but the method is made practicable by the use of forward references.

The first five chapters of the book together with Chapter VIII on correlation are suitable reading for the beginning student with but little knowledge of mathematics. In these chapters elementary statistical methods are explained for the benefit of biologists, economists, educators, psychologists and others who use statistical data in their work. Moreover the treatment is so well illustrated by concrete examples as to make an appeal to those who have data to analyze, and the book will tend to promote a higher standard of statistical practice in this country. While the beginner will thus find the book of interest, a large part of the book is planned for the advanced student, and he will find here a wealth of material for his purposes, whether his main interest be in the theory of statistics or in applications to fields such as economics, biology, psychology or education.

The book presents a good many new formulas and adaptations of known formulas to particular purposes. In the preface the author expresses the very commendable view that he can see "no value except at times a slightly greater ease of manipulation, in using a measure whose probable error cannot be calculated if one with a known probable error and serving the same purpose exists." In harmony with this view the book gives a large number of determinations of probable errors. The determinations of these probable errors was surely a very difficult undertaking on the part of the author and he should be complimented on his courage. The author requests the critical analysis by fellow statisticians of his determinations of probable errors, and "such charity in reporting shortcomings as may be due one who has acted upon the policy that as shrewd an estimate as possible of the probable error of a statistical constant is better than no estimate at all." While there is much to commend this attitude in discussing applied problems, it would seem very important to indicate the fact when there is any considerable doubt as to the validity of the formulas for the probable error.

On page 91, there is a statement which implies that the arithmetic mean and the median are the two most important averages. It would seem safer to limit such a statement to certain purposes. For example, the geometric mean is surely very important for some purposes. On page 154 there seems to be an oversight in the usage of the word "slope". The slope being the tangent of the angle with the positive direction of the x-axis, we note that the slope of the one line would be r and of the other $1/r$ when standard deviations are equal. At the top of page 159, a conclusion seems to be drawn without a clear statement of the hypotheses. It would seem better to say: "It can be easily shown that the deviations x and $y - r(\sigma_2/\sigma_1)\, x$ are uncorrelated. If we assume that these uncorrelated deviations are independent in the probability sense, then the probability of the joint occurrence of the assigned deviations is $z = z' z''$. In the chapter on frequency curves, the book follows the ideas of Pearson on generalized frequency curves rather closely. On pages 146–150 the author gives his views of the relation of moments to certain unstable types of distribution.

Nearly half of the book is very properly given to the subject of correlation. The treatment begins with Galton's original ideas, but is extended in many directions and includes the recent developments of the subject. Dr. Kelley's own contributions to the treatment of correlation have naturally given color to the treatment of certain of the advanced aspects of the subject. Special attention should thus be directed to the methods which facilitate the computation of partial and multiple correlation coefficients, including the method of successive approximation devised by Dr. Kelley for finding the regression coefficients when a large number of variables are involved.

The Kelley-Wood table of the normal probability functions with respect to assigned areas is given at the end of the book and will be found useful. The following typographical errors occur:

p. 31, line 13 from bottom, Table I should read Table VII.
p. 31, line 6 from bottom, 65.5 should read 66.5.
p. 33, lines 5 and 6, the words "ordinates" and "abscissas" should be interchanged.
p. 40, chart 16, block labeled 264 should be labeled 246.
p. 77, line 12, .26215 should read .26315.
p. 79, formula 22. The radical sign should extend over the delta square.
p. 79, formula 23a, 5th line from the bottom $2\,\mu^2$ should read μ_2.
p. 89, line 5, $n(n-1)\,(n-2)$ should read $n(n-1)$.
p. 91, line 26, 78 5°—80°.5 should read 78°.5—83°.5.
p. 92, line 21, "o" should read "of".

p. 95, formula 46, the last equality sign and the middle minus sign should be deleted and a minus sign should be placed before the last x.
p. 184, line 10, "by formula [12]" should read "by formula [121]".
p. 207, table 39, "test Y reliability .2" should read "test Y reliability .4".
p. 210, formula 161c. The exponents of the r's under the radicals should be deleted.
p. 215, line 19, 5.0—should read 4.0.
p. 240, formula 198, kappa minus 1 in the denominator should read kappa minus 2.
p. 258, formula 213, .6457 should read .6745.
p. 294, table 61, the constants .2200 and .2399 are incorrect.

The reviewer has received most of these corrections from the author.

H. L. RIETZ

Einführung in die mathematische Behandlung der Naturwissenschaften. Tenth edition. By W. Nernst and A. Schoenflies. Berlin, R. Oldenbourg, 1922. xii + 502 pp.

Physics and astronomy are no longer the only pure sciences which depend largely on mathematics. At least thirty years ago the necessity of a knowledge of mathematics for work in other natural sciences had come to be recognized by investigators. Research in theoretical chemistry can no longer be carried on nor understood by one ignorant of the ideas of the calculus, and in other fields the need of a mathematical formulation of problems is increasing.

The book by Professors Nernst and Schoenflies is a text and reference manual for the use of scientists in general and chemists in particular. It represents the mathematical equipment beyond trigonometry which, in the opinion of the authors, should be the possession of the modern chemist. The first edition, which appeared in 1895, was a text on elementary calculus with as much analytic geometry as was necessary for the purpose. In the sixth edition in 1910 there was added some explanation of analytic geometry of space, vectors, foundations of analytic mechanics, and partial differential equations. The tenth edition, appearing about a year ago, contains further material on the theory of heat, relativity, and crystal structure.

From the point of view of the teacher of mathematics the book is written with much less care than are most American texts. But it is altogether probable that a meticulous mathematician could not write a book which would make the same appeal to a student of the natural sciences. The wealth of illustrative material which is possible when it is assumed that the reader has studied some physics and a considerable amount of chemistry gives to the mathematical ideas a reality and vitality which they cannot possibly have otherwise. The examples in analytic geometry are built around the laws of Boyle, Mariotte, Gay-Lussac, and van der Waals. Although the first problem formulated in symbols of the differential calculus is that of finding the tangent to a parabola, it is preceded by an introduction giving examples of discoveries in natural science which were made possible by calculus, and by a general explanation of the type of problems in science which lead to the idea of rates and the methods of the calculus. The inversion of raw sugar is used as an example to show the meaning of integration before any technique has been developed, and there is an abundance of illustrative material for an indefinite integral. The idea of a discontinuous function is presented by recalling the behavior of the volume of a substance which is being heated. The volume varies continuously up to the melting point, then takes a sudden increase, after which it again varies continuously.

The teacher of college mathematics in America is familiar enough with the complaints of his colleagues in the departments of physics, chemistry, and engineering. They assert, probably with ample justification, that the students cannot use the mathematics which they are supposed to have learned. One remedy for this difficulty might be to give the calculus, perhaps for a second time, at a later stage in the college course when the students had acquired enough background of scientific knowledge to appreciate the meaning of the mathematics. It seems impossible that a student able to understand the illustrative examples in Nernst and Schoenflies should fail to grasp the significance of the mathematical ideas or be unable to use the tools put before him.

At the end of the book is a list of problems for drill work and a collection of formulas beginning with elementary algebra and continuing through the calculus.

Only one error has been noted. On page 117, in equation (8), the constant of integration has not been properly determined. The same equation, however, is given correctly on page 125.

W. R. LONGLEY

Kartenkunde. By Dr. M. Groll. Neubearbeitet von Dr. Otto Graf. Vol. II, *Der Karteninhalt.* Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 133 pp.

The second volume of Groll's Kartenkunde deals with the classification of maps and their contents. Particular attention is given to the various graphic methods of map construction and topographical representation. Copious historic and literary references and illustrations of various methods in various countries add to the interest and value of the second volume. On the whole it deserves the same praise as volume I, which the writer reviewed in this BULLETIN (vol. 29, No. 2 (February 1923), pp. 89–90).

ARNOLD EMCH

Naturwissenschaften, Mathematik und Medizin im klassischen Altertum. By J. L. Heiberg. (Aus Natur und Geisteswelt, No. 370.) Leipzig and Berlin, B. G. Teubner, 1920. 104 pp.

Mathematics and Physical Science in Classical Antiquity. By J. L. Heiberg. Translated from the German by D. C. Macgregor. New York, Oxford University Press, American Branch, 1922. 110 pp.

This little volume from the well known series *Aus Natur und Geisteswelt* (No. 370) maintains the high standards set by the previous volumes of the series. The name of the distinguished author is sufficient guarantee that the contents is authoritative. It is also, however, of peculiar interest, in spite of its brevity and avowedly popular character. The readers of the Bulletin are doubtless all more or less familiar with the main features of ancient Greek mathematics; they are probably acquainted also with the principal results Greek astronomy and the physical speculations of the great philosophers of classical antiquity. It may be doubted, however, if many have ever read a connected account of scientific development as a whole during this period. The reviewer at any rate admits joyfully that much of the contents of this monograph (it is little more than that) was a revelation to him and a very interesting and illuminating one. This must be the justification, if any is needed, for giving space to such an elementary popular little book.

The chapter headings (taken from the English translation) are as follows: I. Ionian Natural Philosophy; II. The Pythagoreans; III. Medicine in the Fifth Century. Hippocrates; IV. Mathematics in the Fifth Century; V. Plato. The Academy; VI. Aristotle. The Lyceum; VII. The Alexandrians; VIII. The Epigoni; IX. The Romans; X. Greek Scientific Literature of the Empire. Byzantium. It will be noted that the arrangement is chronological rather than topical, so that the reader is given for each period a general view of scientific thought at that time. One of the omissions that seems a bit strange is the fact that no mention whatever is made of Lucretius in the chapter on the Romans or anywhere else for that matter.

The English translation is the second volume of the series *Chapters in the History of Science,* which is appearing under the general editorship of Professor Charles Singer. The translation is free rather than literal, and is throughout readable. The reviewer personally dislikes the word "researcher" which appears at frequent intervals, "goings-on" would sound more familiar to American ears than "on-goings" and the "propugnacula" of exact research are a little terrifying, but the original verdict stands: The translation is well done and may be recommended to any one who desires to spend a couple of hours in pleasant reading of interesting material.

J. W. YOUNG

NOTES

The second number of volume 25 of the TRANSACTIONS OF THIS SOCIETY (April, 1923) contains the following papers: *Developments associated with a boundary problem not linear in the parameter,* by R. E. Langer; *Invariant points of a surface transformation of given class,* by J. W. Alexander; *Applications of analysis to the arithmetic of higher forms,* by E. T. Bell; *On the second derivatives of an extremal-integral with an application to a problem with variable end points (supplementary paper),* by A. Dresden; *Abstract group definitions and applications,* by W. E. Edington; *On the integrals of elementary functions,* by J. F. Ritt; *Invariants of the linear group modulo p^k,* by M. M. Feldstein; *On certain polar curves with their application to the location of the roots of the derivatives of a rational function,* by B. Z. Linfield; *Orthogonal systems of hypersurfaces in a general Riemann space,* by L. P. Eisenhart; *Ruled surfaces with generators in one-to-one correspondence,* by E. P. Lane; *Symmetric tensors of the second order whose first covariant derivatives are zero,* by L. P. Eisenhart.

The second and third numbers of volume 45 of the AMERICAN JOURNAL OF MATHEMATICS (April and July, 1923) contain: *A class of numbers connected with partitions,* by E. T. Bell; *Note on a new type of summability,* by N. Wiener; *On mediate cardinals,* by D. Wrinch; *Periodic oscillations of three finite masses about the Lagrangian circular solutions,* by H. E. Buchanan; *On certain chains of theorems in reflexive geometry,* by F. D. Sutton; *A poristic system of equations,* by L. B. Robinson; *Systems of two linear integral equations with two parameters and symmetrizable kernels,* by M. Buchanan; *The asymptotic expansion of the functions $W_{k,m}(z)$ of Whittaker,* by F. H. Murray; *Some geometric applications of symmetric substitution groups,* by A. Emch; *On elliptic cylinder functions of the second kind,* by S. Dhar.

The second number of volume 24, series 2, of the ANNALS OF MATHEMATICS contains: *Spherical representation of conjugate systems and asymptotic lines,* by W. C. Graustein; *On a short method of least squares,* by B. H. Camp; *On the convergence of the Sturm-Liouville-series,* by J. L. Walsh; *The functional equation $g(x^2) = 2ax + [g(x)]^2$,* by J. H. M. Wedderburn; *On convergence factors in triple series and the triple Fourier's series,* by B. M. Eversull; *On the minimizing of a class of definite integrals,* by P. R. Rider; *A pythagorean functional equation,* by E. Hille.

At the meeting of the British Association for the Advancement of Science at Liverpool in September, 1923, Sir David Bruce was elected president for the year 1923—24, to preside at the meeting of the Association in Toronto, in the summer of 1924. The address of the retiring president, Sir Ernest Rutherford, was on *The electrical structure of matter,* and the address of Professor J. C. McLennan, of the University of Toronto, president of Section A, was on the *Origin of spectra.* On the occasion of this meeting honorary degrees were conferred by the University of Liverpool on Sir Ernest Rutherford, Professor Niels Bohr, and Professor J. C. McLennan.

On the occasion of the centenary of Pasteur, the following French mathematicians and mathematical physicists were made officers or chevaliers of the Légion d'Honneur: Andoyer, Brillouin, Cartan, Cotton, Hadamard, Langevin, Vessiot, officers; Baire, Drach, Fréchet, Lebesgue Ollivier, Riquier, Valiron, Villat, chevaliers.

The decision of the first committee of award of the Bôcher Prize of this Society was announced December 28, 1923, at the annual meeting in New York. The award was made to Professor G. D. Birkhoff, of Harvard University, for his memoir entitled *Dynamical systems with two degrees of freedom,* published in the TRANSACTIONS OF THIS SOCIETY, volume 18, pages 199-300.

The prize of $1000 offerred by a member of the American Association for the Advancement of Science for a notable contribution to science reported at the Cincinnati meeting has been awarded to Professor L. E Dickson, of the University of Chicago. The papers upon which the award was based were presented at sessions of the western meeting of this Society, held in connection with the A. A. A. S.; their titles were: Algebras and their arithmetics, On the theory of numbers and generalized quaternions, and Quadratic fields in which factorization is always unique. The first of these papers, which gives a resume of the work of others and of the authors' own recent work in this field (see p. 65 of this issue), will appear in an early number of this BULLETIN, as will also the third of these papers.

At the annual meeting of this Society in New York, President Veblen announced that the fellowships administered by the National Research Council for physics and chemistry by means of funds from the Rockefeller Foundation had been extended to include mathematics, and that the grant had been increased from $100,000 to $125,000 per annum, effective July 1, 1925. By an action taken since this announcement was made, the inclusion of mathematics as one of the possible subjects will take effect at once, on the fund now operative.

On account of proposed absence from this country, Professor E. B. Van Vleck has resigned as a representative of this Society on the National Research Council. The Council of this Society has selested

Professor H. F. Blichfeldt to complete the term which expires on July 8, 1924, and also to represent this Society for the succeeding period of three years.

At the request of the American Section of the International Mathematical Union, the Council of this Society has selected four additional representatives on the Section, for the nine months including the International Congress, namely, Professors A. B. Coble, Arnold Dresden, L. E. Dickson, and Virgil Snyder.

At the inauguration of Dr. G. J. Trueman as president of Mount Allison University, Sackville, New Brunswick, Canada, the honorary degree of doctor of laws was conferred on Professor R. C. Archibald, of Brown University.

Professor L. A. Howland, head of the department of mathematics at Wesleyan University, has been made acting president of that institution.

Assistant Professor Mary Curtis Graustein, of Wellesley College, has returned to the college after two years' leave of absence.

At St. John's College, Annapolis, Maryland, Professor B. H. Waddell and Assistant Professor T. L. Gladden have resigned, and Mr. G. A. Bingley, instructor in mathematics at the United States Naval Academy, has been appointed assistant professor.

At the Carnegie Institute of Technology, Professor O. T. Geckeler has been made head of the department of mathematics; Assistant Professor H. S. Lightcap has been promoted to an associate professorship, and R. P. Johnson, E. A. Whitman, and E. A. Starr have been promoted to assistant professorships.

At Colgate University, Mr. H. A. Dobell has been granted leave of absence for study at the University of Pennsylvania.

Professor H. P. Kean, of Illinois Wesleyan University, has been appointed assistant professor of mathematics at Wittenberg College, Springfield, Ohio.

At Purdue University, Mr. T. E. Raiford has been granted a year's eave of absence; Dr. W. E. Edington has been promoted to an assistant professorship.

At the University of Georgia, Associate Professor D. F. Barrow has been promoted to a full professorship.

Mr. E. F. Weinberg has been appointed head of the department of mathematics at Rollins College, Winter Park, Florida.

Professor C. H. Richardson, of Georgetown College, Kentucky, has been granted leave of absence for the current year, and is studying at the University of Michigan. Mr. Guy Stevenson is acting for the year as professor and head of the department.

Associate Professor S. Lefschetz, of the University of Kansas, has been promoted to a full professorship of mathematics.

Mr. A. S. Hathaway has been appointed professor of mathematics at Friends University, Wichita, Kansas.

At Iowa State College, Assistant Professor E. C. Keifer has resigned to become head of the department of mathematics at James Millikin University.

Professor D. A. Lehman, of Goshen College, has been appointed professor of mathematics at Bluffton College.

At the Colorado Agricultural College, Mr. C. W. Young has been promoted to an assistant professorship, and Mr. A. G. Clark, instructor in mathematics at the University of Wyoming, has been appointed to an assistant professorship.

Associate Professor J. W. Calhoun, of the University of Texas, has been promoted to a full professorship of applied mathematics.

Mr. W. M. Whyburn, of the University of Texas, has been appointed professor of mathematics at South Park College.

The following have been appointed instructors in mathematics:
Carnegie Institute of Technology, Mr. J. H. Simester;
Colgate University, Messrs. J. C. Polley and G. O. Hohl;
Dalhousie University, Dr. F. H. Murray;
University of Georgia, Messrs. F. Cumming and E. M. Everett;
Georgia School of Technology, Messrs. T. P. Branch, J. G. Griffin, J. G. Evans, W. W. Purks, J. W. Taylor;
Hamline University, Mr. E. L. Michelson;
Iowa State College, Mr. B. A. Rogers;
University of Kentucky, Messrs. T. Andrew and J. N. Nixon;
Purdue University, Mr. C. E. Barr;
Rockford College, Miss Martha P. McGavock;
University of South Carolina, Mr. S. T. Sparkman;
Texas Agricultural and Mechanical College, Mr. H. A. Robinson.

Lady Shaw, wife of Sir Napier Shaw, died September 22, 1923. She was at one time lecturer in mathematics at Newnham College, Cambridge.

Professor Malcolm McNeill, who had been head of the department of mathematics at Lake Forest College for thirty-five years, died suddenly October 5, 1923.

Professor H. F. Stecker, of Pennsylvania State College, died October 30, 1923, at the age of fifty-six years. Professor Stecker had been a member of the American Mathematical Society since 1894.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ARCHIMEDES. Ueber Paraboloide, Hyperboloide und Ellipsoide. (Ostwald's Klassiker der Exakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1923. 73 pp.

BJERRUM (N.). See BRODÉN (T.).

BORCHARDT (L.). Gegen die Zahlenmystik an der großen Pyramide bei Gise. Berlin, Behrend, 1923. 40 pp.

BOUVIER (R.). La pensée d'Ernest Mach. Essai de biographie intellectuelle et de critique. Paris, chez l'auteur, 1922. 370 pp.

BRODÉN (T.), BJERRUM (N.) och STROMGREN (E.). Matematiken och de exakten naturvetenskaperne under det nittonde århundradet. Stockholm, Norstedt, 1922. 248 pp.

DUMOULIN (A.). See MANSION (P.).

DURELL (C. V.). See FAWDRY (R. C.).

FAWDRY (R. C.) and DURELL (C. V.). Calculus for schools. London, E. Arnold, 1923. 8 + 300 + 20 pp.

VAN GEER (P.). See WOLFF (J.).

LORENZ (H.). Einführung in die Elemente der hoheren Mathematik und Mechanik. München, Oldenbourg, 1923. 176 pp.

MANSION (P.). L'essence de la trigonométrie. Avec le portrait de l'auteur et une notice biographique par A. Dumoulin. Paris, Gauthier-Villars, 1922.

NATUCCI (A.). Il concetto di numero e le sue estensioni. Torino, Fratelli Bocca, 1923. 8 + 474 pp.

ROSE (W. N.). Mathematics for engineers. 2d edition. Part 2. London, Chapman and Hall, 1923. 14 + 423 pp.

RUTLEDGE (G.). Fundamental topics in the differential and integral calculus. Boston, Ginn, 1923. 8 + 252 pp.

SMITH (D. E.). Mathematics. (In the series Our Debt to Greece and Rome.) Boston, Marshall Jones Company, 1923. 10 + 175 pp.

STROMGREN (E.). See BRODÉN (T.).

TRIPIER (H.). Les fonctions circulaires et les fonctions hyperboliques étudiées parallèlement en partant de la définition géométrique. Paris, Librairie Vuibert, 1923. 4 + 58 pp.

WEIDEMANN (-.). Zauberquadrate und andere magische Zahlenfiguren der Ebene und des Raumes. Leipzig, Oskar Leiner, 1922. 83 pp.

WITTING (A.). Abgekurzte Rechnung nebst Einfuhrung in die Rechnung mit Logarithmen. (Mathematisch-Physikalische Bibliothek, Band 48.) Leipzig, Teubner, 1922.

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PART II. APPLIED MATHEMATICS

Abadie-Dutemps (—.). Notice sur l'hypsonome. Instrument qui donne les plans cotés sans calculs, les hauteurs des points ou la situation de points dont la cote est donnée. Paris, Gauthier-Villars, 1923. 64 pp.

Appell (P.) et Chappuis (J.). Leçons de mécanique à l'usage des élèves des classes de mathématiques A et B. 3e édition, entièrement refondue. Paris, Gauthier-Villars, 1923. 9 + 416 pp.

Auerbach (F.). Entwicklungsgeschichte der modernen Physik, zugleich eine Uebersicht ihrer Tatsachen, Gesetze und Theorien. Berlin, Springer, 1923. 8 + 344 pp.

Babini, (J.). Termodinámica gráfica. Buenos Aires, Revista del Centro de Estudiantes de Ingenieria, 1923. 56 pp.

Bateman (H.). See Swann (W. F. G.).

Blanc (A.). Le chef mécanicien-électricien. Tome I: Mathématiques pratiques. Paris, Librairie Desforges, 1923. 16 + 319 pp.

Bohr (N.). Les spectres et la structure de l'atome. Traduit par A. Corvisy. Paris, Hermann, 1923. 8vo. 152 pp.

Borel (E.). See Campbell (N. R.).

Bouasse (H.) et Carrière (Z.). Diffraction. Paris, Librairie Delagrave, 1923. 22 + 480 pp.

Büchler (R.). Ueber die Einsteinsche Relativitätstheorie. Aachen, 1922. 4 pp.

Campbell (N. R.). Les principes de la physique. Préface de M. E. Borel. Paris, Alcan, 1923. 19 + 200 pp.

Carrière (Z.). See Bouasse (H.).

Cermak (P.). Die Rontgenstrahlen. Leipzig, Barth, 1923. 130 pp.

Chappuis (J.). See Appell (P.).

Coleman (G. S.). Calculations in heating and ventilation. London, Longmans, 1923. 19 + 255 pp.

Collins (A. F.). The book of the microscope. New York, Appleton, 1923. 15 + 245 pp.

Corps (—.). La simultanéité générale et le temps universelle. Paris, Gauthier-Villars, 1923.

Corvisy (A.). See Bohr (N.).

Croft (T.). Steam-turbine principles and practice. London, McGraw-Hill, 1923. 11 + 347 pp.

Dull (C. E.). Essentials of modern physics. London, Harrapp, 1923. 11 + 525 pp.

Eason (A. B.). The prevention of vibration and noise. (Oxford Technical Series.) London, H. Frowde and Hodder and Stoughton, 1923. 12 + 163 pp.

Feldman (W. M.). Biomathematics; being the principles of mathematics for students of biological science. London, C. Griffin, 1923. 19 + 398 pp.

Fischer (V.). Eine Darstellung des Nernstschen Wärmetheorems. 2ter Teil: Verallgemeinerung des Theorems. Frankfurt a. M., Selbstverlag, 1923. 44 pp.

Gerlach (W.). Atomabbau und Atombau. Die physikalische Analyse des Atoms. Jena, G. Fischer, 1923. 52 pp.

Gleichen (A.). Die Theorie der modernen optischen Instrumente. 2te, neubearbeitete Auflage. Stuttgart, Enke, 1923.

Grimsehl (E.). Lehrbuch der Physik. Band 2. 5te, vermehrte und verbesserte Auflage, herausgegeben von W. Hillers und H. Starke. Leipzig, Teubner, 1923.

Groume-Grjimailo (W. E.). The flow of gases in furnaces. Translated from the French by A. D. Williams. New York, 1923. 399 pp.

ter Heerdt (J.) Overzicht van de theorie en de toepassingen van gassen waarin de onderlinge botsingen der moleculen kunnen verwaardloosed woorden. Utrecht, N. V. Dekker, 1923. 7 + 324 pp.

Hillers (W.). See Grimsehl (E.).

Hobart (H. M.). Electric motors, their theory and construction. 3d edition, revised throughout. Volumes 1–2. London, Pitman, 1923. 16 + 412 + 15 + 384 pp.

Huber (M. T.). The theory of rectangular anisotropic plates, with particular reference to reinforced concrete plates. (In Polish.) Lemberg, 1921.

Kelley (T. L.). Statistical method. New York, Macmillan, 1923. 11 + 390 pp.

Kennard (E. H.). See Swann (W. F. G.).

Kosack (E). Elektrische Starkstromanlagen. 6te, durchgesehene und ergänzte Auflage. Berlin, Springer, 1923.

Le Besnerais (M.). Théorie du navire. Tome 1. Paris, Armand Colin, 1923. 16mo. 162 pp.

Leigh (C. W.). Practical mechanics and strength of materials. London, McGraw-Hill, 1923. 8 + 293 pp.

Lorain (P.). L'hélice propulsive. Etude théorique et expérimentale du propulseur marin. Applications pratiques. Paris, Dunod, 1923. 196 pp.

Loschner (H.). Taschenbuch für praktische Geometrie. Berlin, Vereinigung wissenschaftlicher Verleger, 1922. 10 + 147 pp.

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Maccall (W. T.). Alternate current electrical engineering. London, University Tutorial Press, 1923. 8 + 493 pp.

Mewes (R.). Über die spezifische Warme der gas- und dampfformigen flüssigen und festen Stoffen. Berlin, Verlag der Arbeitsgemeinschaft deutscher Naturforscher und Philosophen, 1922. 8 + 124 pp.

Mortara (G.). Lezioni di statistica metodologica. Città di Castello, 1922. 11 + 562 pp.

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RICE (J.). Relativity. A systematic treatment of Einstein's theory. New York and London, Longmans, 1923. 16 + 397 pp. $6.00

SCHOTT (S.). Statistik. 3te Auflage. Leipzig, Teubner, 1923. 128 pp.

SCHMIDT (H.). La prima conoscenza della relatività dell'Einstein accessibile a tutti. 3a edizione italiana riveduta ed ampliata. (Manuali Hoepli.) Milano, Hoepli, 1923.

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SHIELDS (S.). The cause of the rotation of the earth, planets, etc. London, Pitman, 1923. 7 + 266 pp.

STARKE (H.). See GRIMSEHL (E.).

SWANN (W. F. G.), TATE (J. T.), BATEMAN (H.) and KENNARD (E. H.). Electrodynamics of moving media. (Bulletin of the National Research Council.) Washington, National Academy of Sciences, 1922. 172 pp. $2.00

TATE (J. T.). See SWANN (W. F. G.).

THOMSON (J. J.). Les rayons d'électricité positive et leurs applications aux analyses chimiques. Paris, Hermann, 1923. 8 + 223 pp.

VERDURAND (A.). Théorie simplifiée de la télégraphie et de la téléphonie sans fil. 3e édition. Paris, Dunod, 1923. 6 + 56 pp.

VILLEY (J.) Les divers aspects de la théorie de la relativité. Paris, Gauthier-Villars, 1923. 8vo. 96 pp.

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WEISS (J. B.). Gliding and soaring flight. London, Sampson Low and Company, 1923. 20 + 164 pp.

WILLIAMS (A. D.). See GROUME-GRJIMAILO (W. E.).

WINANTS (M.). La permutation tournante. Liège, Imprimerie Bénard, 1921. 33 pp.

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THE SIXTEENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION

The sixteenth regular meeting of the Southwestern Section of this Society was held at the University of Missouri on Saturday, December 1, 1923. The total attendance was thirty-one, including the following eighteen members:

E. F. Allen, Ashton, Blumberg, Brenke, Chittenden, Decherd, Dunkel, E. R. Hedrick, Ingold, Lefschetz, J. V. McKelvey, U. G. Mitchell, Rider, Eugene Stephens, Stouffer, J. S. Turner, R. A. Wells, Westfall.

Professor Hedrick occupied the chair, being relieved during the sessions by Professors Ashton and Brenke. The morning session was devoted to the reading of the papers listed below.

During the afternoon session Professor Henry Blumberg gave a special lecture by invitation of the program committee on *Properties of unrestricted functions*. This lecture will appear in full in an early issue of this BULLETIN.

It was voted to hold the next meeting of the Southwestern Section at Iowa State College, at Ames, Iowa. The following program committee was elected: Professors E. R. Smith (chairman), R. L. Moore, and E. B. Stouffer (secretary).

The titles and abstracts of the papers read are given below. Professor Dickson's paper was read by Professor Hedrick. The papers of Mr. Michal and Professor Meacham were read by title.

1. Professor L. E. Dickson: *Integral solutions of the equation* $x^2-my^2=zw$.

This paper appeared in the December number of this BULLETIN.

2. Professors E. R. Hedrick and Louis Ingold: *Extensions of Beltrami's equations.*

The equations of Beltrami, which are sometimes used to define a complex function on a two-dimensional surface, occur in connection with other two-dimensional problems. In this paper these problems and the corresponding Beltrami equations are extended to higher dimensions, in a manner consistent with the paper by the same authors presented to the Society at the Summer Meeting.

3. Professor Louis Ingold: *A symbolic treatment of the geometry of hyperspace.*

In previous papers the author has given a vector interpretation of Maschke's symbols and symbolic differential parameters, with applications to the properties of tangent varieties to arbitrary spaces characterized by a quadratic differential form. In this paper the symbolic method with the vector interpretation is used to study the normal and curvature properties of such spaces.

In addition to the geometric results, a number of new relations are obtained connecting various differential parameters and invariants.

4. Mr. A. D. Michal: *Integro-differential invariants of one-parameter groups of Fredholm transformations.*

The author finds necessary and sufficient conditions that an analytic functional of a function y and its derivative be invariant under a given arbitrary continuous one-parameter group of linear functional transformations of Fredholm type of the argument y. Further, it is shown that if the kernel of the infinitesimal transformation is of a certain type, such invariant functionals always exist. The calculation of the invariants in this latter case is then effected.

The functional is assumed to involve y and its derivative between the values 0 and 1, and, considered as a functional of these two arguments, may, for a given functional value of the derivative of y, be assigned arbitrarily as a functional of the other argument.

5. Professor S. Lefschetz: *The Kronecker-Poincaré index for certain manifolds.*

In this paper it is shown that the Kronecker-Poincaré index of two manifolds representing in function S_4 the zeros of two functions of two complex variables is equal to their number of intersections. The theorem and its generalization give rise to important and varied applications in algebraic geometry.

6. Professor J. S. Turner: *Rational triangles in which one angle is a rational multiple of another.*

In this paper, a simple method is described for finding the relation between the sides of a triangle ABC when $A = nB$, where n is rational. The relation is computed, and the complete integral solution (which may also be interpreted as the complete rational solution) is given, when $n = 2, 3, 4, 5$ and $3/2$.

7. Dr. E. F. Allen: *The jacobian of a contact transformation.*

If $x_1 = X(x,y,p)$, $y_1 = Y(x,y,p)$, $p_1 = P(x,y,p)$ is a contact transformation, it satisfies the Pfaff differential equation $dy_1 - p_1 dx_1 = \varrho(dy - p\,dx)$, where ϱ is in general a function of x, y and p. It is shown that the jacobian of this transformation is equal to ϱ^2. This is generalized to a space of $n+1$ dimensions. It is also shown that the surface $\varrho = 0$ in x, y, p space is transformed into a curve in x_1, y_1, p_1 space. This is also generalized to a space of $n+1$ dimensions.

8. Professor E. B. Stouffer: *On polynomials expressed as determinants with linear elements.*

Dickson has determined the types of general homogeneous polynomials which can be expressed as determinants with linear elements. In the present paper the author obtains Dickson's main results by entirely different means, use being made of the independence of sums of principal minors of determinants.

The method here used gives a simple proof of the well known theorem: *A sufficiently general cubic surface can be generated by three projective bundles of planes.*

9. Professor E. D. Meacham: *Surfaces whose osculating ruled surfaces belong to linear complexes.*

The necessary and sufficient conditions that a ruled surface belong to a linear complex have been given by Wilczynski. Making use of these conditions, the author of this paper investigates those non-ruled surfaces whose osculating ruled surfaces are contained in such linear complexes.

Certain properties of these surfaces and of the linear complexes involved are established, and a geometric construction of the surfaces is given.

10. Professor E. D. Meacham: *Notes on the geometric characterization of the two lines common to four linear complexes associated with a point of a non-ruled surface.*

The author proves the following theorem: At a point P on a non-ruled surface S, the two lines common to the four fundamental complexes are the diagonals of a skew quadrilateral whose sides are the four flecnode tangents and whose vertices are the focal points on these sides.

11. Professor E. W. Chittenden: *Nuclear and hypernuclear points in the theory of abstract sets.*

A point p is a nuclear point of a set E of power μ if every neighborhood of p contains a subset of E of power μ; hypernuclear if there exists for every neighborhood V of p a subset of E which is *interior* to V. The paper states in terms of these concepts necessary and sufficient conditions that a compact set in a space V be perfectly compact, and that a self-compact set possess the property of Borel-Lebesgue. The second theorem generalizes theorems of Fréchet and Kuratowski-Sierpinski, and is of special interest because it is shown by an example to be independent of the closure of derived sets.

E. B. Stouffer,
Secretary of the Section.

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PROBLEMS IN INVOLUTORIAL TRANSFORMATIONS OF SPACE*

BY VIRGIL SNYDER

1. *Introduction.* A report of great value and of general interest was presented to this Society at the Chicago meeting of April, 1922. While only a limited number had the opportunity to hear Professor Coble on that occasion, fortunately his message has reached a much wider public, by appearing in this BULLETIN (vol. 28 (1922), pp. 329–364). On account of the wider purpose there in hand, it was impossible to treat in detail all the many ramifications of the theory, or to show all their interrelations. My present purpose is to comment more fully on one narrow phase of this report, namely, that of involutions.

In the plane the problem is almost completely solved. It was shown by Bertini(1)† that there are four types to one of which every plane involutorial transformation can be reduced; they are the harmonic homology (H), the Geiser(2) (G) of order 8 with 7 triple points, the Bertini(1) (B) of order 17 with 8 six-fold points, and the Jonquières(3) (J) of order n with one fundamental point of order $n-1$ and $2n-2$ simple ones. Of these, all but the last are individual types, but (J) exists for every positive integral value of n. For special (J), JH is also an involution, but this can always be reduced to an H. Moreover, all these involutions are rational, that is, the pairs of conjugate points can be mapped rationally upon a plane (x') such that between (x') and the given plane (x) there exists an algebraic (1, 2) point correspondence.

Thus, that associated with (H) may be expressed in the form $x_2'x_3-x_3'x_2=0$, $x_1'x_2x_3-x_2'x_1^2=0$. The invariant or coincident points constitute the line $x_1 = 0$. That associated with (G) has the form

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*Presented to the Society at the Symposium held in New York City, December 28, 1923.

†The numbers refer to the papers listed at the end of this article.

$$x'_1 C_1 + x'_2 C_2 + x'_3 C_3 = 0,$$
$$x'_1 x_1 + x'_2 x_2 + x'_3 x_3 = 0,$$

in which each $C_i = 0$ is a general conic. Each cubic of the net formed by linear combinations of $x_i C_k - x_k C_i = 0$ is transformed into itself. The curve of invariant points is the jacobian of the net, $K_6(r_1^2 \cdots r_7^2)$, in which r_i is a simple basis point. The curve of branch points in (x'), in (1, 1) correspondence with K_6, is the general quartic L'_4, which expresses the condition that the line in (x) touches the associated conic. Every line of (x) contains one pair of conjugate points. From this standpoint the system of the bitangents, the contact conics, cubics, etc. of L'_4 can all be determined directly by elementary methods. Their two images in (x) are conjugate as to (G). That associated with (B) may be expressed by the equations

$$x'_1 \varphi_1 + x'_2 \varphi_2 = 0,$$
$$x'_1 f + x'_3 \varphi_1 (k_1 \varphi_1 + k_2 \varphi_2) = 0,$$

in which $\varphi_i = 0$ is a general cubic curve, and $f = 0$ is a proper sextic having eight of the points h_i common to $\varphi_1 = 0$, $\varphi_2 = 0$ for double points. The curve of invariant points is the jacobian of the system $\varphi_1 = 0$, $\varphi_2 = 0$, $f = 0$. It is of order 9, and has eight h_i as triple points, $K_9(h_1^3 \cdots h_8^3)$. The curve of branch points is also a sextic L'_6, having three branches touching each other at a common point. An arbitrary line in (x) contains four pairs of conjugate points; i. e., the simplest form of (B) is of class 4. Finally, the (1, 2) correspondence of form (J) can be expressed by $x'_1 x_2 - x'_2 x_1 = 0$, $x'_3 M_1 - u' M_2 = 0$, in which $M_i = 0$ is a curve of order m with an $(m-2)$-fold point at (0, 0, 1); u' is linear in (x'). The jacobian is the curve of coincidences, a hyperelliptic curve having an $(n-2)$-fold point at (0, 0, 1). This involution is of class zero. This is the easiest way to find and to classify the plane involutions of order 2; to find all possible ways in which the curves of a net can have two variable points of intersection. The question of reducibility or of the equivalence of two given cases is not more difficult by this method than by that of Bertini. The seven basis points r_i have for images in (x') seven bitangents of L_4'; a point of a bitangent has two images, one of which is the basis point r_i, and the other describes the cubic of the net having a node at r_i. The point and the nodal cubic are conjugate in the involution (G).

Indeed plane involutions of every order were later shown to be rational, by Castelnuovo(4). For particular positions of the basis points the types assume special forms, but they are all included in the same (or simpler) categories(5). An interesting case of (J) is that in which all the curves of the net have contact of the second order on each branch at the multiple point. Its equation is

$$\lambda_1(-u_{n-1}x_3 + u_n) + \lambda_2 u_{n-1} x_1 + \lambda_3 u_{n-1} x_2 = 0$$

in which u_i is binary in x_1, x_2 of order i. All the basis points are now coincident at $(0, 0, 1)$.

The curves of coincident points characterize the type of the involution. Hence it follows that any surface mapable on a double plane is, or is not, rational according as its curve of branch points can be reduced to one of the above curves(6). Another possibility is that involutions may be compounded. This is particularly useful in studying irrational algebraic curves. A curve is hyperelliptic if it has a linear series g_2^1. Thus every hyperelliptic curve is invariant under at least one involution. But a curve may be cut by the curves of a pencil in such a way that each curve cuts more than one pair of conjugate points from the given one. Such involutions, possible only on curves of genus greater than one, are usually irrational; they exist for any given genus(7). No algebraic curve of genus greater than 1 can belong to an infinite group, either continuous(8) or discontinuous. An elliptic curve is invariant under an infinite series of operations which do not form a group. It has an infinite number of central involutions each with four fixed points, and three without fixed points. A rational curve is invariant under a three-parameter group, within which are ∞^2 involutions.

2. *Transformations in Space.* Of the linear transformations, only two are involutorial, the central and the axial involution. In the former we have a plane of invariant points, in the latter, two skew lines. The quadratic transformations may be defined as in the plane. They were all found in the early papers of Cayley(9), Cremona(10), and Noether(11). An elementary derivation is given by Snyder and Sisam(12). The involutorial forms are either perspective, such that every line of a bundle remains invariant, or the product of the perspective inversion and a central homology. A full discussion is given by Doehlemann(13). An application to the theory of electric images was given by Liouville(14). For the theory and the literature, see Doehlemann(13). One great application of the birational transformations of space is to the study of systems of curves lying on given surfaces. Thus, on the quadric, every algebraic curve can be expressed by means of an (a, b) correspondence, and only those plane curves which can be thus expressed are projections of space curves of the same order lying on a quadric. Since the surfaces of a homaloidal web are all rational, the transformation furnishes an immediate method of mapping the surface on a plane. On the other hand, this method does not furnish general criteria for determining whether a birational transformation is involutorial or not. Nearly all of the earlier involutions were found directly from particular geometric conditions imposed by the problem considered, and none have been obtained by following the procedure employed by Bertini for the plane involutions.

The surfaces of the system must satisfy two fundamental conditions; they must form a linear system with four independent surfaces; any three intersect in one variable point. Hence the surfaces must have points or curves common to all. But these criteria are common to all birational transformations.

3. *Rational Involutions.* Consider in space (x), a linear system of ∞^3 surfaces, a web, having the property that

any three not in a pencil intersect in two variable points. Let the web be $\sum a_i \varphi_i = 0$. By putting $x'_i = \varphi_i$ $(i = 1, 2, 3, 4)$ the surfaces can be mapped upon the double space (x'). This was first studied by De Paolis(15).

To the planes of (x') correspond the surfaces of the web $|\varphi|$, each of which remains invariant under the involution of associated points, images of the same point of (x'). To the lines of (x') correspond the curves (φ_i, φ_k), also invariant under I. These curves are either hyperelliptic or belong to sub-types of hyperelliptic curves, of genus 1 or 0. Each surface of the web $|\varphi|$ has then ∞^2 hyperelliptic curves, any two of which intersect in two variable points. The locus of points in (x') having the property that the two images in (x) coincide is the surface L' of branch points. The coincident points in (x) define a surface K, in (1, 1) correspondence with the points of L'. These surfaces may be replaced by curves or by points. The system of surfaces in (x'), images of the planes of (x), do not form a linear system, but have certain points and curves in common which are fundamental for the correspondence. A fundamental point P' may have one or both its images in (x) fundamental(16). If its images are a point P and a curve p, then P is a fundamental point of the associated involution I in (x). If both images are fundamental, the associated curve in (x) may be fundamental in different ways.

A complete enumeration of rational involutions would involve the various webs of surfaces with two variable intersections. While these are of course infinite, it is not known to what extent they can be reduced to a small number of families. Thus, in the plane, if we exclude the hyperelliptic curves with common multiple points there are only two nets of curves with two variable points, one of genus 1 and the other of genus 2. The equations concerning a Cremona net are then sufficient to complete the enumeration. When the point P' describes a locus (curve or surface), the images P_1, P_2 will each describe a locus. These two loci may coincide, or if distinct, they are conjugate in I. The necessary and sufficient condition that the two loci p_1, p_2 are distinct is that p' touch L' at every non-fundamental point(17). From this standpoint all the systems of bitangent planes, of contact quadrics, cubics, etc. of the general L' can be found immediately without the use of transcendental methods. In particular the system of contact surfaces of the Kummer surface can be derived from the web of quadrics through 6 points. The numerous formulas derived by De Paolis are not in the main of greatest usefulness on account of being expressed in too many unknowns. When the web is regular, and the basis elements independent, the Riemann-Roch theorem for surfaces,

$$r = p_a + n - \pi + 1,$$

applies, in which r, the dimension of the system of curves in which a fixed surface of the system is cut by the other surfaces of the web, is 2, n is the number of variable intersections of two curves of the system, and π is the genus of the variable curve. Hence this genus exceeds the arithmetic genus of the general surface of the web by unity. Since each curve is hyperelliptic, the number of coincidences is $2\pi + 2$, hence the order of L' is also $2\pi + 2$.

The general discussion of De Paolis was followed by Schoute(18), who confined his discussion to simple cases. Reye(19) gave a detailed synthetic study of the web defined by quadrics through 6 points, which has been treated by a number of writers since. Hudson(20) developed it from the dual of the previous standpoint; Eberhard(21) followed various systems of curves on the surface, to which I added several(22).

An involution is determined on a quartic surface with five basis points by the space cubics through these and a point P, but this may not be birational for all of space. In the case of the Weddle surface the points P, P' are collinear with the sixth node. Other cases are treated by Sturm(23), by Marletta(24), and by Baldus(16). The latter also discussed (1, 2) correspondences between irrational ruled surfaces, in which fundamental elements appear which are not found in planar involutions. A particular (1, 2) correspondence associated with a net of cubics through a space quartic curve γ_4 and five coplanar basis points was studied by Pieri(25), primarily for the properties of L'. An interesting property appears here, as the variable curves are of genus 2. The system of curves is irregular since the basis points are not independent. Three cases cited by Pieri as belonging to distinct types are now known to be reducible to particular cases of the types considered by him. Another important feature is the appearance of an irrational fundamental curve. In the plane the image of a fundamental point is always a rational curve of order equal to the multiplicity of the point. But the net of surfaces of the web which pass through an arbitrary point of the plane π all pass through the cubic curve of the pencil determined by this point, the basis points A, and the four points of intersection of π and γ_4. The image of this cubic in (x') is a point, and the point describes a straight line as the cubic curve describes the pencil. The other image of P' is a point R in (x) which depends upon how P' is approached. To an arbitrary plane through P' corresponds a linear g_2^1 on the cubic curve. This property accentuates the difficulty of defining a type, as an irrational curve may be transformed into a point.

Another web of cubics was considered by Romano(26), that having three skew lines and four points for basis elements. This presents six fundamental lines of the second kind, such that in the associated involution the image of any point on one of them is the whole line passing through it.*

In every case, when simple basis points of the web appear, the image of each in (x') is a tangent plane to L'; the conjugate, in the associated involution, of the basis point is the surface of the web having a double point at that point. When the web has a simple basis curve, each point of the curve goes into a straight line in (x'), and the conjugate in I is a rational curve. These curves generate a surface, conjugate of the given curve, which is a multiple basis curve in the involution.

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*In the later discussion of the same case by Sharpe and Snyder, TRANSACTIONS OF THIS SOCIETY, vol. 21 (1920), p. 56, these lines are overlooked, so that the characteristics of the involution as there given are incomplete. Similar omissions occur in the later types defined by a web of cubic surfaces. Romano's paper was not known to the authors at that time.

But when the section of the surfaces of a web by a fixed surface is completely accounted for by fundamental elements, various paradoxes may appear. This has not been completely worked out, especially when the section is made on an irrational non-ruled surface.

The problem of the existence of these webs depends upon formulas of postulation involving contact of higher order at multiple points. In the original memoir of Noether(27) the cases are excluded in which the tangent cones to the surfaces of the web at a multiple basis point are composite, and contacts of different sheets along branches of the curve is not considered. Notwithstanding the recent valuable additions made by Hudson(28), by Severi(29), and by Tummarello(30), little progress can be made in the problem of reducibility and the determination of the equivalence of two given involutions until more is known on this subject.

When all the points of intersection of a line or a curve with surfaces of the web are fixed, or are on fixed basis curves, the line is called a parasite by De Paolis. Since a net of surfaces pass through it, its image in (x') is a point. If the given curve belongs to a linear system, each curve of which is a parasite, the image point describes a straight line.

A satellite is a line or point of contact which insists on coming in, unbidden, when certain conditions are imposed. Consider the conical surface having a cuspidal edge $x_1^2x_3 = x_2^3$, and the rational cone $x_1x_3^{n-1} + Ax_2^kx_3^{n-k} + \cdots = 0$ with undetermined coefficients. The number of lines of contact approaching coincidence with $x_1 = 0$, $x_2 = 0$, in the plane $x_1 = 0$ is always an odd number. If we impose the condition that one more line be counted, a second insists on accompanying it. This is true for a cone having any superlinear branch.

Satellites and parasitic branches will have to be considered in view of fundamental elements of new and as yet little understood forms. The explanation given by De Paolis(15) is incorrect, and the relation between a line and its parasite may violate the theorem of order and multiplicity, as has recently been shown by Montesano(31) and by Tummarello(30).

4. *Involutions by Means of* (1, 2) *Correspondences.* The first comprehensive study of the (1, 2) correspondence from the standpoint of classification of associated involutions was made by Sharpe and me(32). After collecting the necessary formulas of postulation and equivalence we applied them to webs of surfaces of orders 3, 4, and 5. Two kinds of restrictions were made at the outset. Surfaces with higher point singularities, including monoids, were excluded, and the basis elements were assumed to be independent. The first restriction was deliberate. When the surfaces are monoids or, more generally, rational, then each surface of the web can be mapped birationally upon the plane, and the curves of intersection with the other surfaces can be treated by means of plane involutions. The two extreme cases of monoids, those having one non-composite basis curve, and those having a fixed tangent cone at the vertex have been determined since, but the results are not yet published. The former are not monoidal transformations, and most of them cannot be reduced to the monoidal type. Those of the other form are always monoidal, of a very particular type. The other restriction was less deliberate. In the plane the basis elements are independent, and the fundamental curves are uniquely fixed by the position and multiplicity of the basis points through which they are to pass. If the elements are chosen in a particular way, for example three collinear points among the seven which define a Geiser net, the resulting involution can be reduced to a simpler form in which the elements are independent.

Within these limitations, the webs of cubic, quartic and quintic surfaces have been determined. Among them appear the focal surfaces of all the line congruences of order 2 and class 2 to 7, and several related surfaces, together making an unbroken sequence. Incidentally, configurations of curves on these surfaces can be determined much more directly by this method than by those by which they were first discovered. The category also includes surfaces invariant under infinite discontinuous birational groups, both those known previously, as the Kummer surface and the Fano surface, and new surfaces not previously studied.

What was not accomplished was to determine how many of the involutions appeared more than once in a transformed form. It was proved, for example, that there are different involutions having a Kummer surface for surface of branch points L' in (x'). Each focal surface of a line congruence of order 2 appears at least twice. Of course one will meet, in the study of webs of higher order, all the involutions associated with simpler webs, which can be transformed birationally into webs of the given order. Thus, Osborn(33) has studied the webs of order 6, and has refound many of the existing types. The webs of singular cubics have recently been studied by C. Moffa(34) by the method of plane mapping without the use of the double space, and the results in the other webs confirmed by the same method. Miss Moffa shows that fundamental lines of the second kind must exist in every case. This paper, together with the correction to the result obtained by Pieri(25), completes the list of involutions which leave each surface of a web of cubics invariant.

5. *Bases of Classification.* Another scheme of classification is to divide the involutions into families, expressed as a function of the order n, like the Jonquières forms in the plane. The first attempt of this kind was made by De Paolis(35), who classified all the forms that leave a line congruence of order one invariant. The lines may belong to a bundle, or meet a line and a rational curve of order m, with $m-1$ points on the line, or be the bisecants of a cubic curve. The second and third cases are treated exhaustively, but from the standpoint of classification the results are of little

importance, as all of them can be reduced to the monoidal form. The involution of order 7 defined by a web of quadrics through six basis points belongs in this category. Most of its properties are much more easily obtained, however, in the form of the web of quadrics, than expressed in terms of the nodal projection of the Kummer surface upon itself.

Another important family was mentioned by Noether(36) and more fully treated by Montesano(37). In this family the images of the planes of space are surfaces of order n with a common $(n-2)$-fold line and a basis curve of order $3n-4$. This will be discussed later. Another family, also discussed by Montesano(38), consists of surfaces of order $2n+1$, having as basis elements a space elliptic quartic to multiplicity n, and $2n$ of its bisecants for simple lines. This case was also previously cited by Noether(39). It includes a number of particular cases previously mentioned by others. Thus, the type discussed by Pieri(25) is this one when $n=5$. The case in which the conjugates of planes are surfaces of order n with a fixed $(n-1)$-fold line was also studied by Montesano(40).

A number of families of involutions, generalizations of the preceding, were found by Sharpe and me(17), the results of which can perhaps be best expressed by the theorem: There exists an involution of order $12n+5$ having a fundamental quartic γ_4 of genus 1 to multiplicity $6n-1$ and a fundamental curve β_{6n+1}, genus $9n-3$, to multiplicity 4, meeting γ_4 in $12n-4$ points. It has also $12n+16$ simple fundamental lines meeting γ_4, β_{6n+1} each twice and $6n-2$ fundamental double conics meeting γ_4, β_{6n+1} each in 4 points. In symbols,

$$s_1 \sim s_{12n+5} : \gamma_4^{6n-1} + \beta_{6n+1}^4 + (12n+16)u_1 + (6n-2)u_2^2,$$

$$[\gamma_4, \beta_{6n+1}] = 12n-4; \; [u_i, \gamma_4] = [u_i, \beta_{6n+1}] = 2.$$

Similarly,

$$s_1 \sim s_{4n+9} : \gamma_1^{4n+1} + \beta_{5n+5}^{3} + (15n+10)u_1 + 2u_2^2,$$

$$s_1 \sim s_{6n+9} : \gamma_1^{2n+6} + \beta_8^{2n+1} + \beta_{7n}^2,$$

$$s_1 \sim s_{6n+17} : \gamma_4^{2n+7} + \gamma_5^{2n+2} + \beta_{7n+1}^2,$$

$$s_1 \sim s_{6n+10} : \gamma_2^{2(n-1)} + \gamma_7^{2n+2} + \beta_{7n-3}^2.$$

These various types are not reducible to each other.

6. *Monoidal Transformations.* The monoidal transformations were discussed by De Paolis(41) and the product of the perspective monoidal and a harmonic homology was treated by Martinetti(42), who thought there could be no other non-perspective types. All the forms were derived synthetically by Montesano(43), who showed that every type could be reduced to one in which the bundle of lines through the vertex remains invariant, and that the lines of the bundle are either invariant (perspective type) or are interchanged in pairs according to one of the four ternary involutions H, G, B, J. Of these, the latter contains two parameters, the order n of J, and the order m of the perspective monoid. The equations of all the types were derived by me(44).

Let $O = (0, 0, 0, 1)$ be the vertex of the monoid. Since the bundle of lines O is transformed into itself, and is involutorial, it must be either the identity, or H, G, B, or J. In the first case, called the perspective, the conjugate of any point P is on OP and is harmonic with regard to P and the pair of points in which OP meets a fixed surface

$$u_{n-2}x_4^2 + 2u_{n-1}x_4 + u_n = 0,$$

u_i being ternary in x_1, x_2, x_3 of degree i.

The equations of the transformation are

$$rx_i' = x_i,\ i = 1, 2, 3; \quad rx_4' = \frac{-(u_{n-1}x_4 + u_n)}{u_{n-2}x_4 + u_{n-1}}.$$

Every plane through O is invariant and contains a perspective J involution from O. If u_{n-2} is identically zero, the fixed surface is itself a monoid, and P' is the harmonic conjugate of P as to O and the residual point of intersection with OP. The basis curve consists of the lines common to the two cones $u_{n-1} = 0$, $u_n = 0$, which are fundamental of the second kind, that is, the image of any point on any one of them is the whole line passing through it. The image of a point on a k-fold basis line is the whole line counted k-fold.

Of the other types, any one is expressible in the form

$$rx_i' = \varphi_i,\ i = 1, 2, 3; \quad x_4' = \frac{x_4\varphi u + v}{wx_4 - ux},$$

in which the φ_i define an involutorial Cremona net of cones with common vertex at O, φ is a linear function of the φ_i and x is the same linear function of the x_i. The u, v, w are algebraic polynomials in functions k_i that are invariant under the ternary involution.

In particular, take BM. The φ_i are of order 17, and have eight basis lines each to order 6. Now suppose $n = 6$, and v, w are identically zero. The result is of order 18, the image of O is the plane $x = 0$ passing through O, and every line joining a pair of conjugate points meets the fundamental line m: $x = 0, x_4 = 0$. The conjugate of an arbitrary line is a rational curve of order 18, which passes simply through O. If the given line meets m, then the image of the point of intersection is a component of the conjugate; the proper image is a plane curve of order 17 in the plane of m and the given line. It is a plane B transformation. The line meets the invariant cone $R_9 = 0$ of B in 9 points, which also lie on the conjugate C_{17}. Hence the line meets C_{17} in four pairs of conjugates. In this particular monoidal transformation any line joining a pair of conjugate points belongs to a special linear complex and contains three other pairs of conjugate points. This probably includes irrational involutions.

In connection with the monoidal types of birational transformations we have an immediate example of the existence of (n, n^2) types, found synthetically by Tummarello(45) for every value of n, although earlier writers supposed they could not exist for $n > 3$(46).

The equations of the transformation are of the form

$$x_i' = \varphi_i, \ (i = 1, 2, 3), \quad x_4' = u_{n-1}x_4 + u_n,$$

in which $|\varphi|$ is a Cremona net of cones of order n with vertices at $(0, 0, 0, 1)$.

7. *Classification.* The first proposal was to classify involutions, like other birational transformations, according to the order of the surfaces conjugate to the planes of space. This was done for those of order 2 in the earliest

memoirs. It was soon found that those of order 3 could be expressed by means of three symmetric bilinear equations in the coordinates of two variable conjugate points(47). The types arising from three polarities have been established by Morris(48), but a number of interesting additional types appear when one or two of the polarities are replaced by linear complexes, either one or both of which may be general or special. These types offer no new difficulty, but no enumeration is complete without them.

But this method was early recognized to be impracticable, as no general criteria are known, to insure the involutorial character of the transformation when $n>3$. A real advance was that of classifying involutions according to the genus of a general plane section of the conjugate surface of a plane(49). The first category is that of genus zero. The conjugate surface must be either a quadric, a Steiner quartic, or a ruled surface of order n with an $(n-1)$-fold line. Those of the first category were first obtained by Aschieri(50) and by Martinetti(42), those of the second by Montesano(51), and those of the third by Montesano(40). This method has not been developed further, on account of lack of knowledge of forms of rational surfaces with plane sections of given genus. Those of genus 1 have recently been made by Nobile(80).

8. *Line Complex containing the Involution.* The next procedure is that of considering the system of lines obtained by joining conjugate points. This system may be either a congruence or a complex. In the former case each line of the congruence contains an infinite number of pairs of conjugates. These congruences are always of order 1(35), and the transformations reducible to monoidal types. When the lines form a complex, each line contains one or more pairs of conjugate points. The linear complex, each line containing one pair, was considered by Montesano(52).

This does not give an extensive list of involutions, but each involution is determined by a particular curve of order 10 and genus 11. Given a hyperelliptic curve C_6^3 of order 6 and genus 3. Two quartic surfaces through it determine a C_{10}^{11} through which pass ∞^4 quartic surfaces. Of this system, the web passing through a point P also intersects in its conjugate P'. The lines PP' generate a linear complex. An arbitrary plane meets C_{10}^{11} in 10 points lying on a cubic curve. The curve may break up into a particular(53) C_7^5 and a cubic. The involution is of order 11, and may be particularized to orders $10 \cdots 3$. Types belonging to a special linear complex were later studied by Pieri(54), but not all such types were found. They can all be expressed by a single formula, given by Sharpe and me(17). All of the monoidal types *JM* are included in this category, as well as the simple *M*.

Involutions belonging to the tetrahedral complex were derived by Montesano(55), and later from another standpoint by Wimmer(56). This type occurs incidentally in the discussion of Sharpe and Snyder(17). The involution is of order 19. Each surface of a web of quartics having a basis C_{11}^{14} is left invariant by the involution. Other cases of involutions belonging to special quadratic complexes were studied by Pieri(57), and of involutions belonging to complexes of tangents by Pieri(58), and by Montesano(59). They can all be reduced to monoidal types. The question now arises whether an involution belongs to every quadratic complex, and the answer is No. (60) The study of the complex to which an involution belongs is of great service, but it does not furnish a satisfactory basis for classification. Montesano(61) showed that involutions in which each conic of a linear system is self conjugate belonged to the type studied by De Paolis(35), hence reducible to monoidal forms.

In all these cases, each line of the complex contains a single pair of conjugate points, but involutions exist in which each line of the complex contains two, three, or four pairs of conjugate forms(44). Montesano has proposed to me to classify involutions according to the minimum product of the order of the complex to which it belongs and the number of pairs of points on each line of the complex.

9. *Irrational Involutions.* It is known that irrational involutions in space exist. The first example was given by Enriques(62), and later simplified by Aprile(63). But both of these are of order higher than two, and it is not known whether irrational involutions of order 2 exist or not. But there are three that make it seem very probable. The first was discussed by Montesano(64). Consider the group of order 32, $x_i' = \pm x_i$ which leaves the equations $\sum x_i^2 = 0$, $\sum a_i x_i^2 = 0$ invariant. Think of the x_i as line-coordinates, and map the quadratic complex defined by them on the points of space. The central perspective transformations go into six perspective monoidal ones of order 3, all the invariant cubic surfaces passing through a common space quintic of genus 2. The product of two of the central-perspectives corresponds to cubic involutions which leave every plane through the line joining the centers invariant, and every point of an elliptic space quartic remains fixed, the curve being the residual to the quintic, intersection of the invariant cubic surfaces of the monoidal components. The ten products by threes $(123 = 456)$ are more complicated. This depends upon the representation of the lines upon the point space, and the configuration is included in the next case, also considered by Montesano(37). The following outline is based upon a study being made by Professor Sharpe and me, not yet completed. Consider the involution I in which the conjugate of a plane is a surface of order n, having a line p to multiplicity $n-2$. There is also a basis curve of order $3n-4$ meeting p in $3n-7$ points. Planes through p are transformed into planes through p, either each into itself, or the pencil is in involution, with two invariant planes. If p is defined by $x_1 = 0$, $x_2 = 0$ we may write

$$x_1' = Mx_1, \quad x_2' = -Mx_2, \quad x_3' = E, \quad x_4' = F.$$

Since the transformation is birational and involutorial, it follows that x_3, x_4 are connected with x_3', x_4' by a bilinear equation. It is now easy to get the equation of the complex to which the involution belongs. It is always a particular form, being most nearly general for $n = 3$. It has been shown by Fano(65) that the general cubic complex is irrational. In some particular cases the spatial, surface, and arithmetic genera were all found to be zero, but these conditions to not insure the rationality of the involution, since a three-dimensional variety may have all its genera zero and still be irrational(65). The third illustration is furnished by the general cubic variety V in four space(67). Pass a plane through any point P on V and any line l lying on it. The plane section consists of l and a conic meeting it in two points, A, B. The lines AP, BP are tangent to V at points on l. Conversely, given any tangent to V at a point of l. It meets V in a residual point P, hence there is a (1, 2) correspondence between the points of V, and its tangents at points of l. If the tangents are now mapped on the points of an auxiliary three space, we obtain an involution belonging to V(68). It is of order 6, and belongs to the type JM, hence also to a linear complex, but every line joining a pair of conjugate points contains one other pair. Thus it is mapped on a special linear complex doubly. But we have practically no criteria of rationality of varieties which can be mapped doubly on rational three spaces(66).

10. *Curves and Surfaces Invariant under Involutions.* In every rational involution every surface of a web remains invariant, and also the curves of intersection of pairs of surfaces of the web. In this way many properties of curves can be obtained which belong to given irrational involutions of any given genus. Associated with every curve of (x'), not a contact curve of L', is a curve of (x) which is invariant under the involution, the pairs of conjugate points belonging to the given curve in (x'). And we may have surfaces belonging to more than one web, each of which defines an involution. On this surface, the two involutions generate a group which may be finite or infinite. Numerous examples of such surfaces have been given(69), but many of them are

disposed of by the theorem of Enriques(70) that such a surface has a pencil of elliptic curves, except when $p_a = P_2 = 1$. In every case thus far found, all the operations can be expressed in terms of Cremona transformations. A surface having $p_a = P_2 = 1$ but not containing a pencil of elliptic curves was found by Fano(71), and its group discussed by Severi(72). That it belongs to two webs connected with (1, 2) correspondence was shown by Sharpe(32) and me. A second example was found by us(73), having similar properties, defined by the web of quartic surfaces through a space curve of order 8 and genus 2. A detailed discussion of all these types, with particular reference to their groups, was given by Fano(74).

But another quartic surface exists (73), for which $p_a = P_2 = 1$, which has no elliptic curves, and is invariant under an infinite discontinuous group, but not generated by involutions (75). The operations of this group may also be expressed by Cremona transformations. The surface is defined by containing a sextic curve of genus 3.

PROBLEMS.

1. What is the definition of a type of space involution?
2. Can rational involutions be classified in a finite number of types?
3. Do irrational involutions of order two exist?
4. Do all surfaces invariant under infinite discontinous groups belong to Cremona groups?

   For example, given a quartic surface with five nodes. The space cubic determined by them and a point P on the surface meets it again in P', defining an involution under which the surface is invariant. Can this be expressed as a Cremona transformation? Now take a quartic with six nodes. There are six such involutions. Do they generate an infinite group? These illustrations may be trivial, since the surface contains pencils of elliptic curves.
5. Can the equivalence of two given involutions be expressed in terms of rational invariants?

This is a short list, but will probably suffice for some time. Encouraged by recent contributions to the theory of postulation and equivalence, as given by Hilda Hudson(76) and by Severi, and illustrated by Tummarello and by Miss Moffa, and by the important theorem of Montesano(77) concerning multiple parasitic lines, let us continue to hope that a final answer may be found to some of them. The recent researches of Chisini(78) on the singularities of algebraic surfaces, and the proposal of Severi(79) concerning classification of space curves will be of fundamental assistance.

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A GENERALIZATION OF THE SYLLOGISM*

BY B. A. BERNSTEIN

The syllogism is the proposition:

If x is y and y is z, then x is z.

In the language of boolean algebra this proposition is:

$$\text{If } xy' = 0 \text{ and } yz' = 0, \text{ then } xz' = 0,$$

where the usual notations are used, with the prime indicating negation. There thus exists in boolean algebras a universal relation† R such that

$$\text{(1)} \qquad \text{If } xRy \text{ and } yRz, \text{ then } xRz.$$

I propose to find the most general boolean relation R which satisfies (1), and to show the connection between this relation and the syllogism relation $xy' = 0$.

I start with the fact that any universal relation between two boolean elements x, y is given by an equation of the form

$$\text{(2)} \qquad Axy + Bxy' + Cx'y + Dx'y' = 0.$$

Let (2) be a relation R satisfying (1). Then the *discriminants* A, B, C, D must be such that from

$$\text{(i)} \qquad Axy + Bxy' + Cx'y + Dx'y' = 0,$$
$$\text{(ii)} \qquad Ayz + Byz' + Cy'z + Dy'z' = 0,$$

we may conclude

$$\text{(iii)} \qquad Axz + Bxz' + Cx'z + Dx'z' = 0.$$

That is, the discriminants A, B, C, D must be such that equation (iii) is the result of eliminating y from equations (i) and (ii). Now (i) and (ii) together are equivalent to the single equation

$$\text{(iv)} \quad (Ax + Cx' + Az + Bz')y + (Bx + Dx' + Cz + Dz')y' = 0.$$

The result of eliminating y from (i) and (ii) is, then, the result of eliminating y from (iv). This result is

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†That is, a relation given by a universal proposition.

$$(Ax + Cx' + Az + Bz')\ (Bx + Dx' + Cz + Dz') = 0,$$

or

$$A(B + C)xz + (B + A)\ (B + D)xz' + (C + A)\ (C + D)x'z + (C + B)Dx'z' = 0,$$

or

$$\text{(v)} \quad A(B + C)xz + (B + AD)xz' + (C + AD)x'z + D(B + C)x'z' = 0.$$

Equating corresponding discriminants of (iii) and (v), we get, as necessary and sufficient conditions that (2) be a relation R satisfying (1),

$$A = A(B + C), \quad B = B + AD, \quad C = C + AD, \quad D = D(B + C),$$

or

$$AB'C' = 0, \quad AB'D = 0, \quad AC'D = 0, \quad B'C'D = 0,$$

or

$$(3a) \quad A + D < B + C, \quad \text{and} \quad (3b) \quad AD < BC.$$

We therefore find that the *most general boolean relation* R *which satisfies the proposition* (1) *is given by the relation*

$$(4) \quad Axy + Bxy' + Cx'y + Dx'y' = 0, \qquad A + D < B + C, \quad AD < BC.$$

We find also that *the syllogism relation* $xy' = 0$ *is the relation* (4) *in which* $A = C = D = 0$, $B = 1$.

We note that the relation of equivalence, which is $xy' + x'y = 0$, is the relation (4), with $A = D = 0$, $B = C = 1$.

The statement that the relation (4) satisfies (1) is true even if the relation (4) be non-existent. Thus, the relation $xy + xy' + x'y + x'y' = 0$, i. e., the relation $1 = 0$, is non-existent. But it satisfies (4) and it is true that *If* $1 = 0$ *and* $1 = 0$, *then* $1 = 0$. If we impose on (2) the condition that it be an *existent* relation, we must have

$$(5) \qquad ABCD = 0.$$

Since condition (3b) can be written in the form

$$AD = AD \cdot BC,$$

conditions (3b) and (5) together are equivalent to

$$(6) \qquad AD = 0,$$

and conditions (3a), (3b) become

$$A+D<B+C, \qquad AD=0. \tag{7}$$

Hence we find that *the most general existent boolean relation R which satisfies* (1) *is given by* (2) *and* (7).

Our main results may also be stated in the following form: *The totality of transitive universal relations in a boolean algebra is given by* (4). *The totality of existent transitive universal relations is given by*

$$Axy+Bxy'+Cx'y+Dx'y'=0, \quad A+D<B+C, \quad AD=0.$$

THE UNIVERSITY OF CALIFORNIA

ON CERTAIN QUINARY QUADRATIC FORMS*

BY E T. BELL

1. *Introduction.* Except the classical theorems on the total number $N_5(n)$ of representations of the integer n as a sum of five integer squares, no explicit results on numbers of representations in quinary quadratic forms seem to have been obtained.† In general $N_5(n)$ is not expressible as a function of the real divisors alone of a single integer, but when n is a square, $N_5(n)$ is so expressible. This remarkable fact was found inductively by Stieltjes for $N_5(p^2)$, p prime, and proved for $N_5(n^2)$ by Hurwitz,‡ who showed that if $n=2^a m$, m odd, $N_5(n^2)=10\zeta_3(2^a)\,H(m)$, where

$$H(m)=[\zeta_3(p^a)-p\zeta_3(p^{a-1})]\,[\zeta_3(q^b)-q\zeta_3(q^{b-1})]\ldots,$$

$\zeta_r(n)$ being the sum of the rth powers of all the divisors of n, and $m=p^a q^b\ldots$ the prime factor resolution of m; by convention $H(1)=1$. In the course of his proof he showed that

$$\zeta_1(m^2)+2\zeta_1(m^2-2^2)+2\zeta_1(m^2-4^2)+\cdots=H(m).$$

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* Presented to the Society, December 27, 1923.

† Cf. Bachmann, *Zahlentheorie*, vol. 4, pp. 565–594.

‡ COMPTES RENDUS, vol. 98 (1884), pp. 504–7; cf. Dickson's *History of the Theory of Numbers*, vol. 2, p. 311. For quadratic forms in $n>4$ variables, cf. ibid., vol. 3, chap. XI.

This identity makes it possible to find numerous theorems for the total number $N(2^{\alpha}m^2)$ of representations of $2^{\alpha}m^2$ in a quinary quadratic form other than a sum of five squares similar to Hurwitz' for $N_5(2^{2\alpha}m^2)$, viz., $N(n)$ is a function of the real divisors alone of a single integer when and only when* $n = 2^{\alpha}m^2$, m odd, $\alpha \geqq 0$, and similarly for the number of proper representations. The results are so unexpectedly simple that we shall take space in § 3 to give 19 of them, or, counting the separate cases according to α, 72.

2. *Theorems on Representations.* Henceforth the variables x, y, z, t, u may take any integer values $\gtreqqless 0$; m, n, α, k_r are constant positive integers, m is odd, $n, \alpha \geqq 0$ are arbitrary, $k_r = 0$ when $r < 0$, and a, b, c, d, e are constant integers $\lessgtr 0$.

THEOREM I. *If the total number of representations of* $2^{\alpha}m$ *in the form* $ax^2 + by^2 + cz^2 + dt^2$ *is* $k_{\alpha}\zeta_1(m)$, *then the total number of representations of* $2^{\alpha}m^2$ *in the form* $(a, b, c, d, \alpha) \equiv ax^2 + by^2 + cz^2 + dt^2 + 2^{\alpha+2}u^2$ *is* $k_{\alpha}H(m)$.

Let $N'(n)$, $N(n)$ denote respectively the total numbers of representations of n in the forms $ax^2 + by^2 + cz^2 + dt^2$, $ax^2 + by^2 + cz^2 + dt^2 + eu^2$. Then, the sum continuing so long as the argument of N' is positive, evidently

$$N(n) = N'(n) + 2N'(n - e1^2) + 2N'(n - e2^2) + \cdots.$$

If $n = 2^{\alpha}m^2$, $e = 2^{\alpha+2}$, the argument of each N' is an odd multiple of 2^{α}. Hence

$$N(2^{\alpha}m^2) = k_{\alpha}[\zeta_1(m^2) + 2\zeta_1(m^2 - 2^2) + 2\zeta_1(m^2 - 4^2) + \cdots],$$

and the theorem follows by Hurwitz' identity.

As customary, a representation, in which the G.C.D. of x, y, z, t, u is 1 is called proper. To state the number of proper representations when the total number is given by Theorem I, we need $J_2(m)$, the Jordan totient of order 2 of m. If $m = \prod p_{\alpha}$ is the resolution of m into powers of distinct primes >1, $J_2(m) = m^2\prod(1 - p^{-2})$, $J(1) = 1$ by convention, and $J_2(m)$ is the number of sets of two equal or distinct positive integers $\leqq m$ whose G.C.D. is prime to m. Applying the general method of a former paper* to Theorem I, we find readily the following result.

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* Because in each case it can be shown that $N(n) = cN_5(n)$, where c is a constant.

THEOREM II. *If the total number of representations of* $2^{\alpha}m^2$ *in the form* (a, b, c, d, α) *is* $k_{\alpha}\ H(m)$, *the number of proper representations of* $2^{\alpha}m^2$ *in the same form is* $(k_{\alpha}-k_{\alpha-2})m\ J_2(m)$.

It is obvious that precisely similar considerations can be applied to any quadratic form in any number of variables, also to such forms when certain of the variables are restricted (examples in § 3 (2), (5)). The discussion is limited here to (a, b, c, d, α) because the application of the method to the deduction of explicit results for other forms presupposes a study of the general class of identities of which Hurwitz' is the second simplest example. The first such identity concerns the function which gives the number of representations of an integer as a sum of two squares. In both this and Hurwitz' identity only odd divisors enter; in the general case there appear only those divisors which are prime to the prime $p > 2$.

3. *Numbers of Representations.* From the way in which Theorems I, II are stated it is sufficient for any given form (a, b, c, d, α) to record the values of k_{α} and $k'_{\alpha} \equiv k_{\alpha}-k_{\alpha-2}$. The values of k_{α} are those which appertain to the quaternary form obtained from (a, b, c, d, α) by omitting the term in u. They can be found by referring to the papers of Liouville and Pepin.†

The results for quinary forms are given in the following list, in which the asterisk $(*)$ denotes any k with a suffix greater than that of the last k in the symbol in which the asterisk occurs. Thus (11) states that the total number of representations of $2^{\alpha}m^2$ (m odd) in the form

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* ANNALS OF MATHEMATICS, (2), vol. 21 (1920), p. 170, § 8.

† JOURNAL DES MATHÉMATIQUES, (2), vol. 6 (1861), pp. 409, 440; vol. 7 (1862), pp. 1, 76, 80, 107, 112, 115, 119, 154, 156, 160, 163, 167; (4), vol. 6 (1890), pp. 5-67.

$$x^2+2y^2+2z^2+4t^2+2^{\alpha+2}u^2$$

is 2, 4, 8, or 24 times $H(m)$ according as

$$\alpha=0,\ 1,\ 2 \text{ or } > 2.$$

(1) $(1, 1, 1, 4, \alpha):[k_0, k_1, k_2, *] = [6, 12, 8, 24]$.
(2) $(1, 1, 1, 4, \alpha)$, $x+y$ odd: $k_1 = 8$.
(3) $(1, 1, 1, 16, \alpha):[k_2, k_3, k_4, *] = [6, 12, 8, 24]$.
(4) $(1, 1, 2, 2, \alpha):[k_0, k_1, *] = [4, 8, 24]$.
(5) $(1, 1, 2, 2, \alpha)$, xy odd: $[k_1, k_2, *] = [4, 16, 0]$.
(6) $(1, 1, 2, 8, \alpha):[k_1, k_2, k_3, *] = [6, 12, 8, 24]$.
(7) $(1, 1, 4, 4, \alpha):[k_0, k_1, k_2, *] = [4, 4, 8, 24]$.
(8) $(1, 1, 4, 16, \alpha):[k_2, k_3, k_4, *] = [6, 12, 8, 24]$.
(9) $(1, 1, 8, 8, \alpha):[k_1, k_2, k_3, *] = [4, 4, 8, 24]$.
(10) $(1, 1, 16, 16, \alpha):[k_2, k_3, k_4, *] = [4, 4, 8, 24]$.
(11) $(1, 2, 2, 4, \alpha):[k_0, k_1, k_2, *] = [2, 4, 8, 24]$.
(12) $(1, 2, 2, 16, \alpha):[k_2, k_3, k_4, *] = [6, 12, 8, 24]$.
(13) $(1, 2, 4, 8, \alpha):[k_1, k_2, k_3, *] = [2, 4, 8, 24]$.
(14) $(1, 2, 8, 16, \alpha):[k_2, k_3, k_4, *] = [2, 4, 8, 24]$.
(15) $(1, 4, 4, 4, \alpha):[k_0, k_1, k_2, *] = [2, 0, 8, 24]$.
(16) $(1, 4, 4, 16, \alpha):[k_2, k_3, k_4, *] = [6, 12, 8, 24]$.
(17) $(1, 4, 8, 8, \alpha):[k_1, k_2, k_3, *] = [0, 4, 8, 24]$.
(18) $(1, 4, 16, 16, \alpha):[k_1, k_2, k_3, k_4, *] = [0, 4, 4, 8, 24]$.
(19) $(1, 16, 16, 16, \alpha):[k_2, k_3, k_4, *] = [2, 0, 8, 24]$.

When combined with the theorems on the number of representations of an integer as a sum of two and of three squares, all of these give interesting binary quadratic class number relations of a new kind, examples of which are presented in another paper (not yet published). To save space we shall omit the numbers of proper representations, which can be written down from the above by Theorem II; e. g., for the form (1) we get

$$(k_0', k_1', k_2', k_3', k_4', *) = (6, 12, 2, 12, 16, 0).$$

THE UNIVERSITY OF WASHINGTON

THE INVARIANTS OF FORMS UNDER THE BINARY LINEAR HOMOGENEOUS GROUP G_6 MODULO 2*

BY O E. GLENN

1. *Introduction.* The transformation of an arbitrary binary quantic whose coefficients are written without binomial multipliers,

$$(1)\quad f=(a_0,\ldots,a_m)(x_1,x_2)^m=\prod_{i=0}^{m}(r_2^{(i)}x_1-r_1^{(i)}x_2),$$

by the formulas

$$T\colon\ x_1=\lambda_1x_1'+\mu_1x_2',\ x_2=\lambda_2x_1'+\mu_2x_2',$$

in which λ_i, μ_i are such residues of a prime p that T ranges over the total group G of order $(p^2-p)(p^2-1)$, leads to the formal modular concomitant system of f. For various reasons $p=2$ gives rise to exceptions in this theory; thus quadratic congruence theory becomes very special when $p=2$, and also certain types of modular concomitants exist† for the even modulus that do not exist for $p>2$.‡

2. *Analogies.* It is a known result of algebraic (nonmodular) invariant theory that every concomitant of (1) is a polynomial in determinants of two types, viz. $(r^{(i)}r^{(k)})$, $(r^{(i)}x)$, i. e., linear forms themselves and resultants of pairs of them. Also the complete system of covariants of any number of quadratic quantics,

$$(2)\quad f_1=(a_0,\ldots,a_2)(x_1,x_2)^2,\ldots,\ f_r=(l_0,\ldots,l_2)(x_1,x_2)^2,$$

is a set of concomitants that can be formed as transvectants of forms f_i taken in pairs.§ The dyadic combinations therefore furnish the complete seminvariant systems, also, but for the invariants it is found that the eliminants of the triadic combinations of the forms $f_1,\ldots,f_r$ are to be added.

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* Presented to the Society, December 28, 1923.

† TRANSACTIONS OF THIS SOCIETY, vol. 19 (1918), p. 110.

‡ Dickson, *The Madison Colloquium Lectures*, Lecture III, p. 33-64.

§ Grace and Young, *Algebra of Invariants*, 1903, p. 161.

The facts are more complicated when the transformations form the group G_6 (mod 2). Then a simultaneous system of covariants of only two linear quantics

$$(3) \qquad f = a_0 x_1 + a_1 x_2, \qquad g = b_0 x_1 + b_1 x_2,$$

is composed of sixteen concomitants. These are tabulated below. The methods for their derivation are exemplified later in this paper in the corresponding problem for two quadratics although the two problems are not without differences in respect to detail. We shall use the abbreviations

$$(4) \qquad E_1 = a_0^2 \frac{\partial}{\partial a_0} + a_1^2 \frac{\partial}{\partial a_1},$$

$$E_2 = b_0^2 \frac{\partial}{\partial b_0} + b_1^2 \frac{\partial}{\partial b_1}, \qquad w = x_1^2 \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2}$$

and, if two linear covariants are

$$C = c_0 x_1 + c_1 x_2, \; D = d_0 x_1 + d_1 x_2,$$

$[CD]$ is the covariant

$$(5) \qquad [CD] = (c_0 d_0 + c_0 d_1 + c_1 d_0) x_1 + (c_1 d_1 + c_0 d_1 + c_1 d_0) x_2.$$

The covariant R in the list is led by the invariant (ab), viz.:

$$(6) \quad R = (a_0 b_1 + a_1 b_0) x_1^3 + (a_0 b_1 + a_0 b_0 + a_1 b_1) x_1^2 x_2 + (a_1 b_0 + a_0 b_0 + a_1 b_1) x_1 x_2^2 + (a_0 b_1 + a_1 b_0) x_2^3.$$

Covariants of two linear forms under G_6:

Invariants: (ab), $I = a_0^2 + a_0 a_1 + a_1^2$, $J = b_0^2 + b_0 b_1 + b_1^2$,
$L_1 = a_0^2 a_1 + a_0 a_1^2$, $L_2 = b_0^2 b_1 + b_0 b_1^2$.

Linear Covariants: f, g, $E_1 f$, $E_2 g$, $[fg]$, $[gE_1 f]$,
$[fE_2 G]$, $[E_1 f E_2 g]$.

Quadratic covariants: wf, wg, $Q = x_1^2 + x_1 x_2 + x_2^2$.

Cubic covariants: R, $L = x_1^2 x_2 + x_1 x_2^2$.

Thus the independent covariants of a set of linear quantics, under G_6, form an extensive set which increases rapidly as the number of quantics is increased. If we add a third form $h = c_0 x_1 + c_1 x_2$ to the set f, g we shall have to consider concomitants formed from ground-forms in triadic combinations, as is proved by the existence of the following irreducible seminvariant:

$$(7)\qquad C = a_0b_0c_1 + a_0b_1c_1 + a_1b_0c_1 + a_1b_1c_0.$$

Similar considerations hold, under more complicated circumstances, for the simultaneous systems, for G_6, of a set of quadratic quantics. The rest of this paper concerns the system of two quadratics.

3. *Seminvariants.* Let Γ be the linear transformations upon a_0, a_1, a_2 which are induced by transforming f by

$$T\colon\ x = x_1' + x_2',\quad x_2 = x_2',$$

and suppose Γ' to be that which corresponds by transformation of g by T, where

$$f = a_0x_1^2 + a_1x_1x_2 + a_2x_2^2,\quad g = b_0x_1^2 + b_1x_1x_2 + b_2x_2^2.$$

Then,

$$(8)\qquad \begin{cases} \Gamma\ :\ a_0' = a_0,\ a_1' = a_1,\ a_2' = a_0 + a_1 + a_2, \\ \Gamma'\colon\ b_0' = b_0,\ b_1' = b_1,\ b_2' = b_0 + b_1 + b_2. \end{cases}$$

With suitable restrictions we connect the problem of the seminvariants of f and g with a simpler problem, previously solved, by taking $a_0 = 0$, temporarily. We then have two simultaneous groups (mod 2), viz.,

$$(9)\qquad \begin{cases} \Gamma_1\colon\ a_1' = a_1,\ a_2' = a_1 + a_2, \\ \Gamma_1'\colon\ b_0' = b_0,\ b_1' = b_1,\ b_2' = b_0 + b_1 + b_2, \end{cases}$$

concerning which it is known that a fundamental system of universal concomitants consists of six quantics, as follows:[*]

$$(10)\qquad a_1,\ b_0,\ b_1,\ \psi_1 = a_2^2 + a_1a_2,\ s = (b_0 + b_1 + b_2)b_2,$$
$$\varrho_1 = a_1b_2 + a_2(b_0 + b_1).$$

We desire six seminvariants of the set f, g such that the forms (10) are respectively residual to the six when $a_0 \equiv 0$ (mod 2).

The seminvariant of f of the type of s is $a_0a_2 + \psi_1 = \sigma$. The leading coefficient of $[fg]$[†] is $a_0b_0 + (fg)$, where (fg) is the invariant

$$(fg) = (a_0 + a_1)(b_1 + b_2) + (b_0 + b_1)(a_1 + a_2) + a_1b_1.$$

The form ϱ_1 is the residue (mod a_0) of $(fg) + (b_0 + b_1)a_1 = x$.

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[*] TRANSACTIONS OF THIS SOCIETY, vol. 21 (1920), p. 293.

[†] The symbolism (fg), $[fg]$, $\{fg\}$, $\{\overline{fg}\}$, explained in (19) in the present article, was first defined in PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 5 (1919), p. 107.

Assume, in arbitrary form, a seminvariant S of the set f, g,

$$S = S(a_0, a_1, a_2, b_0, b_1, b_2); \tag{11}$$

then, by (10),

$$\begin{aligned} S_1 = S(0, a_1, a_2, b_0, b_1, b_2) \\ \equiv F(a_1, b_0, b_1, \psi_1, s, \varrho_1) \pmod 2, \end{aligned} \tag{12}$$

where F is a polynomial in its arguments with integral coefficients. Hence we can arrange S as an expression of the form

$$\begin{aligned} S \equiv F(a_1, b_0, b_1, \sigma, s, x) \\ + a_0 F_1(a_0, a_1, a_2, b_0, b_1, b_2) \pmod 2, \end{aligned} \tag{13}$$

and F_1 is evidently a seminvariant of the set f, g. This process of reduction can be applied successively until we reach a coefficient quantic F_r which is free from a_0 (explicitly), it being, therefore, a polynomial in

$$a_1, b_0, b_1, \sigma, s, x,$$

i. e.,

$$S \equiv F + a_0 F_1 + a_0^2 F_2 + \cdots + a_0^r F_r \pmod 2. \tag{14}$$

THEOREM. *A fundamental system of seminvariants of the set f, g* (mod 2) *consists of the seven forms*

$$a_0, b_0, a_1, b_1, \sigma, s, (fg). \tag{15}$$

4. *Syzygies.* The expressions $k = a_0\sigma$, $\varkappa = b_0 s$, a_1, b_1, $q_1 = \sigma + a_0^2 + a_0 a_1$, $q_2 = s + b_0^2 + b_0 b_1$, (fg) are pure invariants. The following syzygies can be verified:

$$\left\{ \begin{aligned} & a_0^3 + a_0^2 a_1 + a_0 q_1 + k = 0, \\ & b_0^3 + b_0^2 b_1 + b_0 q_2 + \varkappa = 0, \\ & \varrho^2 + (a_0 + a_1)(b_0 + b_1)\varrho + (a_0^2 + a_1^2)s + (b_0^2 + b_1^2)\sigma \\ & \qquad + (a_0 b_0 + a_1 b_1)(a_0 b_1 + a_1 b_0) = 0, \end{aligned} \right. \tag{16}$$

where $\varrho = (fg) + a_1 b_1$.

These syzygies may be employed as literal moduli of reduction for the purpose of reducing the arbitrary seminvariant $S(a_0, a_1, a_2, b_0, b_1, b_2)$ to a polynomial of finite order in a_0, b_0. We have immediately the following theorem.

THEOREM. *The arbitrary seminvariant S of two quadratic forms f, g can be represented in the finite form*

$$\begin{aligned} S = \Phi_0 + \Phi_1 b_0 + \Phi_2 b_0^2 + a_0(\psi_0 + \psi_1 b_0 + \psi_2 b_0^2) \\ + a_0^2(\chi_0 + \chi_1 b_0 + \chi_2 b_0^2), \end{aligned} \tag{17}$$

in which Φ_i, ψ_i, χ_i *are polynomials in the invariants*

$$q_1,\ q_2,\ k,\ \varkappa,\ a_1,\ b_1,\ (fg), \tag{18}$$

and the highest power of (fg) *which occurs is the first.*

5. *A Method in Covariants.* It is known that a complete concomitant scale,* for the modulus 2, of a covariant,

$$C = C_0 x_1^M + C_1 x_1^{M-1} x_2 + \cdots + C_M x_2^M,$$

of a quantic f_m of order m, for the reduction of all concomitants of degree unity in the coefficients of C and of order > 3, is composed of

$$\left\{\begin{aligned} &C,\ (C) = C_1 + C_2 + \cdots + C_{M-1}, \\ &\qquad\qquad [C] = (C_0 + (C))x_1 + ((C) + C_M)x_2, \\ &\{C\} = C_0 x_1^2 + (C) x_1 x_2 + C_M x_2^2, \\ &\{\overline{C}\} = C_0 x_1^3 + J_1 x_1^2 x_2 + J_2 x_1 x_2^2 + C_M x_2^3, \\ &\qquad\qquad (J_1 + J_2 \equiv (C) \pmod{2}). \end{aligned}\right. \tag{19}$$

The latter covariant is existent only when M is an odd number. This scale produces concomitants of f_m from any covariant of the latter by the principle of copied forms.

Another method, not previously described, for the construction of covariants of f_m, to any prime modulus p, is to make an appropriate selection of a primary quantic,

$$P_0 = q_0 x_1^\alpha + q_1 x_1^{\alpha-1} x_2 + \cdots + q_\alpha x_2^\alpha, \tag{20}$$

of given degree-order (i, α) and apply to it, simultaneously, the substitutions of the group upon the variables generated by $x_1 = x_1' + x_2'$, $x_2 = x_2'$, and the corresponding substitutions of the induced group upon the coefficients. We thus obtain p quantics

$$P_0,\ P_1 = (q_0', \ldots, q_\alpha')(x_1, x_2)^\alpha,\ \ldots, \\ P_{p-1} = (q_0^{(p-1)}, \ldots, q_\alpha^{(p-1)})(x_1, x_2)^\alpha, \tag{21}$$

and, if we assume that the primary quantic has been properly selected, any symmetric function of $P_0, P_1, \ldots, P_{p-1}$ is a covariant, modulo p, of f_m. Not many rules, other than empirical ones, for the determination of primary quantics, are known to the writer, but examples of the method are shown in the next paragraph. Obviously P_0 should be such that the assumed symmetric function is invariantive under the other two generators of the total group (mod p), i. e., $x_1 = x_1'$, $x_2 = \lambda x_2'$; $x_1 = x_2'$, $x_2 = -x_1'$.

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* TRANSACTIONS OF THIS SOCIETY, vol. 19 (1918), p. 110; vol. 20 (1919), p. 155.

6. *Fundamental Covariants.* We are evidently able to reduce all covariants in terms of those of orders 0, 1, 2, 3 led by seminvariants $a_0^i b_0^j$, $(i, j = 0, 1, 2)$ and by the invariants of which Φ_0 in (17) is a function. A formula showing this reduction in general form will be derived.

We find the following covariants with the leading coefficients which are adjoined. Abbreviations employed are

$$E_1 = a_0^2\frac{\partial}{\partial a_0} + a_1^2\frac{\partial}{\partial a_1} + a_2^2\frac{\partial}{\partial a_2},$$

$$E_2 = b_0^2\frac{\partial}{\partial b_0} + b_1^2\frac{\partial}{\partial b_1} + b_2^2\frac{\partial}{\partial b_2}.$$

Linear forms:

$$a_0 + a_1, [f];\ b_0 + b_1, [g];\ a_0 b_0 + (fg), [fg];$$
$$a_0^2 + a_1^2, [E_1 f];\ b_0^2 + b_1^2, [E_2 g];\ a_0 b_0^2 + (fE_2 g), [fE_2 g];$$
$$a_0^2 b_0 + (gE_1 f), [gE_1 f];\ a_0^2 b_0^2 + (E_1 f E_2 g), [E_1 f E_2 g].$$

Quadratic forms:

$$a_0, f;\quad b_0, g;\quad a_0^2, E_1 f;\quad b_0^2, E_2 g;\quad a_0 b_0, \{fg\};$$
$$a_0^2 b_0, \{gE_1 f\};\quad a_0 b_0^2, \{fE_2 g\};\quad a_0^2 b_0^2, \{E_1 f E_2 g\}.$$

There are no linear covariants, in the domain, led by invariants, and the only quadratic covariants whose leading coefficients are invariants are comprised in the formula IQ, where I is an arbitrary invariant. The only invariantive leader of covariants of the third order which we shall be required to consider is Φ_0 (cf. (17)). Let ζ be the operation of applying to a primary quantic P_0 the substitutions

$$x_1 = x_1' + x_2', \quad x_2 = x_2', \tag{22}$$
$$a_0' = a_0, \quad a_1' = a_1, \quad a_2' = a_0 + a_1 + a_2.$$

If the primary is

$$P_0 = (a_0 + a_2)x_1 + (a_0 + a_1)x_2, \tag{23}$$

then

$$P_1 = \zeta P_0 = (a_1 + a_2)x_1 + (a_0 + a_2)x_2,$$

so that

$$(24)\quad \begin{cases} \sum P_0 = (a_0+a_1)x_1+(a_1+a_2)x_2=[f], \\ \Delta_1 = \sum P_0P_1 = [(a_0+a_2)x_1+(a_0+a_1)x_2] \\ \qquad\qquad\qquad \times[(a_1+a_2)x_1+(a_0+a_2)x_2]. \end{cases}$$

If the primary quantic is

$$P_0 = (b_0+b_2)x_1+(b_0+b_1)x_2,$$

we obtain, similarly,

$$(25)\quad \begin{cases} \sum P_0 = (b_0+b_1)x_1+(b_1+b_2)x_2=[g], \\ \Delta_2 = \sum P_0P_1 = [(b_0+b_2)x_1+(b_0+b_1)x_2] \\ \qquad\qquad\qquad \times[(b_1+b_2)x_1+(b_0+b_2)x_2]. \end{cases}$$

The respective seminvariant leading coefficients of the covariants $[f]\Delta_1$, $[g]\Delta_2$ are the invariants

$$(26)\qquad a_1q_1+k, \quad b_1q_2+\varkappa,$$

and these may replace k, $\varkappa$, respectively, in the system (18). Instead of (fg) in the fundamental system (15), we may employ $(fg)_1=(fg)+a_1b_1$, which is the resultant of $[f]$ and $[g]$. A cubic covariant led by $(fg)_1$ is

$$(27)\qquad B=[fg]Q+[f]g+a_1[g]Q.$$

There exist no cubic covariants led by any of the invariants k, $\varkappa$, q_1, q_2, due to the fact that all of these invariants contain a term which is left unaltered by the permutational substitution* $(a_0a_2)(a_1)(b_0b_2)(b_1)=S_1$.

The cubic covariants which we require, with their leading coefficients, are listed below.

Cubic forms:
a_0+a_1, $[f]Q$; $a_0^2+a_1^2$, $[E_1f]Q$; b_0+b_1, $[g]Q$;
$b_0^2+b_1^2$, $[E_2g]Q$; $a_0b_0+(fg)$, $[fg]Q$; $a_0^2b_0+E_1(fg)$, $[E_1fg]Q$;
$a_0b_0^2+E_2(fg)$, $[fE_2g]Q$; $a_0^2b_0^2+E_1E_2(fg)$, $[E_1fE_2g]Q$; $(fg)_1$, B.

THEOREM. *An invariant leading coefficient Φ_0 of a cubic covariant of f, g is necessarily congruent modulo 2 to the expression*

$$(28)\qquad C=(a_1q_1+k)\psi_1'+(b_1q_2+\varkappa)\psi_2'+(fg)_1\psi_3',$$

where the quantics ψ_i' are invariants.

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* TRANSACTIONS OF THIS SOCIETY, vol. 19 (1918), p. 111.

To prove this we note that the invariant Φ_0 (cf. (17)) is always of the form

$$F(q_1, q_2, a_1, b_1) + C = \Phi_0, \tag{29}$$

where F is an integral form in its arguments. We have constructed covariants whose leaders are the invariants of the set

$$a_1 q_1 + k,\quad b_1 q_2 + \varkappa,\quad (fg)_1.$$

If G is a covariant whose leader is Φ_0, we have that

$$G + \psi_1'[f]\Delta_1 + \psi_2'[g]\Delta_2 + \psi_3 B \tag{30}$$

is a cubic covariant led by F, and this is an absurdity because every possible form F evidently contains* a term which is left unaltered by S_1. Hence $F \equiv 0 \pmod 2$, $\Phi_0 \equiv C$ (mod 2), which was to be proved.

7. *Reductions of the Arbitrary Linear, Quadratic, and Cubic Covariants.* Let $\sum_1$ represent a covariant of order unity whose seminvariant leading coefficient is S (cf. 17)). We have, identically, $S = I + J$, where

$$(31)\quad \left\{\begin{aligned} I = {} & (b_0 + b_1)\Phi_1 + (b_0^2 + b_1^2)\Phi_2 + (a_0 + a_1)\psi_0 \\ & + [a_0 b_0 + (fg)]\psi_1 + [a_0 b_0^2 + (fE_2 g)]\psi_2 + (a_0^2 + a_1^2)\chi_0 \\ & + [a_0^2 b_0 + (gE_1 f)]\chi_1 + [a_0^2 b_0^2 + (E_1 f E_2 g)]\chi_2; \\ J = {} & \Phi_0 + b_1\Phi_1 + b_1^2\Phi_2 + a_1\psi_0 + (fg)\psi_1 + (fE_2 g)\psi_2 \\ & + a_1^2\chi_0 + (gE_1 f)\chi_1 + (E_1 f E_2 g)\chi_2. \end{aligned}\right.$$

The following covariant is led by I:

$$(32)\quad \left\{\begin{aligned} K_1 = {} & \Phi_1[g] + \Phi_2[E_2 g] + \psi_0[f] + \psi_1[fg] + \psi_2[fE_2 g] \\ & + \chi_0[E_1 f] + \chi_1[gE_1 f] + \chi_2[E_1 f E_2 G]. \end{aligned}\right.$$

Therefore, there would exist a linear covariant led by the invariant J, viz., $\sum_1 + K_1$, unless $J \equiv 0 \pmod 2$. Thus every linear covariant $\sum_1$ is reduced by the formula (32), ($\sum_1 = K_1$).

Let $\sum_2$ represent an arbitrary quadratic covariant which is led by S. The following covariant has S for seminvariant leader:

$$(33)\quad \left\{\begin{aligned} K_2 = {} & \Phi_0 Q + \Phi_1 g + \Phi_2 E_2 g + \psi_0 f + \psi_1\{fg\} \\ & + \psi_2\{fE_2 g\} + \chi_0 E_1 f + \chi_1\{gE_1 f\} + \chi_2\{E_1 f E_2 g\}. \end{aligned}\right.$$

Then K_2 differs from $\sum_2$ by some covariant which contains x_2

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* The number $\binom{2m}{m}$ is even for all integers m.

as a factor; but, as no quadratic covariant can be factored thus, we have $\sum_2 \equiv K_2 \pmod{2}$, i. e., the arbitrary quadratic covariant is reduced.

Let $\sum_3$ represent an arbitrary covariant of order 3 with seminvariant leader S. The following is a cubic covariant whose leading coefficient is I (cf. (31)): $K_3 = K_1 Q$. Note therefore that $\sum_3 + K_3 = \Gamma_3$ is a covariant whose leader is J. Hence, by (28) (Theorem),

$$J \equiv C + (fE_2g)_1\psi_2 + (gE_1f)_1\chi_1 + (E_1fE_2g)_1\chi_2,$$

that is,

$$\Gamma_3 = \psi_1'[f]\Delta_1 + \psi_2'[g]\Delta_2 + \psi_3'B + (\psi_2E_2 + \chi_1E_1 + \chi_2E_1E_2)B, \tag{34}$$

where $(fE_2g)_1 = E_2(fg)_1 = (fE_2g) + a_1b_1^2$, (cf. (27)). Note that operations by E_1, E_2 upon B produce only polynomials in covariants already listed. We have now reduced $\sum_3$ to the form

$$\sum\nolimits_3 = K_3 + \Gamma_3 + \Theta L \pmod{2}, \tag{35}$$

since L is the only cubic covariant which contains x_2 as a factor. The quantic Θ is a pure invariant, but it is not known whether it is reducible, in all cases, entirely in terms of the invariants of the set (18). The following theorem has now been established.

THEOREM. *A fundamental system of formal covariants of the set consisting of the two binary quadratics*

$$f = (a_0, a_1, a_2)(x_1, x_2)^2, \quad g = (b_0, b_1, b_2)(x_1, x_2)^2, \tag{36}$$

under the total group G_6, modulo 2, is composed of 21 quantics, namely, seven invariants, q_1, q_2, k, $\varkappa$, a_1, b_1, (fg), eight linear covariants, $[f]$, $[g]$, $[fg]$, $[E_1f]$, $[E_2g]$, $[fE_2g]$, $[gE_1f]$, $[E_1fE_2g]$, five quadratic covariants, Q, f, g, Δ_1, Δ_2, and one cubic covariant, L.

The remaining forms are reducible, as follows:

$$E_1f = [f]^2 + a_1^2Q, \quad E_2g = [g]^2 + b_1^2Q,$$
$$\{fg\} = [f][g] + a_1g + b_1f + a_1b_1Q, \quad \{gE_1f\} = E_1\{fg\},$$
$$\{fE_2g\} = E_2\{fg\}, \quad \{E_1fE_2g\} = E_1E_2\{fg\}.$$

THE UNIVERSITY OF PENNSYLVANIA

ON THE APPLICATION OF THE THEORY OF IDEALS TO DIOPHANTINE ANALYSIS[*]

BY G. E WAHLIN

1. *Introduction.* About three years ago[†] Professor Dickson stated a certain conjectured theorem, and he has recently published a proof of it.[‡]

After having examined a proof of the same theorem by the author of this article, Professor Dickson suggested the investigation of a more general equation than the one which he had considered, and the following pages contain the results of this investigation.

2. *Rings.* Let us consider any algebraic number field $k(\theta)$ of degree n. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be a fundamental system of integers of $k(\theta)$ so that every integer of the field can be represented by the fundamental form

$$x_1\gamma_1 + x_2\gamma_2 + \cdots + x_n\gamma_n, \tag{1}$$

in which the $x_1, x_2, \ldots, x_n$ are rational integers.

By a *ring* R in $k(\theta)$ we understand a set of integers which is closed with respect to addition, subtraction, and multiplication, and which contains the rational integers. Let $\varrho_1, \varrho_2, \ldots, \varrho_n$ be a fundamental system of R. As above, we shall call

$$x_1\varrho_1 + x_2\varrho_2 + \cdots + x_n\varrho_n \tag{2}$$

the *fundamental form* of R. Every element of R is represented once and only once by (2) when the $x_1, x_2, \cdots, x_n$ are given rational integral values.

Since $\varrho_1, \varrho_2, \ldots, \varrho_n$ are integers in $k(\theta)$, they can be represented by (1), and we shall suppose that

$$\varrho_i = r_{i1}\gamma_1 + r_{i2}\gamma_2 + \cdots + r_{in}\gamma_n, \qquad (i = 1, 2, \ldots, n). \tag{3}$$

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[*] Presented to the Society, December 29, 1923.

[†] L. E. Dickson, *A new method in Diophantine analysis*, this BULLETIN, vol. 27, No. 8 (May, 1921), p. 353.

[‡] L. E. Dickson, *Integral solutions of $x^2 - my^2 = zw$*, this BULLETIN, vol. 29, No. 10 (Dec., 1923), p. 464.

The absolute value of the determinant $|r_{ij}|$ of the system (3) shall be called the *index of the ring,* and shall be denoted by Δ. Since any product of integers of R belongs to R the product

$$\varrho_1^{s_1}\cdot\varrho_2^{s_2}\cdots\varrho_n^{s_n},\quad s_i\geqq 0,\qquad (i=1,2,\ldots,n)$$

can be represented by (2), and we shall write

$$\varrho_1^{s_1}\cdot\varrho_2^{s_2}\cdots\varrho_n^{s_n}=C^{(1)}_{s_1s_2\ldots s_n}\varrho_1+C^{(2)}_{s_1s_2\ldots s_n}\varrho_2+\cdots+C^{(n)}_{s_1s_2\ldots s_n}\varrho_n. \tag{4}$$

This equation uniquely defines the rational integers $C^{(i)}_{s_1s_2\ldots s_n}$.

Let us next consider any k integers $\alpha', \alpha'', \ldots, \alpha^{k}$ from R, and let

$$\alpha^{(i)}=a_{1i}\varrho_1+a_{2i}\varrho_2+\cdots+a_{ni}\varrho_n,\qquad (i=1,2,\ldots,k).$$

We may then write the product

$$\alpha'\cdot\alpha''\cdots\alpha^{(k)}=\sum B_{s_1s_2\ldots s_n}\varrho_1^{s_1}\cdot\varrho_2^{s_2}\cdots\varrho_n^{s_n}, \tag{5}$$

where the summation extends over all terms for which $s_1+s_2+\cdots+s_n=k$, and

$$B_{s_1s_2\ldots s_n}=\sum a_{1i_{11}}\cdots a_{1i_{1s_1}}\cdot a_{2i_{21}}\cdots a_{2i_{2s_2}}\cdot\cdots\cdot a_{ni_{ns_n}}.$$

That is, $B_{s_1s_2\ldots s_n}$ is the sum of all possible products formed by taking s_1 elements from the first column, s_2 from the second, s_3 from the third, etc., and so chosen that no two elements belong to the same row in the matrix

$$m=\left\|\begin{matrix}a_{11}, & a_{21}, & \ldots, & a_{n1}\\ a_{12}, & a_{22}, & \ldots, & a_{n2}\\ \cdot & \cdot & \cdot & \cdot\\ a_{1k}, & a_{2k}, & \ldots, & a_{nk}\end{matrix}\right\|.$$

If in (5) we now replace the power-products $\varrho_1^{s_1}\cdot\varrho_2^{s_2}\cdots\varrho_n^{s_n}$ by their expressions as furnished by (4), we have

$$\alpha'\cdot\alpha''\cdots\alpha^{(k)}=A_1\varrho_1+A_2\varrho_2+\cdots+A_n\varrho_n, \tag{6}$$

where

$$A_i=\sum B_{s_1s_2\ldots s_n}C^{(i)}_{s_1s_2\ldots s_n}, \tag{7}$$

the summation extending over all $s_1, s_2, \ldots, s_n$ whose sum is k.

The A_i, for whose computation a definite process is thus given, are rational integers. They will be used in the application whose consideration is the object of this paper.

3. *Ideals in R.* The *conductor of a ring* is an ideal f in $k(\theta)$ such that the product of any integer of $k(\theta)$ by f belongs to R.[*] By a *ring ideal* we shall understand an ideal in R as defined by Bachmann.[†] That is, an ideal $I^{(R)}$ of R is a set of integers of R such that the sum and difference of any two integers of the set belong to the set; the product of any integer of the set by any integer of R belongs to the set; and the greatest common divisor of f and the moduli thus defined is the ring R.

Let $I^{(R)}$ be any ideal of R, and let $\beta_1, \beta_2, \ldots, \beta_n$ be a fundamental system of $I^{(R)}$. Since the β_i belong to R, they can be represented by the form (2). If we write

$$(8) \qquad \beta_i = b_{i1}\varrho_1 + b_{i2}\varrho_2 + \cdots + b_{in}\varrho_n, \qquad (i = 1, 2, \ldots, n),$$

then the *norm of $I^{(R)}$* in R, which we shall denote by $N_R(I^{(R)})$, is the absolute value of the determinant $|b_{ij}|$.[‡]

If $\alpha = e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n$ is an integer in $I^{(R)}$, and if we apply (8), we have $\alpha = a_1\varrho_1 + a_2\varrho_2 + \cdots + a_n\varrho_n$, where

$$(9) \qquad a_i = b_{1i}e_1 + b_{2i}e_2 + \cdots + b_{ni}e_n, \qquad (i = 1, 2, \ldots, n),$$

whose matrix is the conjugate of that of (8).

In order to distinguish between ideals in R and ideals in $k(\theta)$, we shall speak of them as *ring ideals* and *field ideals*, respectively. Principal ring ideals will be designated by $[\lambda]$, where λ is an integer of R, and principal field ideals by $\{\lambda\}$, where λ is an integer of $k(\theta)$.

To every ideal $I^{(R)}$ of R there corresponds a field ideal I obtained by forming the product of $I^{(R)}$ and the unit ideal of $k(\theta)$; and if I is any ideal of $k(\theta)$ which is relatively prime to f, the numbers of I which belong to R constitute a ring ideal $I^{(R)}$ whose corresponding field ideal is I. The norm $N(I)$ of I is equal to the norm $N_R(I^{(R)})$ of $I^{(R)}$ in R.[§] The index Δ of R is divisible by the conductor f of R.[||]

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[*] Bachmann, *Zahlentheorie,* p. 136.

[†] Bachmann, loc. cit., p. 363.

[‡] Bachmann, loc. cit., chapter 2, p. 74.

[§] Bachmann, loc. cit., chapter 9. No. 2.

[||] Bachmann, chapter 4, p. 136.

Two ideals $I_1^{(R)}$ and $I_2^{(R)}$ shall be called equivalent when R contains two integers α_1 and α_2 which are relatively prime to f, and $\alpha_2 I_1^{(R)} = \alpha_1 I_2^{(R)}$.* Equivalent ideals constitute a class.

For any given ring ideal $I^{(R)}$ there is an ideal $J^{(R)}$ such that the corresponding field ideal J is relatively prime to any given field ideal T and the product $I^{(R)} \cdot J^{(R)}$ is a principal ideal in R.

To prove this, we make use of the fact that an ideal $J_1^{(R)}$ exists, such that $J_1^{(R)}(I^{(R)} \cdot T_2^{(R)}) = [\lambda_1]$, a principal ideal in R.† Here T_2 is used to denote the product of the distinct prime factors of T which are relatively prime to f, and $T_2^{(R)}$ is used to denote the corresponding ring ideal.

Let T_1 be the product of the distinct prime factors of T which are divisors of f. Then $J_1^{(R)} \cdot T_2^{(R)}$ is relatively prime to T_1. For, since $J_1^{(R)}$ and $T_2^{(R)}$ are ideals in R, their product is an ideal in R. Hence this product is prime to f, and therefore also to T_1, which is a factor of f. There exists in $k(\theta)$ an ideal J_2 which is relatively prime to T_2, such that $J_2 \cdot I \cdot f = \{\lambda_2\}$.‡ Since λ_2 is divisible by f, it is an integer in R. I is the field ideal corresponding to the given ring ideal $I^{(R)}$.

Since λ_1 and λ_2 both belong to R, their sum $\lambda_1 + \lambda_2$ belongs to R, and since λ_1 and λ_2 are both divisible by $I^{(R)}$, there exists an ideal $J^{(R)}$ such that $I^{(R)} \cdot J^{(R)} = [\lambda_1 + \lambda_2]$.§

That λ_1 is divisible by $I^{(R)}$ is seen by its definition. Moreover, λ_2 is divisible by I and belongs to R and hence also to $I^{(R)}$; it is therefore divisible by $I^{(R)}$. Hence $\lambda_1 + \lambda_2$ is divisible by $I^{(R)}$.

The $J^{(R)}$ thus defined is such that the corresponding field ideal J is relatively prime to T. For, since $I^{(R)} \cdot J^{(R)} = [\lambda_1 + \lambda_2]$, by multiplying both members by the unit ideal of $k(\theta)$, we have $I \cdot J = \{\lambda_1 + \lambda_2\}$. Now λ_1 / I is relatively prime to T_1 and is divisible by T_2, and λ_2/I is relatively prime to T_2 and is divisible by T_1. Hence, if p is any prime factor of T and hence a factor of T_1 or T_2, either λ_1 or λ_2 is divisible by Ip, but not both. Hence their sum is not divisible by Ip, and therefore $\{\lambda_1+\lambda_2\}/I$ is not divisible by any prime factor of T. It follows that J is relatively prime to T. In the application to Diophantine equations this theorem will be used with $T=\{\Delta\}$.

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* Bachmann uses the restricted equivalent in which $sgn N(\alpha_1) = sgn N(\alpha_2)$.

† Bachmann, loc. cit., p. 398.

‡ Bachmann, loc. cit., p. 221.

§ Bachmann, loc. cit., p. 369.

4. *Decomposable Forms.* If $x_1\beta_1+x_2\beta_2+\cdots+x_n\beta_n$ is the fundamental form of the ideal $I^{(R)}$, then $N(x_1\beta_1+x_2\beta_2+\cdots+x_n\beta_n)$ is a form of degree n in the n variables $x_1, x_2, \ldots, x_n$, with rational integral coefficients. The highest common factor of the coefficient of this form is the norm of the ideal $I^{(R)}$ in R. Hence

$$(10)\quad N(x_1\beta_1+x_2\beta_2+\cdots+x_n\beta_n)=N_R(I^{(R)})\,F(x_1,x_2,\ldots,x_n),$$

where $F(x_1, x_2, \ldots, x_n)$ is a unit form decomposable in R.*

The following theorem is a modified form of one given by Bachmann.† The modifications are made so as to apply to ideals in R, and also for the equivalence as we have defined it.

If $J^{(R)}$ is any ideal of R and $F(x_1, x_2, \ldots, x_n)$ is a unit form obtained as above from an ideal $I^{(R)}$ of the class reciprocal to that of $J^{(R)}$, then there exist rational integers e_1, $e_2, \ldots, e_n$, such that $N_R(J^{(R)})=|F(e_1, e_2, \ldots, e_n)|$; and, conversely, any rational integer $F(e_1, e_2, \ldots, e_n)$ represented by the decomposable form $F(x_1, x_2, \ldots, x_n)$ is in absolute value the norm of an ideal of the class reciprocal to that of $I^{(R)}$ provided $e_1\beta_1+e_2\beta_2+\cdots+e_n\beta_n$ is prime to f.

Let $\beta_1, \beta_2, \ldots, \beta_n$ be the fundamental system of $I^{(R)}$ from which the form $F(x_1, x_2, \ldots, x_n)$ is obtained, and let $I^{(R)}J^{(R)}=[\gamma]$. Then $\gamma=e_1\beta_1+e_2\beta_2+\cdots+e_n\beta_n$, and

$$\begin{aligned}|N(\gamma)|=N_R(I^{(R)})\cdot N_R(J^{(R)}) &= N(e_1\beta_1+e_2\beta_2+\cdots+e_n\beta_n)\\ &= N_R(I^{(R)})\,|F(e_1, e_2, \ldots, e_n)|,\end{aligned}$$

and hence

$$N_R(J^{(R)})=|F(e_1, e_2, \ldots, e_n)|.$$

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* Bachmann, loc. cit., chapter 10, No. 6.

† Bachmann, loc. cit., p. 429.

Conversely, if $F(e_1, e_2, \ldots, e_n)$ is any rational integer represented by $F(x_1, x_2, \ldots, x_n)$ corresponding to the ring ideal $I^{(R)}$, then

$$N_R(I^{(R)})F(e_1, e_2, \ldots, e_n) = N(e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n).$$

But if $e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n$ is prime to f, $[e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n]$ is a principal ideal in R; and, since it is divisible by $I^{(R)}$, there exists in the reciprocal class an ideal $J_1^{(R)}$ such that

$$J_1^{(R)} \cdot I^{(R)} = [e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n].$$

Hence we may write

$$|N(e_1\beta_1 + \cdots + e_n\beta_n)| = N_R(J_1^{(R)})\, N_R(I^{(R)}),$$

whence it follows that

$$N_R(J_1^{(R)}) = F(e_1, e_2, \ldots, e_n).$$

If $e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n$ is not prime to F and if I is the field ideal corresponding to $I^{(R)}$, then

$$\{\gamma\} = \{e_1\beta_1 + e_2\beta_2 + \cdots + e_n\beta_n\}$$

is divisible by I, and $\{\gamma\}/I = J_1$ is a field ideal which is not relatively prime to f. The method used above will show that the absolute value of $F(e_1, e_2, \ldots, e_n)$ is the norm of J_1.

5. *Application to Diophantine Equations.* We shall now turn our attention to the application of the foregoing facts regarding rings and ring ideals to the solution in rational integers $\xi_1, \xi_2, \ldots, \xi_n$; $u_1, u_2, \ldots, u_{k-2}$ of the equation

(11) $$N(\xi_1\varrho_1 + \xi_2\varrho_2 + \cdots + \xi_n\varrho_n) = u_1 \cdot u_2 \cdots u_{k-2},$$

where as before $\varrho_1, \varrho_2, \ldots, \varrho_n$ is a fundamental system of any ring R in $k(\theta)$.

Let us suppose that we have a set of integers satisfying (11). Since $\varrho_1, \varrho_2, \ldots, \varrho_n$ is a fundamental system of R, $\gamma = \xi_1\varrho_1 + \xi_2\varrho_2 + \cdots + \xi_n\varrho_n$ is an integer of $k(\theta)$. We shall suppose it resolved into its ideal prime factors. There are s distinct factors. We shall denote them by $p_1, p_2, \ldots, p_s$. Let us suppose that the first s_1 of these are relatively prime to f. Let $p_1^{\lambda_{11}}$ be the highest power of p_1 which is a factor of γ and whose norm is a factor of u_1; $p_1^{\lambda_{12}}$ the highest power of p_1 which is a factor of $\gamma/p_1^{\lambda_{11}}$ and whose norm is a factor of u_2; $p_1^{\lambda_{13}}$ the highest power of p_1 which is a divisor of $\gamma/p_1^{\lambda_{11}+\lambda_{12}}$ and whose norm is a divisor of u_3; and so on, until finally $p_1^{\lambda_{1k-2}}$ is the highest power of p_1 which is a divisor of $\gamma/p_1^{\lambda_{11}+\lambda_{12}+\cdots+\lambda_{1k-3}}$ and whose norm is a factor of u_{k-2}.

We next consider the prime ideal p_2 in the same way, and we let $p_2^{\lambda_{21}}$ be the highest power of p_2 which is a divisor of γ and whose norm is a divisor of $u_1/N(p_1^{\lambda_{11}})$; $p_2^{\lambda_{22}}$ the highest power of p_2 which is a divisor of $\gamma/p_2^{\lambda_{21}}$ and whose norm is a divisor of $u_2/N(p_1^{\lambda_{12}})$; and we continue in this way until finally $p_{s_1}^{\lambda_{s_1k-2}}$ is the highest power of p_{s_1} which is a divisor of $\gamma/p_{s_1}^{\lambda_{s_11}+\lambda_{s_12}+\cdots+\lambda_{s_1k-3}}$ and whose norm is a divisor of $u_{k-2}/N(p_1^{\lambda_{1k-2}}\cdot p_2^{\lambda_{2k-2}}\cdots p_{s-1}^{\lambda_{s-1\,k-2}})$.

We shall now write

$$P_i = p_1^{\lambda_{1i}}\cdot p_2^{\lambda_{2i}}\cdots p_{s_1}^{\lambda_{s_1i}}, \qquad (i = 1, 2, \ldots, k-2).$$

From the construction of the P_i, we see that γ is divisible by the product $P_1\cdot P_2\cdots P_{k-2}$, and that u_i is divisible by $N(P_i)$. We shall therefore write

$$Q = \frac{\{\gamma\}}{P_1\cdot P_2\cdots P_{k-2}},$$

and

$$u_i = \mu_i\cdot N(P_i).$$

Let us next write $Q = P_{k-1}\cdot P_k$ where P_k is the largest factor of Q all of whose prime divisors are divisors of f. We then have

$$\{\gamma\} = P_1\cdot P_2\cdots P_k,$$

where $P_1, P_2, \ldots, P_{k-1}$ are prime to f, and P_k contains no prime factor excepting factors of f. By (11), we have then

$$\begin{aligned} |N(\gamma)| &= N(P_1)\cdot N(P_2)\cdots N(P_k) = |u_1\cdot u_2\cdots u_{k-2}| \\ &= |\mu_1\cdot\mu_2\cdots\mu_{k-2}|\,N(P_1)\cdot N(P_2)\cdots N(P_{k-2}). \end{aligned}$$

Hence

$$N(P_{k-1})\cdot N(P_k) = |\mu_1\cdot\mu_2\cdots\mu_{k-2}|.$$

Let us separate each μ_i into two factors μ_i' and μ_i'' such that

$$\begin{aligned} N(P_{k-1}) &= |\mu_1'\cdot\mu_2'\cdots\mu_{k-2}'|, \\ N(P_k) &= |\mu_1''\cdot\mu_2''\cdots\mu_{k-2}''|. \end{aligned}$$

Since the $P_1, P_2, \ldots, P_{k-1}$, are all relatively prime to f, to each P_i, $(i<k)$, there corresponds a ring ideal whose norm in the ring is equal to the norm of P_i in $k(\theta)$. Let $P_i^{(R)}$ be the ring ideal corresponding to P_i. Let $I_i^{(R)}$ be an ideal from the reciprocal class. According to § 3, $I_i^{(R)}$ can be so chosen that the corresponding field ideal I_i is relatively prime to $\{\varDelta\}$. We shall suppose that $I_i^{(R)}$ has been so chosen. Let $\beta_1^{(i)}, \beta_2^{(i)}, \ldots, \beta_n^{(i)}$ be a fundamental system of $I_i^{(R)}$, and let $F_i(x_1, x_2, \ldots, x_n)$ be the corresponding decomposable form. Then, by § 4, we have

$$N_R(P_i^{(R)}) = N(P_i) = \left| F_i(e_1^{(i)}, e_2^{(i)}, \ldots, e_n^{(i)}) \right|,$$

and hence

$$u_i = \varepsilon_i \cdot \mu_i' \cdot \mu_i'' \, F_i(e_1^{(i)}, e_2^{(i)}, \ldots, e_n^{(i)}), \qquad (i = 1, 2, \ldots, k-2), \tag{12}$$

where ε_i is 1 or -1. Since $I_i^{(R)}$ and $P_i^{(R)}$ belong to reciprocal classes, $I_i^{(R)} \cdot P_i^{(R)} = [\alpha^{(i)}]$. Hence, since $[\alpha^{(i)}]$ is divisible by $I_i^{(R)}$, we have

$$\begin{aligned} \alpha^{(i)} &= e_1^{(i)}\, \beta_1^{(i)} + e_2^{(i)}\, \beta_2^{(i)} + \cdots + e_n^{(i)}\, \beta_n^{(i)} \\ &= a_{1i}\, \varrho_1 + a_{2i}\, \varrho_2 + \cdots + a_{ni}\, \varrho_n, \qquad (i = 1, 2, \ldots, k-1). \end{aligned}$$

Since P_k contains no prime factors except such as are factors of f, let us suppose that J is the smallest field ideal (i. e., the field ideal containing the fewest prime factors) whose product with P_k is divisible by f, and let M_1 be the smallest rational integer which is divisible by J. Let $M_2 = N(I)$, where $I = I_1 \cdot I_2 \cdots I_{k-1}$.

The ideal I is relatively prime to the principal ideal $\{\varDelta\}$. Hence, since $\varDelta$ is a rational integer, M_2 and $\varDelta$ are relatively prime. Since M_2 is divisible by I, we can choose I_k such that $I \cdot I_k = \{M_2\}$, and I_k is then relatively prime to $\varDelta$, and hence also to f, which is a divisor of $\varDelta$.

Since $\{\gamma\} = P_1 \cdot P_2 \cdots P_k$, P_k belongs to the class reciprocal to that of $P_1 \cdot P_2 \cdots P_{k-1}$. But I belongs to the class reciprocal to that of $P_1 \cdot P_2 \cdots P_{k-1}$. Hence P_k and I belong to the same class, and P_k and I_k belong to reciprocal classes. Therefore $P_k I_k = \{\overline{\alpha}^{(k)}\}$.

Let us now write $\bar{\alpha} = \alpha' \cdot \alpha'' \cdots \alpha^{(k-1)} \cdot \bar{\alpha}^{(k)} \cdot M_1$. Since $\bar{\alpha}^{(k)} M_1$ is divisible by JP_k, it is divisible by f, and hence $\bar{\alpha}$ is an integer in R. In fact, $\bar{\alpha}^{(k)} \cdot M_1$ belongs to R, and we may therefore write

$$\bar{\alpha}^{(k)} \cdot M_1 = \bar{a}_{1k}\varrho_1 + \bar{a}_{2k}\varrho_2 + \cdots + \bar{a}_{nk}\varrho_n.$$

Moreover

$$\{\bar{\alpha}\} = P_1 \cdot P_2 \cdots P_k \cdot I_1 \cdot I_2 \cdots I_k \cdot M_1 = \{\gamma M_2 M_1\},$$

and hence $\bar{\alpha}$ and $\gamma M_1 M_2$ differ only by a factor which is a unit in $k(\theta)$. Then let us write $\bar{\alpha} = E\gamma \cdot M_1 \cdot M_2$. But since $\bar{\alpha}^{(k)} M_1$ is divisible by f, $E \cdot \bar{\alpha}^{(k)} \cdot M_1$ is also divisible by f, and hence belongs to R. Then, if we put $E\bar{\alpha}^{(k)} = \alpha^{(k)}$, we may write

$$M_1 E \bar{\alpha}^{(k)} = M_1 \alpha^{(k)} = a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n;$$

and since $\{\bar{\alpha}^{(k)}\} = \{\alpha^{(k)}\}$, we have $P_k I_k = \{\alpha^{(k)}\}$, and

$$\alpha = \alpha' \cdot \alpha'' \cdots \alpha^{(k)} M_1 = \gamma \cdot M_1 \cdot M_2.$$

We therefore have

$$\begin{aligned} M_1 M_2 \gamma &= M_1 M_2 (\xi_1\varrho_1 + \xi_2\varrho_2 + \cdots + \xi_n\varrho_n) \\ &= \prod_{i=1}^{k} (a_{1i}\varrho_1 + a_{2i}\varrho_2 + \cdots + a_{ni}\varrho_n), \end{aligned}$$

and using the notations of § 2, we have

$$\xi_i = \frac{A_i}{M_1 \cdot M_2}, \qquad (i = 1, 2, \ldots, n). \tag{13}$$

Here the A_i are polynomials in the a_{ji}, $(j = 1, 2, \ldots, n;$ $i = 1, 2, \ldots, k)$, which upon application of (9) to the elements standing in the first k—2 rows of the matrix of § 2 as they occur in the A_i, gives an expression for the ξ_i in terms of the parameters $e_j^{(i)}$ which occur in the expressions for u_i, and the elements of the last two rows of the matrix which are implicitely involved in the u_i in the factors μ_i' and μ_i''.

We have thus by means of (12) and (13) obtained a general form for the solution of (11). We shall next see that all such expressions, when the parameters are given rational integral values, constitute a solution of (11).

Substituting for $\xi_1, \xi_2, \ldots, \xi_n$; $u_1, u_2, \ldots, u_{k-2}$ their values as given by (12) and (13) in (11), we have the equation

$$(14) \qquad \frac{N(A_1\varrho_1 + \cdots + A_n\varrho_n)}{M_1^n \cdot M_2^n} = \prod_{\iota=1}^{k-2} \varepsilon_\iota \mu_i' \mu_i'' F_\iota(e_1^{(i)}, e_2^{(i)}, \ldots, e_n^{(i)}).$$

Since

$$M_2 = I \cdot I_k,\ N(I_\iota) = N_R(I_\iota^{(R)}),\ |\mu_1' \cdot \mu_2' \cdots \mu_{k-2}'| = N_R(P_{k-1}^{(R)}),$$

and $P_{k-1}^{(R)} \cdot I_{k-1}^{(R)} = [\alpha^{(k-1)}]$, a principal ideal in R, we can write (14) in the form

$$(15) \qquad N(A_1\varrho_1 + A_2\varrho_2 + \cdots + A_n\varrho_n)$$
$$= M_1^n M_2^{n-1} N_R(I_{k-1}^{(R)}) \mu_1' \cdot \mu_2' \cdots \mu_{k-2}' \prod_{\iota=1}^{k-2} \varepsilon_\iota \mu_i'' N_R(I_i^{(R)})$$
$$\times F_\iota(e_1^{(i)}, \ldots, e_n^{(i)}).$$

But

$$N_R(I_\iota^{(R)}) F_\iota(e_1^{(i)}, e_2^{(i)}, \ldots, e_n^{(i)}) = N(e_1^{(i)}\beta_1^{(i)} + \cdots + e_n^{(i)}\beta_n^{(i)})$$
$$= N(a_{1i}\varrho_1 + a_{2i}\varrho_2 + \cdots + a_{ni}\varrho_n).$$

Also

$$N_R(I_{k-1}^{(R)}) \cdot |\mu_1' \cdot \mu_2' \cdots \mu_{k-2}'| = N_R(I_{k-1}^{(R)}) N_R(P_{k-1}^{(R)})$$
$$= N\{\alpha^{(k-1)}\} = \varepsilon^{k-1} N(\alpha^{(k-1)})$$
$$= \varepsilon_{k-1} N(a_{1k-1}\varrho_1 + a_{2k-1}\varrho_2 + \cdots + a_{nk-1}\varrho_n).$$

Let us suppose that the signs of $\mu_1', \mu_2', \ldots, \mu_{k-2}'$ have been so chosen that the sign of their product is the same as the sign of $N(\alpha^{(k-1)})$ and hence $\varepsilon_{k-1} = +1$. If we now put $\varepsilon = \varepsilon_1 \cdot \varepsilon_2 \cdots \varepsilon_{k-2}$, we may write (15) in the form

$$(16) \qquad N(A_1\varrho_1 + A_2\varrho_2 + \cdots + A_n\varrho_n)$$
$$= \varepsilon M_1^n M_2^{n-1} \mu_1'' \cdot \mu_2'' \cdots \mu_{k-2}'' \prod_{\iota=1}^{k-1} N(a_{1i}\varrho_1 + \cdots + a_{ni}\varrho_n).$$

Since $M_2 = I \cdot I_k = N(I)$, we have

$$M_2^n = N(I) N(I_k) = M_2 N(I_k),$$

and hence $M_2^{n-1} = N(I_k)$. Also $|\mu_1'' \cdot \mu_2'' \cdots \mu_{k-2}''| = N(P_k)$, and therefore

$$(17) \quad M_1^n M_2^{n-1} |\mu_1'' \cdot \mu_2'' \cdots \mu_{k-2}''| = M_1^n N(I_k \cdot P_k) = N\{M_1 \alpha^{(k)}\}$$
$$= N\{a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n\}.$$

For, since $M_1\alpha^{(k)}$ is divisible by f, it belongs to R. As above, we shall now affix signs to the μ_i'' so that the sign of their product is the same as the sign of $N(\alpha^{(k)})$. Then we have

$$M_1^n \cdot M_2^{n-1} \mu_1'' \cdot \mu_2'' \cdots \mu_{k-2}'' = N(a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n).$$

We may now write (16) in the form

$$N(A_1\varrho_1 + A_2\varrho_2 + \cdots + A_n\varrho_n) = \varepsilon \prod_{i=1}^{k} N(a_{1i}\varrho_1 + a_{2i}\varrho_2 + \cdots + a_{ni}\varrho_n),$$

which, by § 1, is seen to be an identity if $\varepsilon = +1$.

Hence, with the signs of the μ_i' and μ_i'' properly chosen, and ε_i so chosen that their product is $+1$, all numbers obtained by (12) and (13) are solutions of (11). We observe, however, that the expressions for $\xi_1, \xi_2, \ldots, \xi_n$ are fractional in form. Hence we must next determine under what conditions they are integral, that is, we must determine what conditions must be imposed on the μ_i' and μ_i'' in order that the solutions shall be integral, or in other words, that γ shall belong to R.

From the development, it follows that, for

$$\gamma = \frac{A_1\varrho_1 + A_2\varrho_2 + \cdots + A_n\varrho_n}{M_1 M_2},$$

$M_1 \cdot M_2 \cdot \gamma$ is an integer in R and

$$\{\gamma\} = \left\{\frac{A_1\varrho_1 + A_2\varrho_2 + \cdots + A_n\varrho_n}{M_1 M_2}\right\} = P_1 \cdot P_2 \cdots P_k$$

is a principal ideal of the field. Since the product of any integer of the field by Δ is an integer of R, we know that

$$\frac{A_1\Delta\varrho_1 + A_2\Delta\varrho_2 + \cdots + A_n\Delta\varrho_n}{M_1 M_2}$$

is an integer of R. Hence it is equal to $C_1\varrho_1 + C_2\varrho_2 + \cdots + C_n\varrho_n$, where $C_1, C_2, \ldots, C_n$ are rational integers. But the representation by the fundamental system is unique; hence

$$\frac{A_i\Delta}{M_1 M_2} = C_i.$$

Since M_2 is relatively prime to Δ, this says that A_i is divisible by M_2; hence (13) will not give fractions with factors of M_2 in the denominator.

We have defined J as the smallest ideal whose product with P_k is divisible by f, and M_1 as the smallest rational integer divisible by J. Let $M_1 = M_1' \cdot M_1''$, where M_1' is the smallest factor of M_1 such that $M_1' P_k I_k = \{\gamma^{(k)}\}$, when $\gamma^{(k)}$ is an integer of R. If we then choose $\alpha^{(k)}$ in $P_k \cdot I_k = \{\alpha^{(k)}\}$ such that $M_1 \alpha^{(k)} = M_1'' \gamma^{(k)}$, which is always possible since

$$M_1''\{\gamma^{(k)}\} = M_1 P_k I_k = M_1' M_1''\{\alpha^{(k)}\},$$

we see that the $a_{1k}, a_{2k}, \ldots, a_{nk}$ are all multiples of M_1'' since $\gamma^{(k)}$ belongs to R. We thus see that the A_i are all divisible by M_1'' and only factors of M_1' can occur in the denominators of the numbers furnished by (13).

In the first part of this article, we have seen that all integral solutions of (11) can be expressed by (12) and (13). In the proof as given, for $i < k$ the

$$e_1^{(i)}\beta_1^{(i)} + e_2^{(i)}\beta_2^{(i)} + \cdots + e_n^{(i)}\beta_n^{(i)} = a_{1i}\varrho_1 + a_{2i}\varrho_2 + \cdots + a_{ni}\varrho_n$$

were all relatively prime to f. Hence the product

$$\prod = \prod_{i=1}^{k-1}(a_{1i}\varrho_1 + a_{2i}\varrho_2 + \cdots + a_{ni}\varrho_n)$$

is also relatively prime to f. The integer $\prod$ belongs to R. Hence if

$$(a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n)\prod = \prod_{i=1}^{k}(a_{1i}\varrho_1 + a_{2i}\varrho_2 + \ldots + a_{ni}\varrho_n)$$

should have all its coefficients, when it is written as a linear function of $\varrho_1, \varrho_2, \ldots, \varrho_n$, divisible by some factor $M_1^{(3)}$ of $M_1' = M_1^{(3)} \cdot M_1^{(4)}$ it would follow that

$$\frac{a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n}{M_1' \cdot M_1''} M_1^{(4)} \cdot \prod$$

would be an integer of R.

Hence the product of

$$\frac{a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n}{M_1' \cdot M_1''} M_1^{(4)}$$

by $\prod$, or by any integer divisible by f, would be an integer of R. But $\prod$ belongs to R, and the principal ideal $[\prod]$

is relatively prime to f. Hence there exists an integer C in R and an integer D in f such that $C \cdot \prod + D = 1$.*

Since $C\prod$ belongs to R, the products of

$$\frac{a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_k}{M_1' \cdot M_1''} M_1^{(4)}$$

by $C\prod$ and by D, and hence also the sum of these products, belong to R. Therefore

$$\frac{a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n}{M_1' \cdot M_1''} M_1^{(4)}$$

is an integer in R. But $\{\alpha^{(k)}\} = P_k I_k$, and

$$M_1' \cdot M_1'' P_k I_k = \{a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n\}.$$

Hence

$$M_1^{(4)} P_k I_k = \left\{ \frac{a_{1k}\varrho_1 + a_{2k}\varrho_2 + \cdots + a_{nk}\varrho_n}{M_1' \cdot M_1''} M_1^{(4)} \right\},$$

where the integer determining the principal ideal belongs to R. But we have assumed that M_1' is the smallest factor of M_1 such that when $M_1' P_k I_k = \{\gamma^{(k)}\}$ the $\gamma^{(k)}$ belongs to R; hence $M_1^{(4)} = M_1'$, i. e., $M_1^{(3)} = 1$.

Therefore M_1' will always occur as a denominator in the numbers furnished by (13), and integral solutions are possible only when $M_1' = 1$. Consequently, in order to have integral solutions, $\alpha^{(k)}$ must be an integer of R.

We have seen that $a_{1k}, a_{2k}, \ldots, a_{nk}$ are all divisible by M_1''. Hence if we put $a_{ik}/M_1'' = C_{ik}$ we shall have

$$\gamma^{(k)} = C_{1k}\varrho_1 + C_{2k}\varrho_2 + \cdots + C_{nk}\varrho_n.$$

The matrix obtained from m in § 2 by replacing the a_{ik} by the C_{ik}, $(i = 1, 2, \ldots, n)$, shall be denoted by m'.

From the theory of the correspondence between ideals and decomposable forms, we know that to a class of ideals corresponds a class of forms. Any form of the class can be obtained by proper choice of the base of the ideal, from any ideal of the corresponding class of ideals by the method of § 4. Moreover, the numbers represented by any form of the class can be represented by every form of the class.

Let γ be any integer of R. Suppose that the principal field ideal $\{\gamma\}$ is separated into the product $T \cdot P_k$ of two field ideals such that T is relatively prime to f, and that P_k contains no prime factors except such as are divisors of f. Let $T^{(R)}$ be the ring ideal corresponding to T. From the reciprocal class in R, select an ideal $I^{(R)}$ whose corresponding field ideal I is relatively prime to $\{\Delta\}$. Let $N(I) = M_2$. Then M_2 is relatively prime to Δ, and hence also to f, which is a divisor of Δ. Therefore $I_k = \{M_2\}/I$ is relatively prime to f, and the corresponding ring ideal $I_k^{(R)}$ belongs to the class reciprocal to that of $I^{(R)}$.

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* Bachmann, loc. cit., p. 367.

Since $I^{(R)}$ and $T^{(R)}$ belong to reciprocal classes in R, we have $I^{(R)}T^{(R)} = [\alpha]$. Multiplying both members of this equation by the unit ideal of $k(\theta)$, we have $I \cdot T = \{\alpha\}$. It is easily seen that I_k and P_k belong to reciprocal classes. Hence $I_k \cdot P_k = \{\gamma^{(k)}\}$. We shall next see that the integer $\gamma^{(k)}$ belongs to R.

We have $T \cdot P_k = \{\gamma\}$ and γ was chosen an integer of R. Since M_2 is a rational integer, $M_2\gamma$ belongs to R. Since $I^{(R)}$ was chosen from the class reciprocal to that of $T^{(R)}$ in R, α belongs to R and is relatively prime to f. We may write

$$I_k P_k = \frac{\{M_2\}}{I} \cdot P_k = \frac{\{M_2\} T \cdot P_k}{I \cdot T} = \left\{\frac{M_2\gamma}{\alpha}\right\} = \{\gamma^{(k)}\},$$

and hence we may write $\gamma^{(k)} = M_2\gamma/\alpha$. Therefore $\gamma^{(k)} \cdot \alpha$ belongs to R; and, if a is any integer of $[\alpha]$, $\gamma^{(k)} \cdot a$ belongs to R. Also if b is any integer of f, $\gamma^{(k)} \cdot b$ belongs to R and therefore $\gamma^{(k)}(a+b)$ belongs to R. But since $[\alpha]$ is relatively prime to f, a and b may be so chosen that $a + b = 1$. Hence $\gamma^{(k)}$ belongs to R.

We may now sum up the result of the investigation as follows. Let $k(\theta)$ be any algebraic number field of degree n, and $\varrho_1, \varrho_2, \ldots, \varrho_n$ a fundamental system of a ring R whose conductor is f and index Δ. Select any integer γ from R and separate the principal ideal $\{\gamma\}$ into two factors $T \cdot P_k$, where T is relatively prime to the conductor f, and where P_k contains no prime factors except divisors of f. Let $T^{(R)}$ be the ring ideal corresponding to T, and $I^{(R)}$ an ideal from the reciprocal class in R whose corresponding field ideal I is relatively prime to $\{\Delta\}$. Let $M_2 = N(I)$, and $I_k = M_2/I$.

Then

$$I_k \cdot P_k = \{\gamma^{(k)}\} = \{C_{1k}\varrho_1 + C_{2k}\varrho_2 + \cdots + C_{nk}\varrho_n\},$$

since, as we have seen above, $\gamma^{(k)}$ belongs to R.

Select $k-2$ rational integers $\mu_1'', \mu_2'', \ldots, \mu_{k-2}''$, such that their product has the same sign as $N(\gamma^{(k)})$, and an absolute value equal to $N(P_k)$.

Next select $k-1$ ideals $I_1, I_2, \ldots, I_{k-1}$ whose product is I. As before, let $I_i^{(R)}$ be the ring ideal corresponding to I_i. Let $F_i(x_1, x_2, \ldots, x_n)$, $(i = 1, 2, \ldots, k-2)$, be the decomposable forms corresponding to the ideals $I_i^{(R)}$, $(i = 1, 2, \ldots, k-2)$. Choose an ideal $P_{k-1}^{(R)}$ from the class reciprocal to that of $I_{k-1}^{(R)}$ and let

$$P_{k-1}^{(R)} \cdot I_{k-1}^{(R)} = \left[\alpha^{(k-1)}\right] = [a_{1k-1}\varrho_1 + a_{2k-1}\varrho_2 + \cdots + a_{nk-1}\varrho_n].$$

Next select $k-2$ rational integers $\mu_1, \mu_2', \ldots, \mu_{k-2}'$ whose product has the sign of $N(\alpha^{(k-1)})$ and the absolute value $N(P_{k-1})$. Then for rational integral $e_j^{(i)}$, $(i = 1, 2, \ldots, k-2;$ $j = 1, 2, \ldots, n)$, and $\varepsilon_i = +1$ or -1, such that $\varepsilon_1 \cdot \varepsilon_2 \cdots \varepsilon_{k-2} = +1$ the numbers

$$u_i = \varepsilon_i \mu_i' \cdot \mu_i'' F_i(e_1^{(i)} \cdot e_2^{(i)} \cdots e_n^{(i)}), \qquad (i = 1, 2, \cdots, k-2),$$

$$\xi_i = \frac{A_i}{M_2}, \qquad (i = 1, 2, \cdots, n),$$

constitute a solution of the equation

$$N(\xi_1\varrho_1 + \xi_2\varrho_2 + \cdots + \xi_n\varrho_n) = u_1 \cdot u_2 \cdots u_{k-2}.$$

The A_i are computed as in § 2 from the matrix

$$m' = \left\| \begin{array}{llll} a_{11}, & a_{21}, & \ldots, & a_{n1} \\ a_{12}, & a_{22}, & \ldots, & a_{n2} \\ & \cdots & \cdots & \\ a_{1k-1}, & a_{2k-1}, & \ldots, & a_{nk-1} \\ c_{1k}, & c_{2k}, & \ldots, & c_{nk} \end{array} \right\|.$$

when the a_{ij} in the first $k-2$ rows of the matrix are obtained from the $e_j^{(i)}$ by means of $k-2$ sets of equations such as (9), one set for each of the ideals $I_i^{(R)}$, $(i = 1, 2, \ldots, k-2)$.

By the same method, all solutions of the given Diophantine equation may be obtained.

THE UNIVERSITY OF ILLINOIS

ARTHUR DUNN PITCHER
IN MEMORIAM*

Arthur Dunn Pitcher, Professor of Mathematics in Adelbert College, Western Reserve University, died on October 5, aged forty-three years. While he had suffered from a weak heart for some time, his sudden death came as an unexpected blow to his family and colleagues. In fact he had met his classes as usual on October 4, and gave every indication of normal health.

He was born and reared in Kansas. He was graduated from the University of Kansas in 1906 and received the degree of Doctor of Philosophy from the University of Chicago in 1910. After that he held positions at the University of Kansas, Dartmouth College, and during the past eight years at Western Reserve University.

At the University of Chicago Professor Pitcher came under the influence of Professor E. H. Moore, under whose direction he wrote his doctor's thesis. After receiving his doctorate he continued the researches he had begun at Chicago in E. H. Moore's *General Analysis*; and these led him into researches in the Fréchet Theory and the Theory of Functions of a Real Variable. By his contributions to these fields of mathematics he attained a high place among American mathematicians.

While his principal interest was in the study of mathematics, nevertheless he performed conscientiously all the tasks that came to him. For the last three years he was a member of the budget committee and of the executive committee of Adelbert College. His work on these important committees gained for him the reputation of being fair minded and having sound judgment. As a consequence he was frequently consulted on matters of university policy.

At all times he maintained a high standard, but because of his patience and sympathy he enjoyed to an unusual degree the loyalty and affection of his students. The attainments of his students and their love for him called forth the admiration of his associates. His happy disposition and his kindly helpfulness to the students, to his associates, and to his friends, are sadly missed.

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* This statement was adopted as a minute by the Faculty of Western Reserve University, and is here published by permission.

EVANS' COLLOQUIUM LECTURES

The Cambridge Colloquium Lectures, 1916. *Part* I: *Functionals and their Applications. Selected Topics, including Integral Equations.* By G. C. Evans. New York, American Mathematical Society, 1918. 136 pp.

The theory of functionals has been in the foreground so much in the last two decades, particularly in the domain of integral equations, that a colloquium on this subject was very timely. The theory of integral equations, however, bears the same relation to the whole theory of functionals as the theory of algebraic equations does to the entire domain of the theory of functions. In fact, so rapid has been the development of the theory of functionals that Paul Lévy has recently devoted a book of nearly four hundred fifty pages to the study of functional analysis alone, leaving aside entirely the theory of complex functionals and touching but slightly the theory of integral and integro-differential equations and a number of other subjects that would ordinarily be included as topics in the theory of functionals. It can be seen from this what a difficult, and what seems to the reviewer an almost impossible, task it would be to essay a clear presentation of almost every domain of this difficult subject in the space of less than one hundred fifty pages. This was the task that Professor Evans set himself.

The author has divided his work into five lectures. The first lecture takes up general considerations of a functional, such as definitions of continuity, Volterra derivatives, and additive and non-additive functionals of plane curves. A rather interesting connection is noted between additive functionals and functions of point sets, viz., "an additive continuous functional of finite variation has a finite derivative (in the restricted sense) at all points except possibly those of a set of measure zero". Another interesting result is the extension of the law of the mean of the differential calculus. Functionals of space curves are then introduced, and the concept of the *flux of a functional,* due to Lévy, is defined, and is used extensively in the next lecture.

In studying the dependence of the Green's function on the boundary of the closed curve, Hadamard was led to an equation involving functional derivatives. These equations are the analogs of total and partial differential equations in n variables. Just as, in the latter, certain conditions of integrability are required, so here also analogous conditions must be satisfied. These are developed very elegantly by means of another concept due to Lévy, that of the *adjoint linear functional.* Various interesting applications are made of the integrability conditions, and, in particular, it is shown that Hadamard's equation is *completely integrable.*

The author now turns his attention to complex functionals, to which he devotes Lecture II. As far back as 1889 a considerable portion of this theory was developed by Volterra, who takes as a starting point the relation of *isogeneity*, which is the extension to functionals of curves in space of the relation that holds between two complex point functions on a surface. This condition of isogeneity is expressed in terms of the normal component to the curve of the vector flux of a functional defined in Lecture I. Although the author summarizes the properties of the linear vector function used in this lecture, it would be highly desirable for one to have some acquaintance with vector analysis before attempting to read it. In case the functionals involved are additive, the condition of isogeneity may be considerably simplified, since in this case the vector flux can be chosen as a vector point function independent of the curve. Important properties of the relation of isogeneity for additive functionals are given, and the analogs of Green's and Cauchy's theorems are proved. All this has been extended by Volterra to additive complex functionals whose arguments are r-dimensional hyperspaces immersed in n-space. For $r = n - 2$, this is a direct generalization of the theory of complex functionals of curves in 3-space.

Although Lecture III has the caption "Implicit Functional Equations", this part of the work is essentially concerned with the study of the linear functional. This study must necessarily precede any existence theorem on implicit functional equations, since the differential, which has a natural origin here, is a linear functional. $T[\Phi]$ is said to be a *linear functional* of Φ if (1) it is a distributive functional of Φ, and (2) if it is a continuous functional of Φ, where Φ is allowed to range over the whole class of continuous functions. Various representations of such a linear functional have been given by Hadamard, F. Riesz, and Lebesgue, using as their means of representation the Riemann, the Stieltjes, and the Lebesgue integrals, respectively. Since, in the definition of a linear functional, one may change either the class of functions Φ, or the type of continuity, one may obtain two distinct extensions of the linear functionals. The first has been made by Fréchet, Lebesgue, Radon, and others. The second extension is to introduce higher orders of continuity. The importance of this concept seems to have been recognized first by Bliss,* who notes that if the calculus of variations is to be regarded as a chapter in the theory of maxima and minima of functionals, the definition of continuity of a functional must be extended in this way.

Fréchet's definition of a differential of a functional is now given, and a theorem on implicit functional equations due to Volterra is proved in which the differential is used. It must be pointed out, however, that Volterra uses a very special kind of differential. In fact, Volterra proved his result some time before Fréchet gave his formal definition.

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* *A note on functions of lines,* PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 1 (1915), p. 173.

Lecture IV opens with the remark that the most desirable approach to the treatment of Laplace's equation is to study, as Bôcher did, the equation

$$\int_c \frac{\partial u(x, y)}{\partial n} \, ds = 0.$$

This relation is supposed to hold for all circles within a given two-dimensional region, and the existence and continuity of u, u_x, u_y are assumed, instead of the existence of u_{xx}, u_{yy}, which have no physical significance. This lecture is practically devoted to the study of an equation which is a generalization of the one given above. Professor Evans calls such equations integro-differential equations of the Bôcher type. By defining the adjoint of such equations, the author is able to prove an extension of Green's theorem for them. He now subjects the equation to an arbitrary real point transformation, and finds that the equation is transformed into a similar one; and, furthermore, that a certain differential quadratic form $T(dx, dy)$ is a covariant of the transformation. The directions defined by $T(dx, dy) = 0$ are called *characteristic directions,* and the solutions of the equation are called *characteristic curves.* The characteristic directions may be real and distinct, real and coincident, or imaginary. The integro-differential equation is said to be hyperbolic, parabolic, or elliptic, respectively, just as in the theory of linear partial differential equations. Normal forms of these types are then obtained, and a rather detailed study of the parabolic type is made. The lecture closes with some remarks on the usual types of integro-differential equations, as originally given by Volterra.

Lecture V, the last of the series, gives an account of the various generalizations of the theory of integral equations. The lecture begins with a statement of some of the more important properties of Stieltjes' integrals, and an application of these to the study of a class of equations involving Stieltjes' integrals. An existence theorem of such equations is given, but with the hypotheses as stated, an extension of the Fredholm theory is not possible.*

In generalizing the theory of the linear integral equation, one may proceed along at least two distinct lines. In the first place, one may, with E. H. Moore, generalize the variables, the classes of functions, and the linear operations, in such a way that the methods of Fredholm apply for the extended concepts. Or one may proceed, as Volterra has done in his *Theory of Permutable Functions*, to build a calculus of composition, i. e., of the operations which produce the iterated kernels. He is able by this means to solve an extensive class of non-linear integral and integro-differential equations. These extensions Professor Evans takes up in the order mentioned, and it should be remarked that an exceptionally clear exposition of Moore's theory is given, and a number of important contacts with the classical theory are pointed out. He notes, for example, that when the general range P is the one-dimensional continuum, and the class M is the totality of continuous functions over this range, Moore's linear operation J reduces to the classical Stieltjes' integral.

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* It might be stated here that such a generalization has recently been given by F. Riesz in a very important paper in the ACTA MATHEMATICA (vol. 41 (1916), p. 71), which unfortunately appeared just too late to be included in these lectures. He uses there the concept of *completely continuous linear functionals*; it was this notion that made the extension possible.

In the presentation of Volterra's theory, Professor Evans makes a considerable advance over the treatment found in Volterra's *Leçons sur les Fonctions de Lignes*. In the first place, he combines the theories of permutable functions of the first and second kinds into a single theory. In the second place, he introduces notations and concepts that simplify materially the proofs given by Volterra. This theory of permutable functions leads immediately to a very interesting extension of every analytic function. This extension carries with it corresponding extension of addition theorems, moduli of periodicity, etc. One such generalized function, called the *Volterra transcendental,* plays an important role in a certain class of integro-differential equations.*

It can be seen from this outline what a multitude of topics has been discussed in these lectures. After reading this book, one feels that the author is profoundly at home with every phase of the theory of functionals, to which he himself has made many important contributions. Although a serious effort has been made to give a clear presentation of these subjects, one cannot help feeling that it might have been wiser to have included fewer topics and to have developed each of them in a more leisurely fashion.

Of the misprints, one may mention those which have been noted already by Professor Evans in this BULLETIN (vol. 25, p. 461) and the following. In the theorem at the top of page 7, change $F'[\varphi(x) \mid \xi]$ to $F'[\varphi_0(x) \mid \xi]$. The left member of formula (24') on page 16 should read $g'[c \mid ABM]$ instead of $g[c \mid ABM]$. On page 69, the right member of formula (33) should read $F[\varphi + \Delta\varphi] - F[\varphi]$.

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* See Schlesinger, *Zur Theorie der linearen Integro-Differentialgleichungen*, JAHRESBERICHT DER VEREINIGUNG, vol. 24 (1915), pp. 84–125.

The author remarks in his preface that the material has been so arranged that "the text in large type ... may be read by itself". Yet we find on page 34 (large type) references to equations (7) and (8), which are in small type. All these, however, are unimportant oversights, and the reviewer turns from this work with the feeling that there is in it a wealth of valuable information on practically every phase of the theory of functionals, with many suggestions for its future development. The bibliographical list that heads each chapter is a feature of not inconsiderable value.

I. A. BARNETT

JUNG ON ALGEBRAIC FUNCTIONS

Einführung in die Theorie der algebraischen Funktionen einer Veränderlichen, by Heinrich W. E. Jung. Berlin, Walter de Gruyter, 1923. 246 pp.

The three great paths in the study of the algebraic functions of a complex variable—the geometric, the analytic, and the arithmetic—have as a common starting point a single algebraic equation, $f(x,y) = 0$. The traveler on one of the roads, once away from the point of departure, is often far out of hailing distance from those on the other paths; yet he is at times agreeably surprised to find he has reached the same point as they. At such times there will be a sign-post telling him and his fellow-climbers that they have reached the Riemann-Roch Theorem, it may be, or the Lückensatz of Weierstrass. Whatever the point to which the various paths converge, it is almost certain to be concerned with *genus*, or *deficiency*, if another language is used.

Multiplicity of dialects is, indeed, characteristic of the study in question. Not only has each path its own vocabulary, but the arithmetic path, with which we are chiefly concerned here, has no single valid language.* In Jung's book, for instance, we miss the mention of *Ring, Führer, Ideal, Modul, Integrabilitatsbereich, Polygon,* although most of the concepts named find a place. On the other hand, certain terms are borrowed from algebraic geometry, in particular, *canonical class* (corresponding to the canonical series) and *adjoint functions* (corresponding to adjoint curves).

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* For a comparison of the content, and to some extent of the language, of the various theories, see Emmy Noether, *Die arithmetische Theorie der algebraischen Funktionen einer Veränderlichen, in ihrer Beziehung zu den übrigen Theorien und zu der Zahlkörpertheorie*, JAHRESBERICHT DER D. MATH.-VEREINIGUNG, vol. 28 (1919).

Although other paths to the study of algebraic functions have probably greater beauty and historical importance, the arithmetic approach can justly claim to be the most logical one. For the arithmetic theory alone concerns itself primarily with the functions, not with curves or integrals. And Jung's book does much to increase the esthetic claims of his route.

Hensel and Landsberg's scholarly treatise on the arithmetic theory* being the only predecessor of Jung's *Einleitung,* a comparison of the two is valuable. The newer book is much the shorter. Brevity is obtained by assuming somewhat greater mathematical knowledge at the outset, and by restricting the scope of the study. The reader is expected to be acquainted with much of the theory of functions of a complex variable, in particular, with facts on development of functions in power series and on the inversion of functions so developed, and with essential singularities. Riemann surfaces are, however, explained from the beginning. Knowledge is assumed of the Lagrange interpolation formula, a little of the theory of matrices, and both point and line (homogeneous) coordinates in the plane.

Chapter I deals with an algebraic function of x defined by a single equation and with its behavior (i. e., development in power-series) in the neighborhood of any point. An important concept introduced here is that of the *Einheit* (we may call it an *E*-function in English), which, at a given point, is finite but not zero.

Chapter II gives methods for computing the power-series for an algebraic function at any point. Puiseux's diagram is clearly and fully explained. We could wish, in both these chapters, for more attention to convergence proofs.

The next chapter is devoted to an admirable exposition of the Riemann surface. Particularly clear are the figures, the account of the identification of branches, and the summary.

The preliminary apparatus having thus been set up, Jung proceeds, in Chapter IV, to the algebraic body (xy); this consists of all rational functions of x and of y, the variables being connected by the algebraic equation $f(x, y) = 0$. The essential properties of the functions of the body are then developed: unique definition on the Riemann surface, equality of numbers of zeros and of poles, representation in power series, etc. The residue of a differential $\eta d\xi$ appears in this chapter (η and ξ are functions of the body), as well as a few theorems preparatory to the reappearance of the residue in the chapter on integrals.

Chapter V is entitled Birational Transformations. There is in the arithmetic treatment no thought of the (x, y) plane, no occasion, accordingly, for one-to-one transformations of that plane. The chapter centers about the theorem that, if ξ is a variable of the body (xy), there exists another variable η of (xy), such that the bodies (xy) and $(\xi\eta)$ are identical.

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* *Theorie der algebraischen Funktionen einer Variabeln und ihre Anwendung auf algebraische Kurven und Abelsche Integrale.* Teubner, Leipzig, 1902.

In Chapter VI we come to what is probably the most characteristic tool of the arithmetic theory of algebraic functions, the *divisor*. Strangely enough, one is never told *what* a divisor is. To every place (Stelle) $\mathfrak{p}$ of the algebraic body "is assigned" a prime factor with the same name $\mathfrak{p}$. The essence of a prime factor is defined as follows. If t is a function with a zero of lowest possible order at $\mathfrak{p}$, and if a function R is the product of t^λ and an E-function at $\mathfrak{p}$, then R is said to be *divisible* by $\mathfrak{p}^\lambda$. A prime factor is, one might say, the minimum amount of zeroness at a point. A divisor is a product of integral powers (positive, negative, or zero) of prime factors. A divisor is determined, for instance, by the zeros and poles of a function. All divisors are grouped in classes; such a class consists of products of a single divisor and of all divisors of functions. The latter constitute, of course, a class by themselves, the principal class; while differentials, $\eta d\xi$, furnish another, the differential or canonical class. Linear independence of integral divisors is ingeniously defined, and the number of linearly independent integral divisors in a class is called its dimension. It is a misfortune, probably irremediable, that this number is one more than the dimension of the equivalent linear series of groups of points on a curve, — that two languages use the same word in senses just different enough to be misleading. In this chapter we notice an uncertainty as to what an α-fold branch point is. On page 71, line 1, a three fold branch point is one where four sheets join; yet on page 73, line 24, α sheets dance around an α-fold branch point. "Exactly divisible", a phrase introduced on page 79, ought to be defined. In this chapter the genus enters, defined by the number of sheets and by the sum of orders of branch points.

Chapter VII is devoted to matrices. It is hard to see why matrices should be called "systems" throughout this chapter, and developed from the beginning, when Jung will later (p. 154) not only talk unconcernedly of a matrix, but assume that the reader knows facts about it not mentioned in Chapter VII. The treatment is, at any rate, very satisfactory. The introduction of "multiplicative" and "additive" matrices for producing, on multiplication, desired changes in given matrices, is decidedly ingenious. Other important types are the "Einheiten", which we might now translate as "E-matrices". They are square matrices of rational functions of x, whose determinants behave as E-functions, either at all points, or at a single one. It seems unfortunate that matrices should be walled in indiscriminately by parentheses and braces. As parentheses are also used by Jung to indicate a class of divisors, braces to indicate its dimension, the double bar before and after would certainly be better, if the printer could be persuaded.

The eighth chapter treats the problem of determining all multiples of a given divisor; in the language of other writers the totality of such multiples is an "ideal". The solution is reduced to the determination of a "base" for the functions divisible by the divisor — a group of functions such that, in linear combination with rational coefficients, they generate the body (xy), whereas with integral rational coefficients they generate the multiples of the divisor. The task of showing that such a base can always be found is a strenuous one, the chapter correspondingly long and difficult; but the work is done in masterly fashion. There are, at the end, two sections on complementary bases: pairs of bases, such that the two matrices of the functions and of their conjugates are complementary.

The final chapter of the purely arithmetic theory is devoted to the Riemann-Roch Theorem. As here given, it is concerned with the dimensions and orders of two "Ergänzungsklassen", i. e., classes such that a divisor of one and a divisor of the other have as product a divisor of the canonical class. The central position of this theorem arises from the information it furnishes concerning the existence of functions with given poles and zeros. Several consequences of the theorem for bodies of genera 0, 1, and 2 are deduced, and the chapter ends with the introduction of Weierstrass points.

Two chapters are now devoted to the geometry of the curve $f(x, y) = 0$. In one of them, non-homogeneous coordinates are used, and in the other the change to homogeneous coordinates is made. Many headings of sections, such as multiple points, form of real branches, number of intersections, Hessian curve, Plucker formulas, are those found in any book on projective geometry. The treatment, however, using the divisor on almost every page, gives to familiar facts unfamiliar clothing. Plücker's formulas, for example, are equations between orders of divisors.

Just as there is, in the chapters on geometry, an adequate transition to the geometric theory, so we find, in the last two chapters, an introduction to the integral theory. They contain, respectively, a treatment of the analysis situs of Riemann surfaces, and a short discussion of the three kinds of abelian integrals. The writing is clear and well-balanced; yet one feels Jung's realization that his chief contributions to the mathematical reading public had been made earlier in the book.

If I had mentioned the book's chief claim to distinction in any single chapter, I should have needed to repeat it in nearly all. About one-fourth of the space is devoted to examples, carefully chosen and clearly worked out at more or less length. The student who merely checks the results given will have a better knowledge of the real meaning of the theory than one who reads, however carefully, the scholarly book of Hensel and Landsberg. Often there are gaps and suggestions enough to furnish the reader with material for real study; and every example worked out can, of course, suggest other problems to be solved ab initio.

Among the misprints and slips which I have noted, the correction of the following will be of importance to the reader.

PAGE	LINE	FORMULA	FOR	READ
9	30		des in ihm liegenden	eines in ihm liegenden
22	13		keine der anderen Strecken	alle anderen Strecken
49	5		*arg* x	*arg* $(x - a)$
72	5		wo t eine Einheit	wo $E(t)$ eine Einheit
80	8		Kap. VIII	Kap. IX
102		9	$g(x) \pm h(x) y$	$g(x) + h(x) y$
106	7		(22)	(18)
165	3rd row, 2nd matrix, middle of page,		0 1 0 0	0 0 1 0
173		5	$\frac{\partial f}{\partial x} = \frac{\mathfrak{d}\mathfrak{z}_\nu}{\mathfrak{l}^{-2}\mathfrak{m}^m}$	$\frac{\partial f}{\partial x} = \frac{\mathfrak{d}\mathfrak{z}_\nu}{\mathfrak{l}^{l-2}\mathfrak{m}^m}$
186	27		die p_k definierende	die p_1 definierende
191	26		Ebenenkoordinaten	Linienkoordinaten
195		30	$- \mathfrak{a}_3 d\mathfrak{a}_1$	$- \mathfrak{a}_3 d\mathfrak{a}_2$
196		38	$\mathfrak{d}$	—
197		39		$\mathfrak{d}$
211	24		$\lim \frac{\xi}{\eta} = -4$	$\lim \frac{\eta}{\xi} = -4$
235	19		$= \lambda$	$= \lambda + p$
238		15	ρ	p

Jung's book is adequately and clearly illustrated, fully and accurately indexed. His style is good. The prospective foreign reader will be particularly encouraged by the fact that the average sentence is but two lines long. It is to be hoped that this book will have many readers in America, in spite of the obstacles which our educators' continuing slight of the German language puts in the way of our next generation's progress in science. A moral may be learned from the fact that Germans have almost a monopoly of the work published on the arithmetic theory of algebraic functions.

All in all, we can say that Jung has chosen a task well worth doing, and has performed it excellently.

E. S. ALLEN

KLEIN'S COLLECTED PAPERS, VOLUME III

Felix Klein: *Gesammelte mathematische Abhandlungen. Vol.* III: *Elliptische Funktionen, insbesondere Modulfunktionen*; *Hyperelliptische und Abelsche Funktionen*; *Riemannsche Funktionentheorie und automorphe Funktionen.* Edited by R. Fricke, H. Vermeil, and E. Bessel-Hagen. Berlin, Julius Springer, 1923. IX + 774 + 36 pp.

The editors of the third volume are three in number; those of the second having continued, and been joined by the labors of a third collaborator. The plan of the preceding volumes has been carefully preserved. To give some idea of the care with which the work of the editors has been done, it may be mentioned that every formula has been rederived, and in cases of any discrepancies in notation or in result, these are fully explained in appropriate footnotes. The volumes are practically free of typographical errors.

The memoirs in the first part, mentioned under the first subtitle, are fourteen in number, being memoirs LXXXI to XCIV inclusive. Of these, the first eight were written during the period 1877–1880, while the author was in Munich; the next four, one from 1881, the other three in 1885, were written in Leipzig, and the last two, 1893, 1896, in Gottingen. This first part occupies 314 pages.

When Klein went to the technical school in Munich from Erlangen, in 1875, his main interest was in the use of elliptic functions in the solution of the quintic equation; in particular, what part the theory of the icosahedron could play in such solution. His courses in the technical school included a repeated cycle in the theory of numbers, elliptic functions, and algebraic equations. In the seminary he had such men as Pick, Gierster, Dyck, Fricke, Hurwitz, and for one semester, Bianchi. Not only did these men contribute material assistance at the time, but except the first and last, wrote their dissertations under Klein's direction, and have very considerably enriched the literature by their own contributions.

The first memoir is an extract of a letter, in Italian, to Professor Brioschi, showing how certain formulas used by Klein could be reduced to those previously used by Kronecker, Brioschi and Hermite. The second, on the transformation of elliptic functions and the solution of the quintic equation, contains the systematic development of the method, and the result, in abstract form, that the general solution of the quintic equation can be expressed in terms of elliptic functions. The author apologizes that his paper contains so much that was already known, but it was necessary to reproduce much of the known theory, to show the relations between the various parts, and in particular the meaning for his method of procedure. The result is a systematic presentation which serves splendidly to introduce the reader to the whole Klein scheme of the theory of functions, in particular the conformal mapping, extended from Schwarz's paper on the hypergeometric series.

The next paper considers the reduction of the modular equation by appropriate transformations. In this paper it appears that transformations of elliptic functions of orders $n = 5, 7, 11$ offer particularly promising results. The details for $n = 5$ are included in the paper itself. Besides the modular equation of order 6 and its known quintic resolvent, we also have, as Galois resolvent of the latter, the icosahedron equation of order 60.

The next paper, transformations of order seven of elliptic functions, brings the case $n = 7$ to the same degree of completion. The Galois resolvent is now of order 168; its roots have the property of remaining invariant under the linear fractional substitutions of the ratio η of the periods of the elliptic function, the coefficients of which are congruent to the identity, modulo 7, and only these. The next problem is to determine the relation between this root and the absolute invariant. We know that the genus of this equation is 3, and that it cannot be hyperelliptic. Hence the equation can be reduced to represent a non-singular plane quartic curve which remains invariant under a birational group of order 168. Moreover, the p adjoints of order $n-3$ go over into the p adjoints of order $n'-3$ by every birational transformation which sends the given curve of order n into one of order n'; hence in our case, the operations of the group are linear. But from Jordan's list of finite groups of ternary linear substitutions it appeared that no such group existed. Either Klein's premises were wrong, or Jordan had made an error. Subsequent consideration established the existence of the group, which should therefore be added to Jordan's list.*

From the singular values of the crossratio, in the linear fractional substitutions of η, it appears that our curve can have but three sets of points which have fewer than 168 conjugate positions; these exceptions appear by sets of 2, or 3, or 7. The first are the sextactic points, the second the points of tangency of the bitangents, and the last are the points of inflexion. But there is only one set of 24 points, hence the residual intersection of each inflexional tangent with the curve must be another point of inflexion. By choosing three associated inflexional tangents for the sides of the triangle of reference, the equation of the curve is readily determined. The arrangements of the double tangents, the points of inflexion and sextactic points complete this part of the problem. The later sections of the paper are devoted to the construction of the Riemann surface, and of the curve itself, the above coordinate system being an equilateral triangle.

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* C. Jordan, *Mémoire sur les équations différentielles lineaires à intégrale algébrique,* CRELLE'S JOURNAL, vol. 84 (1878), pp. 89–215. The error is on page 167, line 8 from below. The expression Ω should be divided by 3φ instead of 9φ.

This paper is a gem of mathematical writing. Although it contains a wealth of new; and many unexpected results, the thought is so developed that each is derived directly and naturally, and each step suggests how the next should be taken. Instead of being lost in a maze of formulas — particularly likely to be the case in this branch of mathematics, — all the results are obtained with a minimum of manipulation, and by steps that leave nothing to be desired as to rigor or conclusiveness. The most striking feature is the ready and easy use of so many different branches of mathematics; here we find projective geometry, binary and ternary invariant theory, theory of groups, elliptic functions, conformal mapping, differential equations, algebraic curves — brought into service, each contributing its part in the general problem. This is everywhere characteristic of Klein's writings, but perhaps it reached its highest manifestation in this and the following papers. The case $n = 11$ was treated immediately, but the generalization to n any odd prime appeared several years later, after the author had been intensely occupied with more general questions.

Each of these memoirs is followed by a few pages of comments, largely concerned with subsequent studies by other authors. These are followed by two general reports on the theory of elliptic functions and modular functions, written near the end of the Leipzig professorship, then follows a short note on the composition of binary quadratic forms, and a longer outline of the autographed notes on the course on the theory of numbers given at Gottingen in 1895–96. It is shown that by proper interpretation of certain non-euclidean concepts of measurement the definite and the indefinite forms can be treated in the same way. Later the forms with negative discriminant are applied to elliptic functions.

The second part consists of three memoirs, two on hyperelliptic functions, and the third on abelian functions. The latter has two rather distinct parts, the first being general, and the second, which carries certain considerations much further, is restricted to $p = 3$. All these memoirs were written in Gottingen, 1886–1889. Beginning with the summer of 1887, Klein lectured two semesters on hyperelliptic functions, and the three following ones on abelian functions. During the last two years at Leipzig Klein had directed seven doctor's theses in this field, among them Fricke, and during the first few years at Gottingen as many more. These assisted materially in providing confirmations of details which Klein felt were correct, but which he had not taken the time or trouble to verify himself. Even then a great number of formulas were left undeveloped, and many numerical factors unprovided, which were supplied later by Burkhardt for the hyperelliptic case in two splendid papers.*

In the case of the abelian functions, the author mentions that he received the greatest assistance from the two students W. F. Osgood and H. S. White. The purpose of these papers is to bring the theory of hyperelliptic and of abelian functions into systematic and natural relation with the theory of invariants. In order to make this possible, the number of homogeneous variables could not be prescribed, and the domain of rationality had to be defined in each case.

The main result in the first paper is a systematic treatment of the theta functions, in particular for $p = 2$, and to determine the theta constants (for zero arguments) as modular functions. The associated hyperelliptic curves can be expressed in a definite canonical form, which suggests and makes possible a uniform treatment. No corresponding form exists for the non-hyperelliptic curves, and a general treatment is probably not possible. In the former, Klein developed the entire theory of the sigma functions; for abelian functions this was possible only for $p = 3$. The general form of the associated plane quartic curve is presupposed, and all the steps are taken in terms of invariants and covariants, rather than as applicable only to a particular system.

In the second part, a great deal of the paper is taken up with the systems of contact lines (double tangents), conics and cubics, and the necessary modifications when the curve has a double point. While this work is very interesting, and perhaps of considerable value in the study of the theta functions, and their geometric meanings, practically all these theorems on the quartic curve can be established directly from the study of the (1, 2) correspondence associated with the Geiser involutorial transformation. An interesting application is to hyperelliptic surfaces of genus 3, as there are 36 non-equivalent systems of canonical sections that make the surface simply connected. The author mentions that he was indebted to H. D. Thompson for appropriate graphical schemes to show the relations between these various systems.

The third division of the volume occupies about three hundred pages.

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* *Beiträge zur Theorie der hyperelliptischen Sigmafunktionen*, MATHEMATISCHE ANNALEN, vol. 32 (1888), pp. 381–442, and *Grundzüge einer allgemeinen Systematik der hyperelliptischen Funktionen erster Ordnung. Nach Vorlesungen von F. Klein*, MATHEMATISCHE ANNALEN, vol. 35 (1889), pp. 198–296. The later publications in volumes 36, 38, 41 of the same periodical represent further developments of Burkhardt's own ideas.

Chronologically, it precedes the one just mentioned, as practically all of it was written while Klein was at Leipzig. We are told in the introduction that Klein regards this as the most productive period of his activity, as to quantity and as to importance. The beautiful results obtained in the study of the transformation of elliptic functions led the author to a more ambitious program of an exhaustive study of the groups of linear fractional substitutions of one variable, and then of the functions which remained invariant under such substitutions. The method was to be that of Riemann, with a minimum of analysis and a systematic application of physical ideas, strengthened by a highly developed intuition, both of spatial relations and of physical phenomena. Although himself a pupil of Plücker, and later in Gottingen greatly assisted and directed by Clebsch, Klein emphasized throughout his career that he was more influenced by Riemann than by any other source.

At Leipzig he at once began a course on the geometric theory of functions, to extend over several semesters. The lecture notes of the first semester were not published, but several copies were in the hands of students, and the dissertations of Gierster, Hurwitz, Staude, Lange and Weichold were begun. The substance of the second semester's lectures was put into book form and published under the title *Über Riemann's Theorie der algebraischen Funktionen und ihrer Integrale* by Teubner in 1882. It is given in full in this third division. There are no additional comments on the text, but at its end appears a note which calls attention to the similarity of methods here developed and those employed by F. Schottky. In reponse to an inquiry of Klein as to the source of his ideas, Schottky replied that it was the study of the circulation of an incompressible fluid—that is, Riemann's idea. Attention is also called to the fact that in Weyl's book (*Die Idee der Riemannschen Fläche*) most of the concepts used by Klein have been confirmed by the rigid requirements of modern analysis.

An algebraic function was defined by means of its Riemann surface, and the latter was interpreted as any closed surface in space. The rational functions associated with the rotations which leave the regular bodies invariant, trigonometric, and elliptic functions were all well known examples of functions belonging to the group to be studied. The modular functions furnished a large advance, as they belonged to infinite groups in which the coefficients are integers with determinant unity.

While these ideas were rapidly maturing, but before the broad lines of the general theory were published, there appeared, in rapid succession, a number of brilliant notes in the COMPTES RENDUS, 1881, by H. Poincaré, on certain phases of the same subject. These notes prompted Klein to write to Poincaré, primarily to protest against the name (*Sur les fonctions fuchsiennes*) and to call to his attention a considerable number of earlier memoirs in which he (Klein) had developed the same ideas. The name *Fuchsian* had been applied by Poincaré to those functions which have a fundamental circle; the matter was not made more pleasant when he proposed and later used the adjective Kleinéens, to describe the functions which do not have a fundamental circle. Now follow twenty-five letters, all written during the years 1881, 1882. Klein invited Poincaré to prepare a synoptic paper on his work, and proposed that he would also prepare a similar paper. They both appear in volume 19 of the MATHEMATISCHE ANNALEN, that of Klein appearing in the present volume, covering eight pages. It was followed, six months later, by the principal memoir Klein wrote on this subject, *Neue Beiträge zur Riemann'schen Funktionentheorie*, which appeared in the MATHEMATISCHE ANNALEN, vol. 21 (1882-83), and occupies 80 pages. Of these, the last six are occupied with modifications and refinements of the existence theorem that on every given Riemann surface there exists one and but one principal function. The proof is not completed; indeed in various parts of the paper the argument is somewhat sketchy. It was at this time that Klein's health broke down. He suspended work for some time, and when he could resume, it was only on a restricted scale, for several years. Apart from a few notes, this is the last memoir on automorphic functions written over his own name, but in the preface to the treatise on *Automorphe Funktionen*, Fricke explained that he was directed by Klein through most of the great undertaking. Near the end of the present volume appears a critical analysis of the various proofs of this theorem, prepared by Klein and Bessel-Hagen, and a similar criticism of the origin and development of the *Automorphe Funktionen*.

In the appendix of 36 pages a number of lists are collected, as follows: University courses and seminar themes, 1871-1923, complete list of the 48 doctor dissertations written under his direction, together with the time and place of publication of each, list of his assistants, and a list of his 151 memoirs, 5 commemorative address, 3 reviews, 10 books, 10 autographed volumes, 33 reports on the organization of the teaching of mathematics, and his work as editor of the MATHEMATISCHE ANNALEN, the ENCYKLOPADIE, the International Commission, and the works of Plücker, of Mobius, and of Gauss.

These three magnificient volumes, clearly printed on high quality paper, filled with instructive personal notes, generously stocked with references to the work of others, even when cherished ideas of the author were first published by others, all skillfully compiled by patient and sympathetic hands, almost completely free from typographical errors, furnish an almost invaluable commentary on the development of mathematics during the last half century.

VIRGIL SNYDER

BIANCHI'S DIFFERENTIAL GEOMETRY

Lezioni di Geometria Differenziale. Terza edizione interamente refatta in due volumi. Vol. I. By L. Bianchi. Pisa, Spoerri, 1921. IV + 806 pp.*

The first and second editions of this classic, whether in Italian or German, are so well known to all students of differential geometry, or should be, that an extended review of this edition would be an anachronism. When one turns over the pages he is reminded of the earlier editions, but when he compares it with the latter he observes much that is new in the development of the topics formerly treated. And so he decides that hereafter, when he wants to see what Bianchi says about a certain subject (and a student of differential geometry cannot afford not to do so), he will use the new edition.

The first ten chapters of the second editions, both Italian and German, are retained in this volume, but they have been developed to such an extent that they occupy half again as many pages.

The problem of reducing the linear element of a surface to the form $ds^2 = du^2 + 2\cos\omega\, du\, dv + dv^2$ is called the *problem of Tschebychef* by Bianchi, and the parametric curves the *lines of Tschebychef.* This problem was discussed briefly on page 401 of Volume 2 of the old edition, but the new treatment is more extensive. In fact, it is shown that such lines exist on any surface, the degree of arbitrariness being that of two arbitrary functions of one variable each (p. 159). More recently Bianchi† has proved that the tangents to either family of these lines, say $v = \text{const.}$, at points of any curve of the other family are parallel in the sense of Levi-Civita with respect to the latter curve; moreover, this is a characteristic property of the lines of Tschebychef.

In Chapter V it is shown that the Christoffel symbol of the second kind formed with respect to the second fundamental form is the arithmetic mean of the corresponding symbols formed with respect to the linear element of the surface and of its spherical representation. Also there are certain results concerning spaces of constant Riemannian curvature of n dimensions (p. 259).

In 1869 Dini solved the problem of geodesic representation of two surfaces upon one another. In 1896 Levi-Civita extended the problem to spaces of any order. The results of Dini and their application to the determination of pairs of quadrics in this relation have been added to Chapter VI.

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* The first German edition was reviewed by J. K. Whittemore in this BULLETIN, vol. 7 (1901), pp. 431–442, and the second German edition by L. P. Eisenhart in vol. 18 (1912), pp. 411–418.

† BOLLETTINO UNIONE MATEMATICA ITALIANA, vol. I (1922), pp. 11–16.

When two surfaces are applicable, the curves on one corresponding to asymptotic lines on the other have been called *virtual asymptotic* by Bianchi. The differential equations determining such lines on any surface were derived by Darboux, who showed that the problems of finding these lines on a surface S and the surfaces applicable to S are equivalent. This theory is set forth at the close of Chapter VII; it appears at the same place in the second German edition but not in the Italian one.

In his beautiful theory of surfaces applicable to a quadric Bianchi made use of the theorem of Chieffi: If at the point of any asymptotic curve C upon a surface S applicable to a ruled surface R one draws tangents to the geodesics g of S which are deforms of the generators of R, the ruled surface R_1, formed by these tangents is applicable to S with g remaining rigid. Bianchi adds in § 153 an analytic proof of this theorem to the geometric proof given on page 4 of volume 3 and in § 126 of the second German edition.

Twelve years ago Sannia published several papers in the RENDICONTI DI PALERMO dealing with the intrinsic definition of rectilinear congruences by means of two quadratic forms. Bianchi devotes §§ 187 and 188 to this theory and to an application to congruences whose mean evolute is a point.

The general theory of relativity has aroused a general interest in the differential geometry of Riemann spaces of any order. The last half of the first volume of the second Italian edition contains an excellent exposition of this theory. This is omitted from the volume under discussion with the exception of the chapter dealing with "pseudospherical geometry and its non-euclidean interpretation", which appears as Chapter XIV. However, the last half of the second volume will be devoted to general Riemann geometry, and we can expect that it will contain some of the recent additions to this live subject.

Chapter XI is devoted to surfaces with plane or spherical lines of curvature, which appeared formerly as Chapter XXI with some omissions and additions. Chapters XXII and XXIII of the second Italian edition, or XIV and XV of the German edition, dealing with minimal surfaces, appear with little change as Chapters XII and XIII.

Bianchi began his career as a student of pseudospherical surfaces, and the theory of these surfaces and their transformations owes to him its present state of perfection. This subject has always seemed particularly fascinating to him. And now here in the last two chapters of this volume we have his exposition of surfaces of constant curvature based upon the knowledge and experience of a lifetime. They are extremely well done.

Anyone who is familiar with the method of the moving trihedral, introduced by Darboux, and who has appreciated its power in dealing with geometrical problems, must wonder why Bianchi has never made much use of it. He introduced the elements of it in his treatment of Weingarten's method for the study of applicable surfaces (vol. II, p. 176), but seems not to have used it elsewhere. It appears as a note at the close of this volume.

Bianchi is a past-master in the art of writing treatises, not only in the field most closely associated with his name, but in many other fields. And this book reveals him at his best. The reader has no grounds for criticizing the book as to clarity of statement, but the student who wishes to follow up a particular subject may regret the lack of references to many of the sources from which the cream has been taken.

L. P. EISENHART

THE IMUK REPORTS AND LOREY ON INSTRUCTION IN GERMAN UNIVERSITIES

Das Studium der Mathematik an den deutschen Universitäten seit Anfang des 19. Jahrhunderts. Von W. Lorey. (Abhandlungen über den Mathematischen Unterricht in Deutschland veranlaßt durch die Internationale Mathematische Unterrichtskommission.* Band III, Heft 9). Leipzig, Teubner, 1916. 12 + 431 pages and 4 plates.

During the deliberations of the fourth International Congress of Mathematicians at Rome in 1908, steps were taken to organize an International Commission on the Teaching of Mathematics, the members of which were to prepare or procure reports on the methods and materials of mathematical instruction in different countries. Most of these reports were ready when the fifth International Mathematical Congress convened at the University of Cambridge in 1912, but several more have appeared since then. At this writing, 18 countries have published 294 reports containing over 13,500 pages. Germany has issued 53 reports with a total of 5571 pages; about one-fourth of this space is required by the United States for its 18 reports, and about one-sixth of the same space by each of the following countries: Austria for 13 reports; Great Britain for 39; Switzerland for 13; and Japan for 16 reports in two volumes. The reports of France cover nearly 700 pages. Of more modest dimensions are, in order of size, the reports from Belgium, Russia (including Finland),

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* A new word *IMUK* has been coined and is used in German writings as an abbreviation for these last three words.

Hungary, Italy, Sweden, Spain, Holland, Denmark, Australia, Argentine, and Roumania (one report of 16 pages).

The German reports have been published under the general heading: Schriften des deutschen Unterausschusses der IMUK. They include Abhandlungen, edited by F. Klein, and Berichte und Mitteilungen, edited by W. Lietzmann, of which the first series appeared in 12 numbers during the years 1909–1916 and extends to 356 pages. Two numbers of the second series are the reports on mathematical instruction in Denmark (1915, 60 pp.) and England (1915, 211 pp.).* The third and last volume (1917, 115 pp.) of this second series contains a complete index to both the Berichte und Mitteilungen and the Abhandlungen.

The Abhandlungen (38) by 33 different authors (1909–1916), and containing over 4800 pages, are now complete and occupy five (in reality nine)† volumes. The general titles of these volumes are as follows: 1. Die hoheren Schulen in Norddeutschland; 2. Die hoheren Schulen in Sud- und Mitteldeutschland; 3. Einzelfragen des hoheren mathematischen Unterrichts; 4. Die Mathematik an den technischen Schulen; 5. Der mathematische Elementarunterricht und die Mathematik an den Lehrerbildungsanstalten. The imposing Abhandlung under review is the last number of the third volume and it was the last one of all the Abhandlungen to be published‡ (preface dated "6. Juli 1916"). It is more than twice as large as any of the others.

For the American mathematician this volume will probably be by far the most interesting of the Abhandlungen. As in nearly all the other reports, the historical side is heavily emphasized; but here movements and developments are associated with institutions and men whose names are more or less familiar. Moreover the biographical outlines (some quite complete) provided in the case of every mathematician mentioned, are occasionally entertainingly supplemented by extracts from published sketches,§ and reminiscences especially prepared for this volume. Only two days before his death Heinrich Weber contributed an account of his student days.

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* This volume is decidedly useful for filling up many lacunae of the reports from Great Britain which are confined almost wholly to an account of conditions in England.

† Each of three Bande is divided into two Teile, and one of the Teile has two Abteilungen.

‡ A number of the countries have made no report on mathematical instruction in their universities, except in connection with account of the training of secondary school teachers.

§ The bibliographical references are numerous and valuable; much material rarely met with has been brought to light and placed conveniently for reference.

Interesting facts are given concerning eminent mathematicians, other than German, who influenced German mathematics; for example, Abel, Euler, Lie, Lagrange, and Steiner. Regarding such a man as Gauss, the formost German mathematician of the nineteenth century, there is very little, since the volume aims to give an account of mathematical instruction in the universities, and not a history of the development of mathematics as a science. Gauss was director of the observatory and professor of astronomy, not mathematics, at the University of Góttingen, from 1807 to the time of his death in 1855, and he "hat immer nur vor einem kleineren Kreise von Zuhorern und nie mit großer Freude gelesen."

The first chapter of 22 pages contains general information about the 22 German universities. Six of these were founded in the fifteenth century, the one at Leipzig, established in 1409, being the oldest. Four universities were founded in the nineteenth century, while that at Frankfort-on-Main was opened in the autumn of 1914. Just half of the universities are in Prussia.

There are sections on the faculties, the teaching force, and the students. Of the seven classes of teachers comprised in the 3724 professors and instructors for the wintersemester of 1914–15, about 3400 were ordinary and extraordinary professors and privatdozenten. The first and third of these groups were almost numerically equal and constituted about four-fifths of the whole. In mathematics there were 49 ordinary and 19 extraordinary professors and 22 privatdozenten. Of these 90 teachers Gottingen had 9, Munich 7, Strasbourg 6, and so on down to Rostock with 2.

The section on students contains two charts; one indicates the number of candidates for the Prussian Staatsexamen, with mathematics and natural science as major, between 1839 and 1913. From about 20 in 1839 there was a fairly constant increase to nearly 160 in 1882; this number dropped to less than 25 by 1893, but increased almost continuously to over 325 by 1913. The second chart shows the number of students of mathematics and natural science at the University of Gottingen from 1868 to the summer of 1913. The numbers increased from about 60 in 1870 to almost 240 in 1881, dropping to less than 80 in 1892 but increasing to over 750 by 1913. The intimate relation between the number of university students and the number of those taking the Staatsexamen is noticeable.

The rest of Lorey's work is divided into five chapters, 2–6. The second (26 pages) deals with the state of mathematical instruction in the universities at the beginning of the nineteenth century, and with vain-trials of methodically arranged new organization in the years 1818–1848. The third chapter (pages 59–111) is entitled: "The development of mathematical instruction in the universities up to the founding of the German Empire." The fourth chapter (pages 111-141) discusses "Special achievements in the period before 1870 with glances at the present time." The fifth chapter (pages 142-211) deals with "The period from 1870 to 1890. From the founding of the Empire to the beginning of the school reform movement." The sixth chapter (pages 211-402) is entitled: "Recent times (from 1890-1914)." Pages 403-8 are filled with "Corrections and additions" and pages 409-28 with a "Personen-Verzeichnis".

Although lectures in mathematics were given in German universities in the latter part of the eighteenth century, they were not universally regarded as feasible even by those who flourished in the early nineteenth century. A notable case is that of Mollweide (a friend of Gauss who characterized him as "ein gründlicher Mathematiker und trefflicher Mensch"), who did not think that higher mathematics could be properly dealt with in lectures "schon weil es dabei zuviel Schreibens an der Tafel gabe!"

In the early part of the nineteenth century the universities at Konigsberg (founded in 1544), Berlin (1810), and Gottingen (1737) were the three chief centers of mathematical inspiration. The first named university, especially, under the guidance of such notables as K. G. J. Jacobi, Bessel, and Franz Neumann, led to a well defined Konigsberg school whose influence throughout Germany was great. For the 17 years 1826-42 Jacobi was constantly giving in his lectures results developed by prolific research. With Neumann he founded in 1834 a mathemathics-physics seminar, concerning which full and interesting details are given in the section on seminars in Chapter 4. About 1843 Jacobi moved to Berlin where he lived as a royal pensioner until his death in 1851. Jacobi's first pupil Richelot, a native of Konigsberg, became a professor in the university and expounded the researches of his teacher and of Steiner. L. A. Sohncke and Otto Hesse were also natives of Konigsberg. While the former served only two years as dozent in the university, the latter taught there as dozent and professor for 15 years. The influence of the Konigsberg school was later extended when Hesse, Gustav Kirchhoff, H. von Helmholtz, and Bunsen were teaching together at Heidelberg University which "in ihrer hochsten Blute stand". It was at this time that Heinrich Weber was a student, and his interesting reminiscences of the period have been preserved by Lorey.

Weber's early training doubtless influenced him in accepting the call to the University of Konigsberg where he was professor for the ten years 1873-83.

In Berlin by 1835, Crelle (1780-1855), Lejeune Dirichlet (1805-1859), Dirksen (1792-1850), a pupil of Gauss, and Steiner (1796-1863) were prominent mathematical members of the Academy of Sciences. A little later Jacobi was also there. It was under Dirksen at the University in 1825 that Jacobi, when only twenty years of age, passed a brilliant doctor's examination. Although Steiner had studied no mathematics and could scarcely even write at the age of 16 years, he rose rapidly into prominence by his publications during the years 1825-34 that he was teacher in the Gewerbeschule at Berlin. In 1832 on the recommendation of Bessel and Jacobi the philosophical faculty of Konigsberg university conferred the honorary degree of doctor on Steiner. He was elected a member of the Berlin Academy in 1833, and became professor at the University of Berlin in 1834. He held this position until his death.

Of the three leaders (Dirichlet, Steiner, and Jacobi) of the first mathematical school at Berlin, Steiner undoubtedly exercised the greatest influence on his students and on geometrical instruction in schools. A great mass of ideas and propositions discovered or employed by Steiner penetrated into the teaching quite quickly, such as the theory of harmonic points and pencils, of the potency of a point, of pole and polar, the properties of the complete quadrilateral, and Pascal's Theorem.

Dirichlet was born in Aix-la-Chapelle. When 16 years of age, after finishing his secondary school education at Cologne, he went to Paris, where he studied mathematics and taught for six years. Through Fourier he met Alexander von Humboldt, whose influence with the Ministry* resulted in his appointment as dozent in the University of Breslau in 1827. In 1829 he was called to the University of Berlin where he finally became ordinary professor in 1839. He remained there until about four years before his death, when he became Gauss's successor at Gottingen.

Crelle was not a professor in the university but a "Geheimer Baurat" who was, for example, responsible for the plans of the railway between Berlin and Potsdam. His numerous mathematical publications were not of the first rank, and his significance for mathematics lay in his energetic organizing activity. With Gergonne's ANNALES DE MATHÉMATIQUES PURES ET APPLIQUÉES as a model, he founded the JOURNAL FUR DIE REINE UND ANGEWANDTE MATHEMATIK in 1826. Success in the undertaking was partly assured through an arrangement which he made with the Kultusministerium (ministry of ecclesiastical affairs and public instruction) whereby he was allowed to add to the title of the JOURNAL the words to be found even in recent volumes: "Mit thatiger Beforderung† hoher Koniglich Preussischer Behorden." This Beforderung consisted on the one hand in issuing strong official recommendation of the JOURNAL, not only to universities and higher schools, but also, for example, to government boards and, through Prussian ambassadors, in foreign countries. On the other hand the Beforderung involved the purchase of a number of copies of the JOURNAL which were distributed to various schools—a custom prevailing until the great war. In this way, Weierstrass, for example, while a gymnasium pupil, received inspiration by discovering an uncut copy of the JOURNAL "mit den schonen Abhandlungen von Steiner, von denen auch ein Primaner etwas verstehen konnte."

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* Although Alexander von Humboldt (1709-1859) was not a mathematician his name comes up several times in the Abhandlung as that of one very especially interested in the subject and wielding his powerful influence for the promotion of its study.

† For "thatiger Beforderung" Lorey has "Unterstutzung".

But in connection with the JOURNAL, the important thing was that Crelle's faith in young collaborators* was not misplaced. "Darin liegt gerade das unvergängliche Verdienst des damals schon bald funfzigjahrigen Crelle, daß er die jungen mathematischen Forscher, die in den zwanziger Jahren des vorigen Jahrhunderts in Deutschland auf einmal hervortraten, an sich heranzog und ihnen in seinem JOURNAL eine leichte Publikationsmoglichkeit verschaffte."

These indications as to the contents of the volume will suffice for suggesting its general character. It is an invaluable work of reference on the subject of which it treats.

R. C. ARCHIBALD

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* The first volume contained seven memoirs by Abel, five by Steiner, and one by Jacobi.

In spite of the title of the JOURNAL very few articles on applied mathematics were ever published in it. In 1829 Crelle founded another periodical JOURNAL FÜR BAUKUNST, which existed until 1851.

SHORTER NOTICES

Tractatus Logico-Philosophicus. By Ludwig Wittgenstein. With an introduction by Bertrand Russell. New York, Harcourt, Brace and Company, 1922. 189 pp.

The final number of Ostwald's ANNALEN DER NATURPHILOSOPHIE (1921) contains an article by Mr. Wittgenstein, a former pupil of Mr. Bertrand Russell, dealing with the nature of logic and with its relations to mathematics, philosophy, and natural science in a manner so original and profound as to make its publication an important event. The book in hand presents that essay in the original German, along with an English rendering of it under the editorship of Mr. C. K. Ogden of Magdalen College, Cambridge. The original and the translation are printed side by side, facing each other, and that is well, for the original contains many sentences that are not sufficiently clear to admit of quite confident translation.

Of the book's 189 pages 23 are occupied by Mr. Russell's introduction, the remainder being equally divided between the German version and the English one. So it is seen that the work proper is physically very small—only 83 pages. But it is far from small scientifically. Not only does it present "a theory of logic which," in Mr. Russell's opinion, "is not at any point obviously wrong" (notwithstanding it rejects as unsound some of the tenets hitherto held by the English logician) but it contrives to deal in a fundamental way with other great matters.

How can so small a work be so big? What is the art involved? The answer is found in a variety of considerations.

One of them is that Mr. Wittgenstein's thinking is confined to fundamentals. His book is addressed to none but the most seasoned of thinkers. The author will be content, he tells us, if only one person reads his book with understanding and pleasure.

Again, there is no index, no table of contents, no division into chapters, no bibliography, no specific acknowledgement of indebtedness to others save that of having been stimulated by "the great works of Frege and the writings of my friend, Bertrand Russell"; there is hardly any comparative criticism, setting the author's thought in relation with the thought of others, whether past or contemporaneous; and no clear indication of such parts of his work as he may deem to be new or original, "because," says he, "it is indifferent to me whether what I have thought has already been thought before me by another."

But the chief secret of his being able to deal effectively with so many great matters in so brief a space, is to be found in the temperamental quality of his style. Wittgenstein is a mystic—a logical mystic—and like the great ones of that kind (Spinoza, for example, or Blaise Pascal), he is at once a slave of the propensity for condensation and master of the art. One may say of his style what Porphyry said of the style of Plotinus: "Dense with thought, more lavish of ideas than of words."

Wittgenstein's style is not admirable. His book is not an exposition; it is rather a conglomeration of insights, often profound, intimately related, wide-ranging, fit material for a magnificent structure, but they are not so ordered and presented as to constitute a luminous whole. In order to understand the book it is necessary to read it again and again, forward and backward, up and down, in and out. Even then, despite Mr. Russell's somewhat helpful introduction, some passages remain ambiguous, indeterminate, obscure; not because the subject is difficult, which it is, but because the author has not taken sufficient pains to be clear. Mr. Russell tells us that Mr. Wittgenstein's theory of logic "is not at any point obviously wrong." But upon the score of obviousness, he might have said with equal justness that the theory is not at any point obviously right. Mr. Wittgenstein deserves to be thanked for producing a book that every mathematical or philosophical logician must read, and to be at the same time reprimanded for allowing his lust for mystic condensation so to obscure his thought as to burden and sometimes to irritate the reader. Such a reprimand is not unjust, for it is of the very essence of the author's teaching that "everything that can be thought at all can be thought clearly" and "everything that can be said can be said clearly."

The central aim of the book is to answer a very important and very difficult question that has seldom been asked: What must be the essential nature of a logically perfect language? To present Wittgenstein's answer fully and clearly would require, as already said, more space than he himself has devoted to it. Yet even a brief review may give a few hints, serving perhaps to orient and stimulate the reader.

Consider the following random propositions: (1) the canary bird in the corner of this room has black eyes and a golden throat; (2) the specific gravity of mercury exceeds that of gold; (3) the velocity of light is greater than any other velocity. The propositions (whether true or false) are "symbols" representing "possible" facts. The facts, which are not propositions, "exist" or do not "exist" according as the propositions are true or false. If a fact, like that symbolized by proposition (1), is composed of two or more facts, it and the corresponding proposition are "molecular;" otherwise, "atomic." Even an atomic fact contains parts, for the fact is a relation among things (objects) and these are its parts. So, too, an atomic proposition has parts, for it is a relation among names (symbols for objects) and these names are its parts. Now, in a logically perfect language, each proposition has a unique and definite meaning, but the meaning of a proposition is determined by the meanings of its parts (the names in it). A name denoting a "complex" object can have such a meaning only when the object has been completely analyzed; but complete analysis is possible, theoretically possible, only upon the assumption that a complex object is composed of "simples." So it is seen that the concept of a logically perfect language involves the hypothesis that every complex object is composed of simples and involves the requirement that each of the simples in the world shall have one and but one name, no name (of a simple) shall denote more than one simple (and of course no complex). Between all other names and the complexes in the world there must be a unique and reciprocal correspondence.

Language is composed of symbols: names symbolize objects, simple or complex; propositions (true or false) symbolize possible facts (existent or non-existent). The role of propositions is that of depiction (Abbildung), representation by means of pictures. How does a proposition symbolize—picture, depict—the fact asserted by it? In the philosophy of logic the importance of that question is fundamental, and Wittgenstein's answer is one of the gravest theses in his book. His answer is, in very brief, substantially as follows: A fact, being a relation among objects, has a certain structure, or form; a proposition (asserting the fact) also has a certain structure, or form, for it, too, is a relation—a relation among the names occurring in it and symbolizing the objects in the facts. Now, says Wittgenstein, the structure or form of the fact and the structure or form of the proposition (asserting the fact) are identical, and that is why the proposition is a picture of the fact. Moreover—and here we touch the nerve of the author's mysticism—a proposition cannot "express" its own structure or form, but can only "exhibit" it: the structure cannot be said, it can only be shown. The inexpressible is the mystical.

Wittgenstein's theory of logical inference is profound and beautiful. Space is lacking to set it forth here, for to do so would require a careful preliminary explanation of certain technical terms. It must be said, however, that the theory in question discriminates sharply between propositions of logic (including mathematics) and all other propositions. The former are true unconditionally, and are known to be true by inspecting them. All other true propositions are only conditionally true, and their truth cannot be recognized by inspecting them. Propositions of logic (including mathematics) are silent about the empirical realities of life and the world. But they are absolutely essential in the process of inferring from propositions that do relate to such realities to other propositions relating to them.

C. J. KEYSER

Was ist Mathematik? Unterhaltungen während einer Seereise. By Lothar Heffter. Freiburg i. Br., Theodor Fischer, 1922. 160 pp.

Les Mathématiques. By Pierre Boutroux. Paris, Albin Michel, 1922. 183 pp.

Problems of Modern Science. Edited by Arthur Dendy. *Mathematics.* By J. W. Nicholson. New York, Henry Holt and Company, 1922. 237 pp.

Judging by the number of books that are appearing which devote themselves to the popular exposition of various branches of science, it would appear that the general public is becoming increasingly interested in the nature of scientific progress. Or is it only that publishers are beginning to realize the existence of this interest and are gradually becoming bold enough to meet the resulting demand? Of the books listed above the first two endeavor to give a popular account of mathematics as a whole, they are attempts to answer for the layman the question he so often asks: "What is mathematics all about, anyway? I know a little something about elementary algebra and plane geometry. But you people (meaning the mathematicians) talk about 'mathematical research', you use a lot of technical terms and a mass of mysterious looking symbols that nobody else can understand. Can't you tell something about it in terms that are intelligible to us?" The two books referred to offer replies to these questions in very different ways. The first is by far the less formal and is concerned primarily with ideas; the second is more formal and is primarily concerned with processes. The former can be more easily read, the latter is more adapted to study. Both cover a wide field.

The little book by Heffter is in the form of a dialog between a mathematician and a merchant, is delightfully written, and admirably accomplishes its purpose within the limits which the author has imposed on himself. "In the beginning was number" is the title of the third chapter (the first two are introductory) and here the author speaks genially of the concept of whole number (positive and negative), calls attention to the fact that there exist an infinitude of such numbers although each one is finite, illustrates the commandment "Thou shalt not divide by zero," discusses prime and composite numbers, congruence and quadratic residues and diophantine equations, not neglecting Fermat's last theorem and the Wolfskehl prize. The next chapter deals with the successive extensions of the concept of number up to and including the introduction of complex numbers, not omitting the Dedekind cut, the concept of transcendental numbers and the squaring of the circle, the fundamental theorem of algebra, and the distinction between the existence of a root of an algebraic equation and its algebraic solvability. In Chapter V, the story of Achilles and the tortoise serves to introduce the notion of infinite series and products and continued fractions.

Chapter VI on "Connected variable numbers" introduces the function concept and the graph (including periodic functions and the depiction of one plane on another) and Chapter VII under the suggestive title of "Snapshot and moving picture" brings the concepts of derivative and integral and the fundamental theorem of the integral calculus, going so far as to introduce the notion of curvature and of differential and integral equations. The next three chapters are devoted to geometry, mainly projective (parallel and central projection, photograph, double ratio, points at infinity, duality, projective transformations and the group concept); but the fundamental notions of analytic geometry, imaginary elements, theory of surfaces (surfaces of constant curvature, non-euclidean geometry, minimal surfaces, lines of curvature and geodesics) are also discussed. In Chapter XI the fairy tale of the king and his five sons serves to introduce a problem in analysis situs and we are give a glimpse of the four-color problem, one-sided surfaces, and of concepts of genus and connectivity. The next chapter treats briefly of the classical mechanics and the last of the theory of relativity. It will be surprising to many that such a wide domain of advanced mathematical doctrine should be capable of informal and yet intelligible exposition to the layman, and yet it has been done and done extremely well. It would be highly desirable if this little book could be made available in an English translation; but would any American publisher be bold enough to publish it?

Boutroux's little book must be treated more briefly, not because it is less meritorious but because it will have a narrower appeal. As has already been suggested, it is more abstract in its treatment and is less easily digested by the lay reader. The author was himself well aware of this characteristic of his work and was apparently quite deliberate in bringing it about. In a ten page preface he discusses at length the difficulties in the way giving a popular exposition of mathematics and reaches the conclusion that in order to give a true picture of mathematics as it is the treatment must necessarily be rather severe. "It is desirable," he says, "that a book treating of mathematics be a bit abstract. If it demands more thought, the impression which it leaves will doubtless be more in conformity with truth." The character of the discussion is foreshadowed by this attitude on the part of the author. The chapter headings are as follows: I, Properties of Numbers; II, Properties of Figures; III, Mathematical Demonstration; IV, Algebraic Computation; V, Analytic Geometry; Functions and Derivatives; VI, Differential Equations; VII, The Theory of Functions. It will be seen that the emphasis is almost exclusively on analysis; there is no treatment of projective geometry or of analysis situs. The book would seem to be admirably adapted to the interested layman, who having read Heffter's little book, desires to go a bit further.

Professor Nicholson's article of thirty-seven pages in the last of the books under review has a different purpose than the other two, although it too is addressed to the layman. It forms one of a series of lectures delivered at King's College of the University of London on problems of modern science and attempts to present to a general audience an account of present activities in mathematical research with special emphasis on those which are related to other sciences. Any one faced with such a task has a perfect right to make his own selection of topics to be presented, and a reviewer has little right to criticise such selection. The present reviewer has no disposition to do so, beyond observing that other men would doubtless have made other selections. As is natural, the problems in vogue in England receive the greater part of the lecturer's attention. We thus hear of the theory of partitions, and of Waring's problem, and of related topics in the theory of numbers. A strange error has crept into the account of Fermat's theorem, when the founding of the Wolfskehl prize is located at Vienna. Dubious also is the statement (p. 19) that Fermat's theorem is one "towards the solution of which an extensive mathematical training is of little help." The last sixteen pages are devoted to some mathematical considerations connected with the theory of relativity and Planck's quantum theory. The lecture is readable throughout. It may be doubted, however, if a general educated audience would carry away more than a very vague notion of what it was all about, especially concerning the latter part of the lecture. More than that ought probably not to be expected from such a lecture, however, and even so, who would be bold enough to deny that the giving of such lectures and their publication is worth while? If they serve to arouse interest and give their audience even a vague notion of the beauty and significance of scientific progress they have accomplished a noble purpose.

J. W. YOUNG

Principes et Premiers Développements de Géométrie Générale Synthétique Moderne. By Emile Bally. Paris, Gauthier-Villars, 1922. VIII + 218 pp.

The preface to this rather ambitious work contains some amusing statements which may be considered as characteristic of the treatise under review. The author says: "This book is dedicated to the friends of geometry. For many among these, simple amateurs as they are, the ingenious geometric deductions are agreeable relishes which they taste without any afterthought and without bothering themselves with the elements of which they may be composed. Others, more distrustful, the *critics*, wish to know what one has put into the relishes; some of these, the *purists*, cannot tolerate certain ingredients, and declare execrable every composition which contains a trace of those ingredients. Although endeavoring to please the first category, (and this book is above all the work of an amateur addressed to amateurs) we have tried to take account of the reasonable exigencies of the critics and purists without pretending to give them throughout full satisfaction; a purist is never satisfied."

The author thinks that his treatise can be understood by every intelligent person even without any mathematical knowledge. He does not give any references and seems to excuse this lack by the fact that the only books at his command are the works of Darboux (*Classe Remarquable de Courbes et de Surfaces*), of Dumont (*Surfaces Cubiques*), of Duporcq (*Géométrie Moderne*), and the ENCYCLOPÉDIE DES SCIENCES MATHÉMATIQUES. He also suspects that here and there a proposition supposed to be new may have been discovered or demonstrated before, but leaves the decision as to priority to those who are better versed in the subject.

In a preliminary chapter, M. Bally treats of ordinal arithmetic, then follows Chapter I on "general notions on geometry", in which he includes the fundamental ideas of higher spaces by introducing some new terms, like *polinarity* for the number of elements necessary to determine geometric forms.

On account of the high cost of printing, Chapters II–XI have not been published in the present volume. It concludes with three chapters XII, XIII, XIV on the hexangle, Pascals hexagramme, and related configurations; an appendix on Chapter XIII, and finally with additions and corrections to the previous work of the author on synthetic geometry on unicursal curves of the third class and the fourth order.

The author shows considerable mathematical ability, but he is seriously hampered by his unfamiliarity with the current literature and the present tendencies of geometric research.

ARNOLD EMCH

Darstellende Geometrie, vol. II. By Theodor Schmid. Second edition. Berlin and Leipzig, Walter de Gruyter (Sammlung Schubert, LXVI). 1923. 340 pp.

Although the work of Professor Schmid was to occupy three volumes, three editions of the first and now a second edition of the second volume have appeared before the third volume could be completed. The first edition of volume 2 was published in 1921, and reviewed in this BULLETIN (vol. 28 (1922), pp. 68, 69). The present one differs only slightly from it, and the remarks there made still apply. Three articles have been lengthened; one discusses a nodal normal section of a tubular surface, another amplifies the anaxomatic representation of the helix, and the third enlarges on the projections used in geographic maps. A section has been added on graphical tables and nomography, together with a good bibliography on this subject.

VIRGIL SNYDER

Die mechanischen Beweise fur die Bewegung der Erde. By R. Grammel. Berlin, Julius Springer, 1922. IV + 71 pp.

The author states in the preface that his reason for writing the book is to make the material accessible to students who have only an elementary knowledge of mathematics and mechanics. His plan is to describe the experiment, state the law of the motion mathematically, and put the derivation of the formulas in small print. The experiments are grouped into (a) those depending on the principle of the motion of the centroid, (b) those depending on the principle of angular momentum.

Such subjects as Foucault's pendulum experiment, the gyroscopic compass, and the effect of the earth's rotation on a projectile and also on a freely falling particle are familiar to students who have had a course in mechanics. Nevertheless a systematic presentation which gives numerous references and also some experiments not so generally known, makes interesting reading. The material is well arranged and the figures are good.

In connection with Fig. 16, which gives the trajectories of two projectiles fired in vacuo with equal velocity and elevation, one toward the east and the other toward the west, one would naturally wonder if numerical data were used in constructing the diagram, or if it is simply a figure drawn with the idea of bringing out certain general properties.

PETER FIELD

Calculus and Probability for Actuarial Students. By Alfred Henry. Published by the authority of the Institute of Actuaries of Great Britain. London, C. and E. Layton, 1922. 152 pp.

The scope of this book is outlined in its introduction.

"Actuarial science is essentially practical in that, whilst it is based on the processes of pure mathematics, the object of the worker must be to produce a numerical result.'

"For this reason it is necessary for considerable prominence to be given, in the curriculum of the actuarial student, to the subject of Finite Differences, and it thus becomes convenient, in the study of those subjects not included under the heading of Algebra, to deal with this part of the syllabus first and, notwithstanding certain theoretical objections, to treat the fundamental propositions of the Differential and the Integral Calculus as being, substantially, special cases of similar propositions in Finite Differences. The subjects enumerated, cover so wide a field that it has been necessary to exercise considerable compression."

A chapter on the theory of probability is also added. This book undoubtedly will be the standard introduction to mathematics necessary for a preparation to actuarial science although it may be found rather difficult reading to many students, as graduated exercises and a fuller discussion of difficulties and of fundamental principles, especially those involving the interrelations between formulas, would clarify the subject. This book coordinates much material that has never been conveniently arranged and should be valuable to students in any field of applied mathematics where interpolation is used or where it is necessary to obtain the approximate summation of the actual data where only typical individual values are known.

J. S. ELSTON

Handbuch der angewandten Mathematik. Herausgegeben von H. E. Timerding. Vol. I. *Praktische Analysis,* von H. von Sanden. Zweite verbesserte Auflage. Leipzig and Berlin, B. G. Teubner, 1923. 195 pp.

The first edition of this book appeared in 1914, and was reviewed in this BULLETIN (vol. 15 (1914–15), pp. 256–259). In the present edition numerous small changes have been made throughout; proofs have been simplified and algebraic work arranged in a manner more easy to follow. A few new sections have been added, in particular: graphical methods now include a discussion of nomography and a larger number of examples, differentiation and integration are derived in connection with formulas for interpolation, expressions for the limit of error in analytic functions are obtained, and a much fuller treatment of numerical integration is added. As in the former edition, both press-work and proof-reading have been well done. Unfortunately the quality of paper used is such that the page has a less pleasing appearance than in the first edition.

VIRGIL SNYDER

Life Contingencies. By E. F. Spurgeon. Published by the authority of the Institute of Actuaries. London, C. and E. Layton, 1922. XXVI + 477 pp.

This book replaces the old Institute of Actuaries Text Book, Part II, which has been the fountain head of knowledge on this subject for thirty-five years in America and England and to some extent in Europe. Of course the new book is more comprehensive, as some additional material is added, the most important being that dealing with functions depending on select lives. The main difference in treatment is that a knowledge of the mathematics covered in the previously reviewed book is assumed throughout, whereas the old book war so arranged that a large part of it could be read without such knowledge. This work will be indispensible to actuaries. As to others, it will be most valuable to anyone interested in statistical subjects connected with the mortality rate.

J. S. ELSTON

NOTES

The third number of volume 24 of the TRANSACTIONS OF THIS SOCIETY (October, 1922) contains the following papers: *On the location of the roots of certain types of polynomials,* by J. L. Walsh; *A note on the preceding paper,* by D. R. Curtiss; *Determination of all general homogeneous polynomials expressible as determinants whose elements are homogeneous polynomials,* by H. S. Everett; *General vector calculus,* by J. B. Shaw.

The Nobel prize in physics for 1923 has been awarded to Professor R. A. Millikan, of the California Institute of Technology.

The Royal Society of London has conferred its Copley medal on Professor Horace Lamb, for his researches in mathematical physics, and its Hughes medal on Professor R. A. Millikan, for his determination of the electronic charge and other physical constants.

The University of Paris has conferred honorary doctorates on Professor S. Arrhenius, of Stockholm, and Sir J. J. Thomson, of Cambridge.

Professor G. D. Birkhoff, of Harvard University, delivered a series of six lectures at the Lowell Institute, Boston, in December, 1923, on *The origin, nature and influence of relativity.*

Professor J. A. Miller, of Swarthmore College, lectured October 25 on *The solar eclipse of* 1923 before the Franklin Institute.

Professor C. W. L. Charlier, of the University of Lund, will lecture at the University of California during the summer of 1924, on *The motion of the stars,* and *The distribution of the stars.*

Dr. J. Trousset, maître de conférences in mathematics at the University of Bordeaux, has been promoted to a professorship.

Mr. R. R. S. Cox has been appointed assistant lecturer and tutor in mathematics at the University of Sheffield.

Dr. B. de Kerékjártó has been appointed lecturer in mathematics at Princeton University.

Mr. H. S. Vandiver, of Cornell University, has been granted leave of absence for research during the second half of the academic year, through a grant from the Heckscher Foundation.

At the University of Vermont, Mr. H. G. Millington has been promoted to an assistant professorship of mathematics in the College of Engineering.

At Swarthmore College, Mr. L. J. Comrie has been promoted from the position of research assistant in the Observatory to an assistant professorship of mathematics and astronomy.

At Lafayette College, Associate Professor W. M. Smith has been granted a six months' leave of absence for study at Oxford and Cambridge.

Mr. F. W. Winters, of Harvard University, has been appointed assistant professor of mathematics at Miami University.

Mr. R. H. MacCullough has been appointed head of the department of mathematics at Defiance College (Ohio).

Dr. H. P. Pettit, of the University of Illinois, has been appointed professor of mathematics and head of the department at Illinois Wesleyan University.

Dr. Marie Whelan, of Johns Hopkins University, has been appointed professor of mathematics at Olivet College (Michigan).

At the Michigan State Normal College, Ypsilanti, Dr. Theodore Lindquist, formerly head of the department of mathematics, State Teachers' College, Emporia, Kansas, has been appointed professor of mathematics, and Associate Professor J. F. Barnhill has been promoted to a full professorship.

At Beloit College, Mr. R. F. Huffer, formerly instructor at Michigan Agricultural College, has been appointed assistant professor.

At Kentucky Wesleyan College, Mr. W. B. Hughes has been appointed to succeed Professor R. G. Demaree as head of the department of mathematics.

At Lander College, Greenwood, South Carolina, Professor W. W. Weber, of Southern College, Lakeland, Florida, has been appointed head of the department of mathematics, and Miss Laura D. Sargent has been appointed associate professor.

Professor C. S. Cox, of Birmingham Southern College, has been appointed head of the department of mathematics and physics at Southern College, Lakeland, Florida.

At Washington University, Assistant Professor Theodore Doll has resigned; Dr. Jessica M. Young has been promoted to an assistant professorship of mathematics and astronomy; and Mr. Edmond Siroky has been appointed assistant professor of applied mathematics.

Associate Professor W. T. Stratton, of Kansas State Agricultural College, has been promoted to a full professorship of mathematics.

At the University of Oklahoma, Professor S. W. Reaves, head of the department of mathematics, has been appointed acting dean of the College of Arts and Sciences. Associate Professor J. O. Hassler has been promoted to a full professorship, and Assistant Professor N. A. Court has been promoted to an associate professorship.

At Oklahoma Agricultural and Mechanical College, Mr. G. B. Grumbine has been appointed professor of mathematics, and Mr. W. C. Payne assistant professor.

At the Agricultural and Mechanical College of Texas, Assistant Professor W. I. Hughes has resigned, and Mr. F. W. Sparks has been appointed assistant professor.

The following have been appointed instructors in mathematics:
Beloit College, Mr. William Oliver and Miss Irene Eldridge;
Case School of Applied Science, Mr. V. L. Benander;
Cornell University, Dr. E. L. Post;
Illinois College, Mr. G. W. Schneider;
Lafayette College, Mr. E. H. Wells;
University of North Dakota, Messrs. S. F. Bibb and J. F. Gates;
University of Tennessee, Mr. Augustus Link;
Agricultural and Mechanical College of Texas, Mr. W. D. Baten and Dr. W. P. Udinski;
Tulane University, Messrs. C. G. Latimer and T. F. Cope;
Union College, Mr. H. N. Rowe;
University of Vermont, Mr. C. H. Nicholson;
Virginia Polytechnic Institute, Mr. T. A. Hatcher;
Washington State College, Mr. O. M. Akey;
Washington University, Mr. H. A. Hoover;
University of West Virginia, Mr. Claire Haskins.

The deaths are announced of Professors P. J. H. Baudet and J. Cardinaal, of the Technical School at Delft.

Professor G. Huber, of the University of Bern, died January 24, 1923, at the age of sixty-five years.

Professor Louis Bertrand, formerly of the College of Geneva, died March 13, 1923, in his eighty-third year.

Charles de Freycinet, member of the Paris Academy of Sciences and of the Académie Française, died May 15, 1923, at the age of ninety-five years. He was the author of a number of mathematical books, and had frequently held high political office.

Professor E. Jablonski, of the Lycée Saint-Louis, Paris, died at Royan in April, 1923.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BOGGIO (T.). Calcolo differenziale con applicazioni geometriche. Volume 1: Funzioni di una variabile. Torino, S. Lattes, 1921. 19 + 611 pp.

BOHR (H.). See ENCYKLOPADIE.

BOREL (E.). See JULIA (G.).

BURALI-FORTI (C.) e MARCOLONGO (R.). Corso di matematica per il 2° biennio degli istituti tecnici. Napoli, F. Perella, 1921. 7 + 213 + 8 + 191 pp.

CRAMER (H.). See ENCYKLOPÄDIE.

ENCYKLOPADIE DER MATHEMATISCHEN WISSENSCHAFTEN. Band II 3, Heft 6: N. E. Norlund, Neuere Untersuchungen über Differenzengleichungen. H. Bohr und H. Cramer, Die neuere Entwicklung der analytischen Zahlentheorie. Leipzig, Teubner, 1922.

FIELD (P.). Projective geometry with applications to engineering. New York, Van Nostrand, 1923. 8 + 98 pp. $2.00

FLAMANT (P.). See JULIA (G.).

HAAG (J.). Exercices du cours de mathématiques spéciales. Tome IV: Géométrie descriptive et trigonométrie. Paris, Gauthier-Villars, 1924. 153 pp.

HAGGE (K.). Die Kunst mathematischer Schönheit und Einfachheit, "Geometrographie". Kiel, 1922. 36 pp.

HAMMER (E.). Lehr- und Handbuch der ebenen und spharischen Trigonometrie. 5te, erweiterte Auflage. Stuttgart, 1923.

JULIA (G.). Leçons sur les fonctions uniformes à point singulier isolé. Rédigées par P. Flamant. (Collection de Monographies sur la Théorie des Fonctions publiée sous la Direction de M. Emile Borel.) Paris, Gauthier-Villars, 1924. 8 + 151 pp.

KERST (B.). Ebene Geometrie. (Mathematisch-Physikalische Bibliothek.) Leipzig, Teubner, 1923.

LIETZMANN (W.). Methodik des mathematischen Unterrichts. 2ter Teil. 2te, durchgesehene und vermehrte Auflage. Leipzig, Quelle und Meyer, 1923.

MARCOLONGO (R.). See BURALI-FORTI (C.).

NÖRLUND (N. E.). See ENCYKLOPADIE.

D'OCAGNE (M.). Notions sommaires de géométrie projective à l'usage des candidats à l'Ecole Polytechnique. Paris, Gauthier-Villars, 1924. 25 pp.

SEVERI (F.). Geometria proiettiva. Padova, La Litotipo, Editrice Universitaria, 1921. 4 + 363 pp.

SHEPPARD (W. F.). From determinant to tensor. Oxford, Clarendon Press, and New York, Oxford University Press, 1923. 127 pp. $2.85

PART II. APPLIED MATHEMATICS

Barr (G.). See Graetz (L.).

Burali-Forti (C.). Geometria descrittiva. Volume 1: Assonometria. Volume 2: Projezione quotata, projezione Monge, prospettiva. Torino, Lattes, 1921–1922. 15 + 170 + 11 + 144 pp.

Campbell (N. R.). Modern electrical theory. Supplementary chapters. Chapter 17: The structure of the atom. Cambridge, University Press, 1923. 10 + 161 pp.

Frischauf (J.). Grundriss der theoretischen Astronomie und der Geschichte der Planetentheorie. 3te vermehrte Auflage. Leipzig, Engelmann, 1922.

Gerlach (W.). Materie, Elektrizitat, Energie; die Entwicklung der Atomistik in den letzten zehn Jahren. Band 7. Dresden und Leipzig, J. Steinkopff, 1923. 195 pp.

Graetz (L.). Recent developments in atomic theory. Translated by G. Barr. London, Methuen, 1923. 11 + 174 pp.

Haas (A.). The new physics. Lectures for laymen and others. New York, Dutton, 1923. 12 + 165 pp.

Hatschek (E.). See Willows (R. S.).

Kopff (A.). Grundzuge der Einsteinschen Relativitatstheorie. 2te Auflage. Leipzig, Hirzel, 1923. 8 + 204 pp.

Lacmann (O.). Die Herstellung gezeichneter Rechentafeln. Ein Lehrbuch der Nomographie. Berlin, Springer, 1923. 8 + 100 pp.

Muller (C. H.) und Prange (G.). Allgemeine Mechanik. Hannover, Helwingsche Verlagsbuchhandlung, 1923. 10 + 551 pp.

Nunn (T. P.). Relativity and gravitation. A elementary treatise upon Einstein's theory. London, University of London Press, 1923. 162 pp.

Prange (G.). See Müller (C. H.).

Russell (B.). The ABC of atoms. London, Kegan Paul, and New York, Dutton, 1923. 175 pp.

Sakellaropoulo (N.). Essai sur la cause de la gravitation. Le Caire, 1923. 12mo. 33 pp.

Schiller (K.). Einfuhrung in das Studium der veranderlichen Sterne. Leipzig, Barth, 1923. 8 + 383 pp.

Townsend (C. M.). The hydraulic principles governing river and harbor construction. New York, Macmillan, 1922. 10 + 189 pp.

Velasco de Pando (M.). Solution générale du problème de l'élasticité. Sevilla, 1922.

Vial (F.). Les arguments de M. Einstein. Paris, Gauthier-Villars, 1923. 16 pp.

Willows (R. S.) and Hatschek (E.). Surface tension and surface energy and their influence on chemical phenomena. 3d edition. London, J. and A. Churchill, 1923. 8 + 136 pp.

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THE BÔCHER MEMORIAL PRIZE

THE FIRST AWARD. REGULATIONS GOVERNING THE SECOND AND THIRD AWARDS.

An interesting and significant event in the annals of the Society was the awarding at the Annual Meeting of 1923 of the first Bôcher Memorial Prize to George David Birkhoff, Professor of Mathematics at Harvard University. The memoir so crowned with the first prize ever given under the auspices of the Society is entitled *Dynamical systems with two degrees of freedom* and appeared in the TRANSACTIONS OF THIS SOCIETY, volume 18, pages 199–300.

The presentation was made at the morning session on December 28, by Professor H. S. White, a member of the committee appointed by the Council to make recommendations concerning the first award. After summing up in happy fashion the history of the foundation of the prize, Professor White gave a brief analysis of the memoir. Professor Birkhoff, in acknowledging the honor done him, spoke of Bôcher's contribution to the mathematical ideals and traditions of the Society.

The Committee on Award, consisting of Professors C. N. Haskins (chairman), T. S. Fiske, and H. S. White, were unanimous in their recommendation of this memoir. They pointed out, however, in presenting their report to the Council, that owing to the delay in printing the TRANSACTIONS, the instructions given them (see this BULLETIN, vol. 28, p. 42) were not self-consistent; the Council then voted to revise and supplement the regulations and to make a second award at the Annual Meeting in 1924.

The revised regulations are as follows:

(1) The Bôcher Memorial Prize of $100 shall from time to time be awarded for a notable research memoir, published

in the TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY by a resident of the United States or Canada.

(2) The age of the recipient of the prize shall not be over forty years, and the prize shall not be awarded twice to the same person.

(3) The age and place of residence of a writer shall be reckoned as of the first day of the month of date of issue of the number of the TRANSACTIONS containing the article in question, or the first part of it.

(4) The second award, to be announced at the Annual Meeting of 1924, shall be made with reference to volumes 19 to 24 inclusive.

(5) The third award, to be announced at the Annual Meeting of 1928, shall be made with reference to the volumes beginning with that issued for the year 1923 and ending with that issued for the year 1927.

R. G. D. RICHARDSON,
Secretary.

SPECIAL MEETING OF THE SAN FRANCISCO SECTION

A special meeting of the San Francisco Section of the American Mathematical Society was held at the University of Washington, Seattle, on Saturday, December 22, 1923. The following seventeen members of the Society were in attendance:

Bell, Daniel Buchanan, A. F. Carpenter, DeCou, Feinler, Gavett, Griffin, Griffiths, Kent, McAlister, W. E. Milne, Moritz, Mullemeister, Neikirk, Smail, Stager, Winger.

Professor A. F. Carpenter, chairman of the Section, presided.

Titles and abstracts of the papers presented are listed below. Professor Cajori's paper was read by title, while the paper of Mr. Kelly and Miss Small was read by Professor Griffin, who introduced these authors to the Society.

1. Professor E. T. Bell: *On certain quinary quadratic forms.*

This paper has appeared in the March-April number of this BULLETIN.

2. Professor Daniel Buchanan: *Asymptotic satellites near the straight line equilibrium points (elliptic case).*

If two finite masses move in ellipses about their common center of gravity, there are five points, called *equilibrium points* or *points of libration,* at which an infinitesimal body will remain fixed, relatively to the moving bodies, if given proper initial projections. Three of these points are collinear and lie on the line joining the centers of gravity of the finite bodies. The other two lie in the plane of motion of the finite bodies at the vertices of the equilateral triangles described on the join of their centers of gravity.

The author determines orbits which approach asymptotically (1) the straight line equilibrium points, and (2) the two- and three-dimensional periodic orbits near these straight line equilibrium points which were obtained by Moulton in his *Periodic Orbits,* Chapter VII. The asymptotic orbits are somewhat similar to those obtained by the author in the corresponding circular case (AMERICAN JOURNAL, vol. 41, pp. 79–110).

3. Professor Daniel Buchanan: *Periodic and asymptotic satellites near the equilateral triangle equilibrium points (elliptic case).*

This paper deals with the periodic and asymptotic orbits near the equilateral triangle points of libration as the finite masses move in ellipses. Two- and three-dimensional periodic orbits are first obtained and later the orbits which are asymptotic to the three-dimensional orbits and to the points of libration themselves. As in the circular case determined by Buck (Moulton, *Periodic Orbits,* Chapter IX) the two-dimensional periodic orbits exist only when the ratio of the finite masses is small. The orbits, however, which are asymptotic to the equilibrium points exist, as in the corresponding circular case (Buchanan, TRANSACTIONS

OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY, vol. 22, No. 15, pp. 309–340) only when the ratio of the finite masses is larger than in the case of the periodic orbits. Owing to the variation in this ratio, it has not been possible according to the methods employed in this paper to obtain orbits asymptotic to the two-dimensional periodic orbits.

4. Professor A. F. Carpenter: *Complex cones and cone-cubics of a ruled surface.*

In this paper four quadric cones are associated with each line of a ruled surface by means of the osculating linear complex. Their intersections, pair at a time, determine four space cubics, called primary cone-cubics, two of which lie upon the osculating quadric. Each of these cubics is shown to be projectively equivalent to the primary flecnode cubic. By an analogous process four secondary cone-cubics are defined. These bear to the primary cone-cubics the relation borne by the secondary flecnode cubic to the primary flecnode cubic. A number of theorems involving these curves are proved.

5. Mr. F. J. Feinler: *On the Bernoulli numbers.*

Adams calculated the Bernoulli numbers up to B_{124} by the help of the von Staudt and Clausen Theorem, requiring the tedious handling of fractions. There is no practical necessity to recalculate this series of valuable constants, except perhaps for checking Adams' results. Yet there remains always the theoretical interest in the genesis and the relations of these numbers. The denominators are found to present a very simple law for the reappearance of the prime number series. But if another slightly modified formula is used for the denominators, the tedious handling of fractions is altogether avoided. The values thus obtained are then reduced by the quotient of the two denominator formulas. The resulting series is found identical with that of Adams.

6. Miss Lois Griffiths: *Contact curves of the rational plane cubic.*

This account of contact curves of the rational plane cubic developed from consideration of the subject of involution as applied to rational curves. The determinate case is that in which the complete intersection of the contact curve of order n and the cubic is two distinct points, one a d-point contact p^d and the other a complementary contact p^{3n-d}.

Representative theorems for curves of this type are (1) All points of the cubic are related in sets of $3n$ each; within a single set d points are of one type, contacts p^d, and the remaining $(3n-d)$ are contacts p^{3n-d}, such that each of the $(3n-d)$ points is a complementary contact for each of the d points, and for no other point of the cubic, and vice versa. Thus all contact curves are related in sets of $d(3n-d)$ each. (2) Superimposed on each set of $d(3n-d)$ contact curves, when and only when n is even, is a non-contact curve of order n, cutting the cubic singly in each of the $3n$ contacts for that set of contact curves.

7. Professor F. C. Kent: *Note on the derivation and reduction of annuity formulas.*

The annuity formulas in which the nominal rate of j is convertible m times a year and the annuity is paid p times a year are reduced to simple standard forms which are readily adapted to computation by means of ordinary annuity tables. Also a simplified method is presented for deriving the standard formulas for the amount and present value of annuities certain.

8. Professor L. L. Smail: *A theorem on convergent factors in summable series.*

By making use of certain results of James (ANNALS OF MATHEMATICS, vol. 21, p. 123) and Schur (JOURNAL FÜR MATHEMATIK, vol. 151, pp. 79–111) on the general theory of summability, a theorem is obtained on convergence factors in series summable $(C, 1)$.

9. Professor Florian Cajori: *The growth of legend about Sir Isaac Newton.*

The author considers the growth of legend in accounts of the discovery of the law of gravitation, of the invention of the reflecting telescope, of the delay in the publication of the *Principia,* of action at a distance, and of the wave theory of light.

10. Mr. C. T. Kelly and Miss Dana Small: *Curves conjugate to conics with respect to lines and circles* (*Type* I).

The authors discuss in some detail the conjugate of given conics with respect to a straight line or circle as the basic curve, in the case where the relation between radii vectores is of the form $\varrho_c = \varrho_g = \varrho_b$ or $2\varrho_z$.

13. Professor R. M. Winger: *Note on the invariants of the ternary icosahedral group.*

The fundamental system of invariants of this group, given by Klein, *Ikosaeder*, includes a conic A and a set of six lines, say F. There is thus a pencil of invariant sextics $A^3 + \lambda F = 0$ which contains Klein's curve B $(\lambda = -1)$, the author's rational sextic R $(\lambda = 27/5)$, AMERICAN JOURNAL, 1916, and the sextic invariant of the Valentiner group (Wiman). The salient properties of the pencil are discussed. In particular it is shown that Klein's C which is a rational curve of order ten and class six is the projective dual of R.

14. Professor L. I. Neikirk: *Some non-associative algebras.*

Several examples of these finite linear algebras are given. Left handed division is always possible (except by zero) and is unique. Right-handed division is not always possible and when possible is not always unique. There may be one, two or four quotients. Some of these algebras have nilpotent elements, also elements with equations of the type $S^3 = S^2$. Here S^3 is a left-handed power and $S^3 = [S\cdot(S\cdot S)]$. Multiplication is distributive on the right and non-distributive on the left in some cases. Multiplication tables are given.

R. M. WINGER,
Secretary for the special meeting.

THE THIRTIETH ANNUAL MEETING OF THE SOCIETY

The thirtieth annual meeting of the American Mathematical Society was held at Columbia University on Thursday and Friday, December 27–28, 1923. Dormitory accommodations were furnished by Columbia University and Barnard College, and attending members were introduced at the Faculty Club. About seventy-five members attended the dinner on Thursday evening, at which Professor Coolidge spoke on the progress of the Endowment Fund. At the close of the sessions it was voted to express the thanks of the Society to Columbia University for its courtesy, continued through more than thirty years. It was voted to thank also the Governors of the Faculty Club, and the Committee on Arrangements for the meeting, which consisted of Professors T. S. Fiske (chairman), G. W. Mullins, and L. P. Siceloff, Dr. G. A. Pfeiffer, and Mr. A. E. Meder.

The attendance included the following ninety-five members of the Society:

C. R. Adams, Alexander, C. R. Ballantine, J. P. Ballantine, Barnum, Bernstein, Birkhoff, Borden, R. W. Burgess, Burwell, Clements, Abraham Cohen, Coolidge, C. F. Craig, Dadourian, Douglas, Edmondson, Eisenhart, Fields, Fine, Fite, Forsyth, Philip Franklin, Fry, Gafafer, Gilman, Glenn, P. H. Graham, Grant, M. C. Graustein, W. C. Graustein, Hammond, W. L. Hart, Harter, Hawkes, Hazlett, Robert Henderson, Hille, Himwich, Hotelling, Huntington, Joffe, Kasner, Kazarinoff, Lamson, Langman, Leib, Linehan, Lynch, McGiffert, MacInness, Mathews, Meder, Mirick, H. H. Mitchell, Frank Morley, Marston Morse, Mullins, Oglesby, Olds, F. W. Owens, Pell, Post, Rainich, Ranum, Raynor, Reddick, R. G. D. Richardson, Ritt, Seely, Seidlin, Shook, Siceloff, Simmons, Slotnick, Simons, C. E. Smith, D. E. Smith, W. M. Smith, Virgil Snyder, M. H. Stone, Elijah Swift, J. S. Taylor, Tracey, Tyler, Veblen, Evelyn Walker, J. L. Walsh, Wedderburn, Weisner, Weiss, H. S. White, E. E. Whitford, Whittemore, R. G. Wood.

The Council elected the following sixty-five persons to membership in the Society:

Professor Olafur T. Anderson, Wesley College;
Professor William Elijah Anderson, Miami University;
Dr. Bernardo Ig. Baidaff, Buenos Aires;
Dr. John Perry Ballantine, Columbia University;
Professor Maurice Baudin, Miami University;

Professor James Marcus Bledsoe, East Texas State Teachers College;
Mr. William Miller Bond, College of the City of New York;
Mr. Orley Edward Brown, Northwestern University;
Dr. Keivin Burns, Allegheny Observatory;
Mr. William Elmer Byrne, Rensselaer Polytechnic Institute;
Professor Theodosia Tucker Calloway, Stephens Junior College;
Professor George B. Clinkscales, Converse College;
Dean James Frost Cross, St. John's College;
Professor Samuel Ridgely-Cruse, University of Arizona;
Dr. Heber Doust Curtis, Allegheny Observatory;
Mr. Alfred Davis, Soldan High School, St. Louis;
Mr. Hannibal A. Davis, West Virginia University;
Professor Samuel Dickstein, University of Warsaw;
Mr. Walter Anthony Dynan, Amityville, N. Y.;
Mr. George Gilbert Entz, Los Angeles;
Mr. Elgin George Fassel, Cleveland Life Insurance Company;
Mr. James Hampton Fithian, Brown University;
Mr. Anselm J. Fleisig, St. Procopius College;
Professor Robert Dale Ford, St. Lawrence University;
Miss Alice Ann Grant, Brown University;
Professor Ruby Mabel Grimes, Northern Normal and Industrial School, Aberdeen, S. D.;
Mr. Duncan Claire Harkins, West Virginia University;
Professor Eli Stuart Haynes, University of Missouri;
Professor Chester Rodney Hillard, College of the Ozarks;
Professor Frederick Chapman Jonah, Acadia University;
Mr. Arthur Daniel Kinsman, Ipswich, Mass.;
Professor Carl Dewey Laws, Hampden-Sidney College;
Dean Thomas Jefferson Leslie, Austin College;
Mr. Locke Ellsworth Lunn, State High School, Heron Lake, Minn.;
Mr. Arthur Louis McCarty, Lowell High School, San Francisco;
Dean Lee Marcus McCoy, Morgan College;
Professor Raymond Earl Manchester, Kent State College;
Mr. Alfred Joseph Maria, Rice Institute;
Professor Gertrude Whittier Mendenhall, North Carolina College for Women;
Mr. Alfred William Muldrew, St. John's Technical High School, Winnipeg;
Mr. James A. Nyswander, University of Chicago;
Dr. Robert Bowie Owens, Franklin Institute;
Mr. George Ambrose Parkinson, University of Wisconsin;
Professor Frederick Malling Pedersen, College of the City of New York;
Miss Sallie E. Pero, American Telephone and Telegraph Company;
Professor Otto J. Ramler, Catholic University;
Professor James Nelson Rice, Catholic University;
Mr. Harold E. Riter, Daniel McIntyre Collegiate Institute, Winnipeg;
Mr. William Haldeman Rittenhouse, Philadelphia;
Miss Georgia Edna Robinson, Stephens College;
Mr. Harry Mance Roeser, Bureau of Standards;
Mr. James Shohat, University of Chicago;
Dr. Charles Albert Shook, Yale University;
Professor Robert Frederic Smith, College of the City of New York;
Dean Julia Spalding, Christian College;
Mr. Arthur A. Sperling, University of Wisconsin, Milwaukee Branch;
Mr. Robert Meldrum Stewart, Dominon Observatory, Ottawa;
Professor Robert Ambrose Thornton, Shaw University;
Professor Ernest R. Tucker, Texas Christian University;

Professor Charles Ambrose Van Velzer, Carthage College;
Mr. Albert Huntoon Wait, University of Wisconsin;
Miss Fannie Watkins, Winthrop College;
Mr. Lawrence E. Widmark, Star Electric Motor Company;
Mr. Edgar William Woolard, United States Weather Bureau;
Mr. Floyd Eugene Young, Oregon Agricultural College.

The total membership of the Society is now 1250, including 83 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 541; the number of papers read was 266. The number of members attending at least one meeting was 331. At the annual election 172 votes were cast.

The report of the Treasurer was received, showing a balance of $9,638.78, exclusive of special funds; of this, $5,726.91 is reserved to secure the life memberships. Sales of the Society's publications during the year amounted to $3,570.03. The report of the auditors (Mr. S. A. Joffe and Professor W. J. Berry) was deferred until the February meeting.

The Board of Trustees adopted a budget for 1924.

The librarian reported that the Library of the Society now contains 6,546 volumes.

At the annual election, which closed on Thursday afternoon, the following trustees and officers and other members of the Council were chosen:

Trustees, Professor G. A. BLISS, Professor W. B. FITE, Mr. ROBERT HENDERSON, Professor R. G. D. RICHARDSON, Professor OSWALD VEBLEN.

Vice-Presidents, Professor E. V. HUNTINGTON (one year); Professors T. H. HILDEBRANDT and J. H. M. WEDDERBURN (two years).

Secretary, Professor R. G. D. RICHARDSON.

Assistant Secretary, Professor ARNOLD DRESDEN.

Treasurer, Professor W. B. FITE.

Librarian, Professor R. C. ARCHIBALD.

Member of the Editorial Committee of the Bulletin, Professor ARNOLD DRESDEN.

Member of the Editorial Committee of the Transactions, Professor EDWARD KASNER.

Members of the Council, Professor H. W. TYLER (one year); Professor J. L. COOLIDGE (two years); Professors HARRY BATEMAN, G. A. BLISS, SOLOMON LEFSCHETZ, C. L. E. MOORE, ANNA J. PELL (three years).

The tellers appointed by President Veblen to count the ballots were Professor R. E. Gilman, Dr. Jesse Douglas, and Mr. A. E. Meder.

An account of the proceedings connected with the presentation of the first Bôcher Memorial Prize and of the regulations governing the second and third awards is contained elsewhere in this issue.

On account of proposed absence from the country, Professor E. B. Van Vleck presented his resignation as representative of the Society on the National Research Council. Professor H. F. Blichfeldt was selected to complete the term which expires on July 1, 1924, and also to represent the Society for the succeeding period of three years. Professors A. B. Coble, Arnold Dresden, L. E. Dickson, and Virgil Snyder were elected as additional representatives of the Society on the American Section of the International Mathematical Union.

A committee on the western meetings of the Society was appointed for a two-year period, consisting of Professors T. H. Hildebrandt (chairman), Henry Blumberg, and Arnold Dresden. Dean H. L. Hodgkins and Professor R. G. D. Richardson were appointed to represent the Society on the Council of the American Association for the Advancement of Science for the year 1924.

The committee on membership was continued, to consist of Professors Abraham Cohen (chairman), E. B. Stouffer, and C. E. Smith (permanent secretary), and such others as President Veblen may appoint.

A letter was read from the secretary of the Physico-Mathematical Society of Japan, thanking the Society for the message of sympathy sent in September and recounting some of the effects of the disaster on mathematical research in that country.

The Council recommended to the Society that the By-Laws be amended so as to include ex-Presidents in the Council for a period of six years after expiration of their presidential terms.

It was announced that Professor H. H. Mitchell had been appointed chairman of the Committee on the next Colloquium, in place of Professor L. E. Dickson resigned; and that Dean H. L. Hodgkins had represented the Society at the inauguration of President Lewis of George Washington University.

Announcement was made by President Veblen, who is also Chairman of the Division of Physical Sciences of the National Research Council, concerning the extension of one group of the Research Fellowships to include mathematics. This group of fellowships which is adminstered by the Research Fellowship Board of the National Research Council and supported by the Rockefeller Foundation now covers mathematics, physics, and chemistry; and, beginning July 1, 1925, the annual income is increased to $125,000. In order that adequate use be made of this remarkable addition to the resources for mathematical research, members are urged to give information concerning talented and well-equipped young men and women.

President Veblen presided at the sessions on Thursday, relieved by Vice-President Tyler; during the sessions on Friday, Vice-Presidents Huntington and Wedderburn alternated in the chair.

At the session on Friday morning a paper was read, at the request of the program committee, by Professor Virgil Snyder, on *Problems connected with involutorial transformations in space.* This paper appeared in full in the March-April issue of this BULLETIN.

Titles and abstracts of the papers read at this meeting follow below. The papers of Professor Bell, Dr. Zeldin, Professor Lipka, Dr. Pfeiffer, Dr. Wiener, Professor Ford, Mr. Michal, and Professor Ritt, and the second papers of Professor Huntington, Dr. Walsh, Dr. Ballantine, and Dr. Taylor, were read by title.

1. Professor W. B. Fite: *The analytic functions defined by a certain functional differential equation.*

In this paper the author considers the solutions in the complex plane of the functional differential equation $y''(z) = A^2 y(\sin z) = 0$. He finds that they are entire functions of z. Some theorems relating to the existence and distribution of the real zeros of these functions are proved. Every such function has at least one real zero. Nothing is determined concerning the complex zeros.

2. Dr. K. W. Lamson: *The field equations of relativity as integrability conditions.*

The condition for the vanishing of the sum of the Riemann (two-dimensional) curvatures determined by μ mutually perpendicular directions in the general Riemann n-space is $K_\mu = 0$, where $K_\mu = \sum g_{ij} g_{ij} R_{ii,jj}$. Here $R_{ii,jj}$ is the Riemann-Christoffel 4-index symbol, and the i's are any μ numbers from the set $1, 2, \ldots, n$, and similarly for the j's. The summation is taken over all permutations of the i's among themselves and of the j's among themselves. The object of the paper is to prove that $K_\mu = 0$ is a necessary and sufficient condition for the existence of a contact transformation which transforms $\sum g^{ij} p_i p_j$ into the sum of squares. This contact transformation is determined by $n - \mu + 2$ relations between the old and new point coordinates. In particular, if $n = 4$, then $K_3 = 0$ is equivalent to the Einstein field equations. The general solution of the field equations is found by algebraic processes.

3. Dr. Jesse Douglas: *Characterization of spaces of constant curvature by the arrangement of their geodesics.*

Necessary conditions are derived that a family of curves in n-space defined by a system of differential equations of the form $d^2x_i/dx_1^2 = G_i(x_j, dx_j/dx_1)$ be such that through every three points a two-spread passes which contains ∞^2 curves of the family. As a special case, a proof is obtained of Schur's theorem (MATHEMATISCHE ANNALEN, 1886) that the existence of a geodesic two-spread through every three points of a Riemann space characterizes the space as having constant curvature. A generalization is developed to the effect that "two-spread" can be replaced by "k-spread".

4. Mr. G. Y. Rainich: *The non-linearity of Maxwell's equations in general relativity theory.*

The Maxwell equations consist of two sets of four equations each which have to be satisfied, respectively, by the covariant and mixed components of the electromagnetic tensor; for vacuum each set is linear. But these two sets are not sufficient to determine the electromagnetic field; we must consider together with them the equations which give the connection between the covariant and mixed components, and these equations involve the g's which are connected with the coefficients of the electromagnetic tensor by the energy relation which is not linear; therefore the whole system is non-linear. In order to be able to consider these things without introducing the restriction connected with the use of the g's, we introduce a new system of representation for curved space*. Each point is characterized by four coordinates as usual, but to represent vectors and tensors we introduce in every bundle cartesian coordinates. We have thus no distinction between co- and contravariant quantities, and the algebra of tensors is simplified. But additional complications arise in the tensor analysis, and they are responsible in this case for the non-linearity of Maxwell's equations.

5. Professor J. L. Coolidge: *A simple proof of Luroth's theorem.*

Lüroth's theorem states that if x and y are rational functions of a parameter t, not both constants, the curve so defined has the deficiency zero. The author gives a new proof based on Zeuthen's correspondence formula, and communicates one by Professor Osgood based on abelian integrals.

6. Professor B. A. Bernstein: *Representation of three-element algebras.*

The author gives a method of obtaining a concrete representation of any binary operation or dyadic relation possible in an algebra consisting of three elements. The representations are all arithmetic and modular. The device used is an extension of that employed by him in the representation of two-element algebras.

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* Cf. the author's paper in PROCEEDINGS OF THE NATIONAL ACADEMY, Dec., 1923.

7. Dr. J. L. Walsh: *On the expansion of analytic functions in series of polynomials.*

Let C be a simple closed finite analytic curve in the z-plane including in its interior the origin. Then there exist two sets of functions, $p_0(z), p_1(z), \ldots, p_n(z), \ldots$ and $q_1(z), q_2(z), \ldots, q_n(z), \ldots$, polynomials respectively in z and in $1/z$, such that if $f(z)$ be any function defined on C and satisfying on C a Lipschitz condition, then $f(z)$ can be expanded in the form $f(z) = a_0 p_0(z) + a_1 p_1(z) + \cdots + a_n p_n(z) + \cdots + b_1 q_1(z) + b_2 q_2(z) + \cdots + b_n q_n(z) + \cdots$, where the former series converges uniformly in the closed region interior to C and the latter series converges uniformly in the closed region exterior to C and vanishes at infinity. The coefficients are given by formulas $a_k = \int_C f(z)\, s_k(z)\, dz$, $b_k = \int_C f(z)\, t_k(z)\, dz$, where the functions $s_k(z)$ and $t_k(z)$ depend on C but not on $f(z)$. The polynomial $p_k(z)$ has precisely k roots interior to C, and the polynomial $q_k(z)$ has precisely k roots exterior to C.

8. Professor E. V. Huntington: *A new set of postulates for betweenness, with proof of complete independence.*

A paper published in 1917 by E. V. Huntington and J. R. Kline (TRANSACTIONS OF THIS SOCIETY, vol. 18, pp. 301–325) gave eleven sets of independent postulates for betweenness, selected from a basic list of twelve postulates, A, B, C, D, 1–8. Of these eleven sets, three contained seven postulates each and eight contained six postulates each. In the present paper, a new postulate is added to the basic list, namely Postulate 9; it is then shown that the following set containing only five postulates, A, B, C, D, 9, is sufficient to define betweenness:

(A) If ABC, then CBA.

(B) If A, B, C are distinct, then at least one of the six permutations, $ABC, ACB, BAC, BCA, CAB, CBA$, is a true order.

(C) If ABC is true, then ACB is false.

(D) If ABC is true, then A, B, C are distinct.

(9) If A, B, C, X are distinct and ABC is true, then at least one of the orders ABX, XBC is true.

These five postulates are shown to be completely independent; and the long list of theorems in the earlier paper is extended so as to cover all the inter-relations among the thirteen postulates of the enlarged basic list. The paper will be offered to the TRANSACTIONS.

9. Professor E. V. Huntington: *A new set of completely independent postulates for cyclic order.*

If Postulate A for betweenness (see preceding abstract) is replaced by Postulate A′: If *ABC*, then *BCA*; then the five postulates A′, B, C, D, 9 will form a completely independent set of postulates for cyclic order. This paper is supplementary to two earlier papers on cyclic order, one presented on December 28, 1916 (see an abstract in the PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 2 (1916), pp. 630–631), and the other on October 27, 1923. It will be offered to the TRANSACTIONS.

10. Professor E. T. Bell: *Representations of integers in certain quadratic forms in* 2, 3, 4, 5 *indeterminates and allied class-number relations.*

When there is but one class in the principal genus of binary quadratic forms of a negative determinant, the number of representations of an integer in the form can be found at once in terms of the devisors of the integer. For certain ternary and quaternary forms there are similar simple theorems. For quinary forms the like exists in several cases if the integer represented is the square of an odd integer times a power of 2. All of the results give class number relations; those relating to quinary forms are of a wholly new type.

11. Dr. J. L. Walsh: *On Pellet's theorem concerning the roots of an algebraic equation.*

This paper furnishes a complete answer to the following question: If there are given the absolute values of the coefficients of an algebraic equation, what can be said of the regions of the complex plane in which the roots of the equation do or do not lie? The only statement which can be made regarding the roots is essentially a theorem proved by Pellet, DARBOUX'S BULLETIN, (2), vol. 5 (1881).

12. Dr. E. L. Post: *The mth derivative of a function of a function*; *calculus of mth derivatives.*

At the October, 1923, meeting of the Society the author presented a paper on a theory of generalized differentiation. The application of this to the Heaviside theory requires a usable knowledge of mth derivatives. The problem of finding the mth derivative of a function of a function, $F(u)$, has always presented great difficulties. Thus, although the Faa di Bruno formula gives $(d^m/dx^m)\,F(u)$ as a finite series in $F^{(m)}(u)$ with coefficients depending wholly on u and its derivatives, the formula for these coefficients is quite unmanageable. The present paper concerns itself specifically with these coefficients, and sets up formulas for the coefficients of a sum, product, and quotient of two functions, and for a function of a function in terms of the coefficients of these functions. Thus in the last case if v is a function of u and u a function of x,

$$(m)v/(n)x = \sum_{l=m}^{n}((m)v/(l)u)\,((l)u/(n)x),$$

where $(m)u/(n)x$ is the mth coefficient in the nth derivative of a function of u. These rules in connection with the coefficients for the simplest elementary functions may be said to constitute a calculus of the coefficients in question, and this in turn, added to the previously known formulas, gives a calculus of mth derivatives.

13. Professor W. C. Graustein: *Invariant directions in the theory of surfaces.*

Given a single surface or a map of one surface upon another, there exist directions, or vectors, which are relatively invariant under a change of parameters. It is the purpose of this paper to determine directions with this property and to demonstrate their usefulness in the theory of surfaces and the theory of maps.

14. Mr. D. K. Kazarinoff: *A property of the curvature of the line of intersection of two surfaces.*

Consider the two surfaces σ_1 and σ_2 and any point P of their line of intersection (σ_1, σ_2), where P is not a singular point for either of the surfaces. Consider at P the two planes τ_1 and τ_2 tangent to σ_1 and σ_2 respectively, and the plane curves of intersections (σ_1, τ_2) and (σ_2, τ_1). We regard the curvature of a curve as a vector of length equal to the absolute value of the curvature and directed as the principal normal from the curve to its center of curvature. We have, then, the following theorem, believed to be new: *The curvature of* (σ_1, σ_2) *at* P *is the geometric sum of the curvatures of* (σ_1, τ_2) *and* (σ_2, τ_1). A proof of remarkable simplicity, based on Meusnier's theorem, can be given.

15. Dr. Louis Weisner: *Simultaneous algebraic equations.*

Let two plane algebraic curves whose equations have their coefficients in a domain R intersect in a finite number of points all of which are in the finite part of the plane. It is shown that the group relative to R of the equation which the abscissas of the points of intersection of the two curves satisfy is the same as that which the ordinates satisfy. A similar theorem holds for any finite number of algebraic equations which are independent and consistent.

16. Professor Virgil Snyder: *On the types of monoidal involutions.*

The purpose of this paper is to unify the theory of monoidal involutions by proving all the known theorems by means of the same method. A number of new types are discussed, and it is shown that if irrational types exist they must be compounded from the perspective monoidal types, either by means of the plane Bertini type or the perspective Jonquières type. The involution which belongs to the cubic variety of four-dimensional space is included in the latter category.

17. Dr. Einar Hille: *On the zeros of the functions of the parabolic cylinder.*

The differential equation of the parabolic cylinder can be brought to one of the two forms $w'' \pm (z^2 - a^2)w = 0$, where a is real in general. The paper contains a study of the set of zeros and extrema of a given solution of one of these differential equations. The problem is considered from three different angles: (1) the asymptotic distribution of the set, (2) the geometric structure of the set, and (3) the variation of the structure with different initial conditions and with the parameter a.

18. Professor Edward Kasner: *Geodesic families and their generalizations.*

In this paper the author discusses various generalizations of the family of straight lines in ordinary n-space or geodesics in a Riemann space, in particular *cubic* families, *extremal* families, and *intersectional* families. Cubic families (which include the natural and velocity types previously studied by the author) are related to the geodesics of

Weyl's general affine connected space or the path geometry of Eisenhart and Veblen. Extremal families are related to the general integral of first order in the calculus of variations. Intersectional families are defined by the curves of intersection of families of surfaces or hypersurfaces (just as straight lines are obtained from planes). The overlapping of the various types is considered.

19. Dr. J. S. Taylor: *A statistical theory of depreciation.*

The principal results obtained are, first, a means of determining more accurately than heretofore the useful life of the average machine of a given type; second, a method of distributing the depreciation charges in such a way that the unit cost of output remains constant (or increases by given amounts); and third, a method of determining when an individual machine should be discarded when that time arrives in which the only conjecture is in the estimates of the costs for a new machine. Finally, the paper points out the interdependence of costs of output and the distribution of the depreciation charges, together with the fact that the useful life of a machine is affected by such distribution and by the current interest rate.

20. Dr. S. D. Zeldin: *A classification of integral invariants in the calculus of variations.*

The curve $y = f(x)$ which maximizes or minimizes the integral $I = \int F(x, y, y', y'', \ldots, y^{(n)})\, dx$ satisfies Lagrange's equation. The integral I is invariant under point and contact transformations. In this paper the author classifies the integrals I invariant under the contact transformations of the plane.

21. Professor Virgil Snyder: *Problems connected with involutorial transformations in space.*

This paper appeared in the March-April issue of this BULLETIN.

22. Professor Joseph Lipka: *Some theorems on trajectories in V_n.*

A study is made of a system of trajectories in a conservative field of force in space of n dimensions, taking, as a basis, n mutually orthogonal congruences and the Ricci coefficients of rotation $\gamma_{\alpha\beta h}$. Among other theorems, the author proves that the sum of the squares of the geodesic curvatures of any n mutually orthogonal trajectories through a point is an invariant, *i. e.*, independent of the n directions chosen; this invariant is equal to $(n-1)$ times the square of the magnitude of the trajectory vector at the point.

23. Professor Joseph Lipka: *On irreversible dynamical systems under a zero force.*

These curves may be defined as the extremals of the variation problem $\int ds + \int d\Phi =$ minimum, where ds^2 is the fundamental quadratic differential form in the space, and $d\Phi = \sum \Phi_i \, dx_i$, the Φ_i being functions of the coordinates. An interesting geometric property of these curves is the following: The locus of the centers of geodesic curvature of the ∞^1 curves of the system which pass through a given point is a hyperplane if the dimensionality of the space is *odd*; on the other hand, the above locus is a quartic hypersurface of a special variety if the dimensionality of the space is *even*. A curve of the system defined above may also be identified with Weyl's "Singularitäts-Kanal" in a field with a given structure.

24. Professor Marston Morse: *Relations between different types of critical points of a real analytic function of n real variables.*

If there be given a real analytic function of n variables, Birkhoff has shown that, under certain boundary hypotheses, the existence of m proper minima $(m > 1)$ implies the existence of other critical points affording neither maxima nor minima. The present paper attacks the problem of determining the relations between general types of critical points each characterized by the signature of the quadratic form making up the terms of second order in the expansion of the given function in the neighborhood of the given critical point.

25. Dr. C. R. Adams: *The general theory of the linear q-difference equation.*

The author uses matrix methods in treating the system

$$g_i(qx, ry) = \sum_{j=1}^{n} p_{ij}(x, y) \, g_j(x, y), \quad (i = 1, 2, \ldots, n),$$

in which q and r are real or complex constants different from zero, the $p_{ij}(x, y)$ are polynomials in x and y, and the $g_i(x, y)$ are the functions to be determined; such a system is equivalent to a single homogeneous equation of the nth order. If $|q| \leqq 1$, $|r| \leqq 1$, the system possesses in general four matrices of formal series solutions (in powers of x and y, $1/x$ and $1/y$, x and $1/y$, $1/x$ and y). The case $|q|, |r| > 1$ is typical; then the elements of two matrices converge to analytic functions in the vicinities respectively of (0,0) and (∞, ∞). Two matrices of solutions are thus defined, one analytic whenever neither x nor y is infinite, and the other analytic save for poles whenever neither x nor y is zero. A matrix of periodic functions is defined by these two solutions, which, after transformation, appears as a matrix of triply periodic functions, a special case of a class of such functions discussed by Cousin. An inverse of the general existence theorem is proved and a method given for handling certain exceptional cases. It is further proved that if $|q| = |r| = 1$, there exists in general no analytic solution.

26. Professor O. F. Glenn: *The invariants of forms under the total group* G_6, *modulo* 2.

This paper appeared in full in the March-April number of this BULLETIN.

27. Professor Olive C. Hazlett: *On associated forms in the theory of formal modular covariants.*

By the aid of the symbolic theory of formal modular covariants, there is proved for such covariants a theorem which is closely analogous to Hermite's well known results for algebraic covariants. If h be any formal modular covariant of the binary form f with respect to the Galois field $GF[p^n]$ of order p^n, then (aside from a power of h) every formal modular covariant of f is expressible as a polynomial in the associated covariants with respect to h and a finite number of formal modular covariants obtained from h in a certain manner.

28. Professor Edward Kasner: *Intermediate curvatures in a Riemann space.*

A result analogous to Schur's theorem on Riemann curvature holds for Ricci curvature and for the intermediate curvatures defined by the author in a previous paper: If any one of the curvatures is a point function (instead of an orientation function) it must be a constant. When this is proved for the Ricci curvature (by making use of Hilbert's identities) Schur's theorem is obtained as an immediate corollary, and by combining these results in a simple manner suggested by Mr. Rainich, the general theorem is obtained. When the curvatures are equated to zero, we obtain spaces which Dr. Lamson has integrated by means of contact transformations.

29. Dr. G. A. Pfeiffer: *A proof of Cauchy's integral theorem for any rectifiable boundary.*

Cauchy's integral theorem in the strong form, namely, if $f(z)$ is analytic in a domain bounded by any rectifiable curve and continuous on the set consisting of that domain and its boundary, then the integral of $f(z)$ over the boundary is zero, was established, so it seems, only recently. E. B. Van Vleck published in 1921 a proof of Green's lemma for any rectifiable boundary. The above theorem follows in the familiar way from that result. A direct proof of said theorem was published this year by S. Pollard, who in deriving certain preliminary results of a semi analysis situs character made use of some special results of de la Vallée Poussin. The present paper gives a proof of the same theorem, based only on well known mathematical facts. The lemma of the paper seems of interest on its own account. It states that the perimeters of all polygons which are within a domain S, bounded by a rectifiable curve, and which are derived from rectangular networks in a well known way, are less than $2\sqrt{2}$ times the length of the boundary of S.

30. Dr. J. P. Ballantine: *A generalization of the calculus of finite differences to include the differential calculus.*

In the calculus of finite differences, one speaks of a difference quotient taken at a set of distinct values. Differential calculus is often regarded as the limiting case when all the values approach equality. In this paper, it is shown that by properly enlarging the notion of difference quotient, differential calculus becomes a special, and not a limiting, case. Other special cases are found to be interesting, and have a practical application to the numerical solution of differential equations.

31. Dr. J. P. Ballantine: *A postulational introduction to the positive integers.*

Certain properties of the numbers 0 and 1, expressible without the use of any other numbers, are supposed. A class of undefined elements, called integers, is postulated, subject to a binary relation, called successor. The successor relation is in the form of a matrix, $\varkappa''$, whose values are 0 and 1 according as the second argument is not or is a successor of the first. It is subject to three axioms: Axiom 1, $S'(1'-S'\varkappa''1')1'=0$; Axiom 2, $S'(1'-S'1'\varkappa'')1'=1$; Axiom 3, $S'\xi'\varkappa''=\xi'\cdot)\cdot\xi'=0'$. It is proved that the successor of each integer is unique, that there exists (uniquely) a first integer, 1, that the set of successive successors of the integer 1 includes all integers except 1, and that the integers are linearly ordered.

32. Dr. J. S. Taylor: *A note on the theory of testimony.*

This paper deals with the question of determining the probability of the occurrence of a given event, whose a priori probability is known, when attested to by one or more observers. The *personal accuracy* of observation of an individual observer is defined as the limit of the accuracy of his observation as the a priori probability of the occurrence of an event approaches zero in a specified way. Given the personal accuracy of observation of an individual observer the probability of the occurrence of an event to which he attests can be determined for any case of a rather general type, and conversely. The results are then extended to the case of n observers. Finally, a method is obtained of determining the personal accuracy of observation of an individual observer to any degree of approximation, thus furnishing a means of obtaining the data upon which the foregoing results depend.

33. Professor C. H. Forsyth: *The determination of how much one would be justified in expending to extend the life and multiply the value of an article.*

Students of the mathematical theory of finance are familiar with the problem of determining how much one would be justified in expending to extend the life of an article a specified period, and the corresponding formula. In this paper the problem is extended and a formula derived for determining how much one would be justified in expending to extend the life and at the same time multiply the value or productivity of the article. In fact, two formulas are derived so that a criterion of the extra expense can be determined whether both the life and the value of the article are increased or only one (either one) is increased and the other decreased.

34. Dr. Norbert Wiener: *A contribution to the theory of interpolation.*

Given a finite set of points S, surrounded by a curve C, the properties are developed of that function, harmonic on the interior of C, which assumes given values over S and has the least integral around C.

35. Professor L. R. Ford: *The numerical integration of differential equations.*

In solving a differential equation by Picard's method of successive approximations, the repeated integrations can be performed by the approximate methods of finite differences or by the use of a few terms of the Euler-Maclaurin expansion. Two questions arise: Does the numerical process converge? If so, what are the errors of the approximate solution? In this paper, it is proved that the process converges under quite general conditions, and a limit is found for the errors that result when the approximate method is repeated over successive intervals.

36. Mr. A. D. Michal: *Invariant functionals of closed plane curves under one-parameter groups of transformations of the plane.*

In this paper the author shows that there always exist functionals of closed plane curves invariant under a given arbitrary continuous one-parameter group of transformations of the plane. The functional of closed plane curves is assumed to be developable in a new type of expansion involving multiple double integrals, the region of integration being that of the area of the closed curves. The necessary and sufficient conditions for the invariance of the functionals in question reduce to the form of certain non-homogeneous linear partial differential equations, of the first order. The symmetric solutions of these partial differential equations yield the sought for integrands found in the assumed expansion of the invariant functionals in question. The actual calculation of the invariants is effected completely in the case of a number of well known continuous one-parameter groups of transformations of the plane.

37. Professor J. F. Ritt: *Note on Dirichlet series with complex exponents.*

This note gives a very simple proof of an extension made by E. Hille of a theorem of the present author's.

38. Professor E. L. Dodd: *Integration in certain n-dimensional fields with the aid of a general discontinuity factor.*

Use is made of $F(t)$, where $F(0) = 0$ and

$$\int_{-\infty}^{\infty} \frac{F(t)\,dt}{t} = K \neq 0, \neq \infty,$$

to effect integration of real functions in fields such as that determined by $-a < X < 0$, where X is a real function defined for all real values of its n variables. The general formula thus obtained, when applied to the case of $n = 1$, with $F(t) = \sin t$, and X linear, yields the Fourier integral equation. The discontinuity-factor method, used in the theory of probability to integrate $\varphi_1(x_1)\,\varphi_2(x) \ldots \varphi_n(x_n)$ between parallel planes, is shown to be valid in case each $\varphi_i(x_i)$ is bounded and integrable and absolutely integrable, and two of these functions are of limited variation. This device of integration, which makes integration proceed to ∞, while keeping the n variables separate, may then be used for almost any bounded function $\varphi_i(x_i)$ that has been suggested for frequency distributions, such as the Pearson types, finite Brun's series, Poisson exponential function, Charlier B-series, Jorgensen function, and Makeham life function.

R. G. D. Richardson,
Secretary.

THE CINCINNATI MEETING OF THE SOCIETY

The twentieth Western meeting of the Society was held at the University of Cincinnati on Friday and Saturday, December 28 and 29, in conjunction with the meeting of the Mathematical Association of America and the seventy-fifth meeting of the American Association for the Advancement of Science.

The meeting was attended by about one hundred persons, among whom were the following seventy-nine members of the Society:

Alexander, R. B. Allen, Archibald, G. N. Armstrong, Atchison, Babb, Bareis, Barnett, E. M. Berry, Borger, Boyd Bradshaw, Brand, Cairns, Cajori, A. D. Campbell, Carmichael, Coble, H. H. Conwell, H. T. Davis, Davisson, Denton, Dickson, Dostal, Dresden, Edington, L. C. Emmons, Eversull, Focke, Fort, Gerst, Hancock, E R. Hedrick, Holl, Dunham Jackson, Kindle, Kuhn, Lefschetz, Le Stourgeon, MacMillan, March, Marshall, T. E. Mason, G. A. Miller, C. N. Moore, R. L. Moore, C. C. Morris, Muehlman, Nassau, Olson, Overman, Palmié, Plant, Rasor, C. N. Reynolds, H. L. Rietz, E. D. Roe, Roever, Rothrock, Safford, Schottenfels, W. G. Simon, Sinclair, Slaught, Edwin R. Smith, I. W. Smith, Spenceley, R. B. Stone, Swartzel, B. M. Turner, Vandiver, Van Vleck, Veblen, J. H. Weaver, Norbert Wiener, Wilczewski, Yanney, J. W. Young, Zehring.

The session of Friday afternoon was held under the joint auspices of the Society, the Mathematical Association, and Section A of the American Association for the Advancement of Science, and was presided over by Professor Hancock, Chairman of Section A. The program consisted of the following three papers:

I. Professor G. A. Miller (as retiring Chairman of Section A): *American mathematics during three quarters of a century.*

II. Professor A. B. Coble (as retiring Chairman of the Chicago Section of the Society): *On the equation of the eighth degree.*

III. Professor L. E. Dickson (by invitation of the Program Committee): *Algebras and their arithmetics.*

At the close of this session Professor Jackson moved and those present unanimously adopted a resolution expressing to the University of Cincinnati and in particular to its department of mathematics, their appreciation of the cordial reception and the hospitality accorded the visitors.

On Friday evening a dinner was held at the Hotel Gibson for mathematicians and astronomers, at which 110 persons were present. Mr. J. A. Shohat, who has recently come from Russia, expressed the appreciation of Russian scientists for the aid received from America. Professor Slaught spoke on the incorporation of the Society, and on the changes made in the By-Laws. Professor Dresden reported on the progress of the campaign for an endowment of $100,000. President Veblen, after supplementing the remarks of the previous speakers, announced the establishment by the National Research Council of Fellowships in Mathematics.

Professor Jackson reported for the committee appointed to make arrangements for a symposium at the meeting of April 1924, that the symposium lecture would be given by Professor H. L. Rietz on *The mathematical theory of statistics.*

At the session of Friday forenoon President Veblen took the chair. On Saturday Vice-President R. L. Moore presided, relieved by Professors A. B. Coble and R. C. Archibald.

The following papers were presented at the sessions of Friday morning, and Saturday morning and afternoon. Dr. Kraupner was introduced to the Society by Professor Hancock, Mr. Merriman by Professor C. N. Moore. Dr. Wilder's paper was presented by Professor R. L. Moore. The papers of Professor Reynolds, Dr. Kraupner, Professors Chittenden, MacDuffee, Wahlin, Cajori, Emch, and Sisam, Mr. Vandiver, Professor Smith, and Miss Schottenfels were read by title.

1. Professor W. E. Edington: *Theory of construction of group generators as substitutions.*

In general, to prove the existence of an abstract group generated by two or more operators obeying certain conditions requires the exposition of substitutions which fulfil all the required conditions on the generators. In this paper a theory for constructing such substitutions is discussed. Several theorems are proved giving general rules for finding substitutions fulfilling in certain cases the conditions

$s_1^m = s_2^n = (s_1 s_2)^k = 1$, and in some cases additional conditions. The theory of transpositions is made the basis of this theory.

2. Miss Bess M. Eversull: *On the summability of the triple Fourier series at points of discontinuity of the function involved.*

In a paper previously presented to the Society (this BULLETIN, vol. 28 (1922), p. 289) it was shown that if a function of three variables is finite and integrable (Lebesgue) its Fourier development is summable ($C1$) to the value of the function at all points within the region of continuity of the function. It is the purpose of the present paper to show by similar methods that the Fourier development of a function satisfying the above conditions is in certain cases summable ($C1$) at points of discontinuity of the function and to determine the values to which the series is summable. Results have been obtained where the point of discontinuity is such that all other points of discontinuity in the neighborhood lie on a plane or curved surface through that point, on two planes or curved surfaces having certain relative positions whose intersection passes through the point, and on three planes parallel to the coordinate planes whose common point of intersection is the point of discontinuity.

3. Mr. G. M. Merriman: *On a general theorem regarding divergent series, and its application to the double Fourier series.*

In the present paper a general theorem is proved which gives sufficient conditions that a double series, known to be summable ($C1$), should also be convergent. By a combination of this theorem with that concerning the summability of the double Fourier series, certain sufficient conditions for the convergence of the double Fourier series are obtained; these conditions are analogous to Dini's conditions in the case of the ordinary Fourier series, and were first introduced by Miss Hilda Geiringer (MONATSHEFTE FÜR MATHEMATIK UND PHYSIK, vol. 29 (1918)). The methods employed in proving the above results are generalizations of those used by Mr. S. Pollard in carrying through a similar investigation in the case of the ordinary Fourier series (PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 15 (1916)).

4. Professor C. N. Moore: *On necessary and sufficient conditions for convergence factors in double series.*

Sufficient conditions that a set of factors introduced into the terms of a double series summable by Cesàro's method should cause the resulting double series to be convergent have been given by the author in a paper that appeared in the TRANSACTIONS OF THIS SOCIETY in 1913. In the present paper some of these conditions are modified in such a manner that the resulting set has the property of being both necessary and sufficient. Furthermore, it is shown that if certain additional restrictions are made, we obtain the necessary and sufficient conditions that the function defined by the series with the convergence factors should approach the value to which the original series is summable when the convergence factors approach unity.

5. Professor C. N. Reynolds, Jr.: *Note on the map coloring problem.*

In this note it is shown that in Dr. Franklin's recent solution of the map coloring problem for any map of n regions where $n<N$ and $26 \leqq N \leqq 42$ (AMERICAN JOURNAL OF MATHEMATICS, vol. 44 (1922)), the N is subject to the narrower restriction $26 \leqq N \leqq 32$.

6. Professor R. L. Moore: *Concerning the prime parts of a continuum which separates its plane.*

In this paper it is shown that if, in a plane S, M is a bounded continuum which has more than one prime part and no prime part of M separates S, then in order that $S-M$ should be the sum of two mutually exclusive connected domains having M as their common boundary it is necessary and sufficient that the set G whose elements are the prime parts of M should be a simple closed curve of prime parts in the sense that it is disconnected by the omission of any two of its elements which are not identical. For a definition of prime part ("Primteil") of a continuum, see Hans Hahn, WIENER SITZUNGSBERICHTE, vol. 130 (1921).

7. Dr. W. Kraupner: *Integral solutions of the diophantine equations* $\xi_1^2+\xi_2^2+\cdots+\xi_n^2=\eta_1^2+\eta_2^2+\cdots+\eta_n^2$ *in the quadratic realm of rationality.*

This paper contains a generalization of the results obtained by Hancock in his paper in Liouville's JOURNAL, (8), vol. 4 (1921), p. 327. It is shown that there exist an infinite number of integral solutions of the quadratic diophantine equations $\xi_1^2+\xi_2^2+\cdots+\xi_n^2=\eta_1^2+\eta_2^2+\cdots+\eta_n^2$. The investigation is based upon the irreducible quadratic equation $x^2-2a_1x-a_2=0$, a_1 and a_2 being rational integers, which defines an algebraic integer in the quadratic realm of rationality.

8. Professor Harris Hancock: *On algebraic equations whose roots are trigonometric functions.*

This paper considers algebraic equations whose roots are trigonometric functions, their discriminants, realms of rationality etc., and a method of derivation of certain reciprocal equations.

9. Professor E. W. Chittenden: *Properties of abstract sets implied by properties of the class of all continuous functions.*

Let P be an abstract class and $\delta(p,q)$ a non-negative function such that $\delta(p,p)=0$. The late Professor A. D. Pitcher studied the class C of all continuous real functions of a variable p with range P and obtained among other results the conditions implying that the space P be a compact, connected, metric space (A. D. Pitcher and E. W. Chittenden, *On the foundations of the calcul fonctionnel of Fréchet*, TRANSACTIONS OF THIS SOCIETY, vol. 19 (1918)).

The present paper proposes to extend and complete the results of Professor Pitcher. It considers the class C in the more general system (P, K) where K is a relation defining the points of accumulation of the subsets of P. The paper contains a set of independent properties of C, a characterization of those spaces P which admit a nonconstant continuous function, and certain generalizations and modifications of previous results. It is found that with respect to the theory of continuous functions the notions point of accumulation and neighborhood (voisinage) are equivalent.

10. Professor A. D. Campbell: *The classification of linear families of conics in various domains.*

The domains considered in this paper are the ordinary complex domain, the real domain, and the Galois fields of order p^n $(p>2)$. The corresponding planes considered are the complex projective plane, the real projective plane, and the Galois field projective plane. The classes of conics and of pencils of conics in these planes are well known; as also the nets and three- and four-parameter linear families of conics in the complex domain, and nets in the above Galois fields.

This paper deals with the nets and three- and four-parameter families in the real domain, also the three- and four-parameter families in the Galois fields. A geometrical description of all these classes is given by means of apolarity. The classes of families in the other domains are obtained as sub-classes of those in the complex domain or by interpretation therefrom. The classes of three- and four-parameter families are obtained by apolarity from the classes of pencils and the types of conics respectively. The classes of nets of conics are obtained by use of the associated cubic curves in the plane of the parameters of the nets.

11. Professor Mary E. Sinclair: *The isoperimetric problem with variable end points.*

Using the derivatives of the extremal-integral, necessary conditions are found for a minimum in the usual isoperimetric problem with variable end points. These conditions, strengthened in the usual manner, are found to be sufficient for a weak minimum. Extension to the isoperimetric problem is made of the Bliss geometric interpretation of critical value of radius of curvature. If r_1, r_2, C_1 and C_2 are radii and centers of curvature of the fixed curves at the two end points, P_1 and P_2, there exists for every value of r_1 a critical value $\bar{r}_2$ of r_2, increasing with r_1, projective to r_1, and such that, according as P_1P_2 and $P_2\bar{C}_2$ have the same or opposite direction, C_2 must or must not lie on $P_2\bar{C}_2$.

Our conditions may be expressed in a form very close to those of Merrill for his special type of problem. Again, as with Dresden's results in the non-isoperimetric case, they can be expressed in terms of particular solutions of the differential equation, $\Psi(w) + \mu V = 0$, and they reduce to Dresden's if the isoperimetric condition is dropped.

12. Professor Mary E. Sinclair: *The hanging chain with end points variable on curves in a plane.*

The conditions for a minimum in the isoperimetric problem with variable end points are applied to the problem of the hanging chain of given length and uniform density. In the cases of one and of two variable end points a geometrical construction is given for the critical value, $\bar{r}_2$, of the radius of curvature.

13. Professor C. C. MacDuffee: *Covariants of differential forms of arbitrary order and degree.*

Covariants of differential forms of arbitrary order and degree have been studied by E. Pascal, ATTI DEI LINCEI, MEMORIE FISICHE, vol. 8 (1910). The definition of differential form used in this paper is essentially equivalent to that given by Pascal, but the subsequent treatment is quite different. A differential form of order h and degree r is defined as a polynomial of the type

$$\sum a_{i_1 i_2 \cdots i_m j_1 j_2 \cdots j_m} d^{i_1} x_{j_1} d^{i_2} x_{j_2} \cdots d^{i_m} x_{j_m},$$

the sum extending over all distinct partitions of r into positive summands $i_1, i_2, \ldots, i_m$, each $\leqq h$, and $j_1, j_2, \ldots, j_m$ being chosen with repetitions allowed from among the integers $1, 2, \ldots, n$, so as to yield all possible distinct terms. The coefficients a are functions of $x_1, x_2, \ldots, x_n$. The principal theorem obtained, an analogue of Hilbert's theorem, is that every relative covariant of weight ϵ in at most n distinct differential operators can be expressed as a function with constant coefficients of certain elementary covariants and their differentials multiplied by the ϵth power of the identical covariant $| d_j x_i |$. These elementary covariants are roughly analogous to the polars of the ground forms in the algebraic theory.

14. Professor G. A. Miller: *Number of cycles of the same order in any substitution group.*

This paper appears in the present number of this BULLETIN.

15. Professor I. A. Barnett: *Note on linear differential equations with constant coefficients.*

This note considers linear differential equations in general analysis whose coefficients are constant with respect to the independent variable. It is shown that in case the coefficient system is either symmetric or skew-symmetric the constituents of the fundamental set of solutions are all of the form of exponentials and not products of exponentials by powers of the independent variable. The result for the symmetric case is given without proof in a course of lectures of Professor Bôcher, and the statement is made that Weierstrass completely solved this case in 1858, ten years before he introduced elementary divisors.

16. Professor H. W. March: *Deflection of a rectangular plate clamped at its edges.*

The solution of the problem of the deflection of a rectangular plate clamped at its edges and under a given load has not previously been expressed in terms of functions which satisfy either the prescribed non-homogeneous differential equation or the corresponding homogeneous equation. W. Ritz (Crelle's JOURNAL, 1909) gave a solution in terms of a series of functions which satisfy the boundary conditions but which do not individually satisfy either differential equation. The classical method of treating such problems has failed in this case because it has not been found possible to combine known solutions of the homogeneous equation with a solution of the non-homogeneous equation in such a way as to satisfy all of the boundary conditions. In this paper an apparently new type of solution of the homogeneous equation is given and the problem in question is solved for the case of a uniform load. It appears that the method can be readily extended to more general cases. The present method gives a result that is much better adapted than that of Ritz to numerical computation and to the discussion of the elastic surface.

17. Dr. Norbert Wiener: *A generalization of the Dirichlet problem.*

It is shown that the Dirichlet problem is a particular case of a problem solvable for any bounded open region and any continuous boundary conditions.

18. Professor Arnold Dresden: *On Brouwer's contributions to the foundations of mathematics.*

This paper has appeared in the January-February number of this BULLETIN.

19. Professor G. E. Wahlin: *On the application of the theory of ideals to diophantine equations.*

This paper has appeared in the March-April number of this BULLETIN.

20. Professor Florian Cajori: *J. H. Rahn's mathematical symbols.*

The symbols in Rahn's Algebra written in German (1659) are compared with those found in the English translation (1668). There are set forth certain incorrect statements due to John Collins and John Wallis regarding John Pell's rôle in the preparation of the original edition as well as of the translation.

21. Professor Dunham Jackson: *A generalized problem in weighted approximation.*

In a recent note in this BULLETIN (June, 1923), the author has discussed the convergence of the trigonometric sums $T_n(x)$ determined as approximations to a given continuous function $f(x)$ by the condition that the integral of the square of the error, multiplied by a weight-function $\varrho(x)$, shall be a minimum. The function $\varrho(x)$ was assumed to be bounded, and to have a positive lower bound. The purpose of the present paper is to deal with the somewhat less simple problem of convergence that arises if the greatest lower bound of $\varrho(x)$ is zero. There is occasion incidentally to develop the theory of the existence and uniqueness of discontinuous approximating functions of a high degree of generality. The discussion is put in such form as to be applicable when the square of the error is replaced by the mth power of its absolute value.

22. Professor E. B. Van Vleck: *On the zeros of polynomials.*

In a very interesting memoir in the ANNALES DE L'ÉCOLE NORMALE SUPÉRIEURE, 1923, Montel shows, among other results, that, when the first $r+1$ coefficients $1, a_1, \ldots, a_r$ of the equation

$$1 + a_1 x + a_2 x^2 + \cdots + a_r x_r + a_{r+1} x^{r+r_1} + \cdots + a_{r+n} x^{r+r_n} = 0 \quad (r_i \geqq i)$$

are given, the r roots of smallest absolute value lie within a circle whose radius depends only upon r and the number of terms $n+r+1$ in the equation, and he determines this radius for $r=1$ and $a_1 = 1$, and for $r=2$, $a_1 = 0$. In this paper these results are supplemented by the determination of such a radius for any value of r when each $r_i = i$. It is also ascertained under what conditions an arbitrarily selected set of r coefficients a_i will limit in absolute value the r smallest roots and their upper limit is then determined when the degree of the equation is given.

23. Professor Arnold Emch: *On the Weddle surface and analogous loci.*

This paper starts from the geometric definition of the Weddle surface as the locus of the vertices of all quadric cones through six points in space. The locus of the vertices of quadric cones through seven points is a sextic S of genus 3. All quadrics through the seven points pass through an eighth fixed point; hence the sextic S is also the locus of vertices of quadric cones through the 8 base points of the net of quadrics. S cuts each of the 28 lines, connecting the base points, in two points. There are 28 Weddle surfaces through S, each having 6 of the 8 base points as nodes. The problem is also investigated for the cubic and Geiser transformations connected with the 8 base points. The locus of the vertices of cubic cones through 10 points B is found to be a surface of order 15 with the B's as cubic nodes, passing through each of the 45 lines joining the B's. The locus of the vertices of cubic cones through 11 points is a curve of order 189 cutting each of the 45 lines in 12 points; its genus is determined. Finally the paper gives the loci of the vertices of $n$ic cones through $(n(n+3)/2)+1$ and through $(n(n+3)/2)+2$ points in space.

24. Professor Tomlinson Fort: *Note on Dirichlet and factorial series.*

In a note published in the TRANSACTIONS OF THIS SOCIETY, January, 1922, the author studied a class of series which included ordinary Dirichlet series and factorial series as special cases. The present paper extends the results of that note.

25. Dr. R. L. Wilder: *On the dispersion sets of a connected point set.*

A proper subset H of a point set M is called a dispersion set of M if $M-H$ is totally disconnected. H is called a primitive dispersion set of M if no proper subset L of H exists such that $M-L$ is totally disconnected. A special study is made of the properties of connected sets that contain primitive dispersion sets. If H is a primitive dispersion set of a connected set M consisting of a finite number of points (the existence of connected sets with primitive dispersion sets consisting of a finite number of points has recently been established by Knaster and Kuratowski, (FUNDAMENTA MATHEMATICAE, vol. 2 (1921), pp. 206–255), it is shown that M is the sum of a finite number of mutually exclusive connected sets, each of which consists of more than one point and contains one and only one point of H. It is also shown that if a connected set M contains a primitive dispersion set H which consists of a finite number of points, then H is the only primitive dispersion set of M.

26. Mr. J. A. Shohat: *The theory of closure of Tchebychef's polynomials for an infinite interval.*

If $\varphi_n(x)$ $(n = 0, 1, 2, \ldots)$ denote a system of Tchebychef's orthogonal and normal polynomials corresponding to the interval (a, b) with the characteristic function $p(x)$ not negative in (a, b), then $\int_a^b p(x)\Phi_m(x)\Phi_n(x)dx$ is equal to unity if m and n are equal, and to zero if m and n are unequal. In the case of a finite interval (a, b), every function $f(x)$ which with its square is integrable in (a, b) satisfies an important *closure equation.*

In this paper, the author investigates the closure equation for an infinite interval (a, ∞), and shows that the closure equation holds if, for x sufficiently large, $p(x) < e^{-|x|^{\lambda}}$, with $\lambda > 1/2$ for $a = 0$, and $\lambda > 1$ for $a = -\infty$. Particular cases are the polynomials of Hermite-Tchebychef and those of Laguerre-Tchebychef. The closure equation is not true in general. A general example is given of a system of Tchebychef's polynomials in $(0, \infty)$ for which the closure equation is not true.

27. Professor C. H. Sisam: *On curves whose first polars contain a pencil of lines.*

The algebraic plane curves for which every right line through a fixed point is a component of a first polar with respect to the curve are completely determined.

28. Professor H. T. Davis: *Integral equations as differential equations of infinite order.*

In this paper an equivalence is pointed out between linear differential equations of infinite order and integral equations of either the Volterra or Fredholm type which have solutions analytic within the interval of integration. An existence theorem for the differential equation of infinite order with constant coefficients in both the homogeneous and non-homogeneous cases is given and the solutions are shown to satisfy a singular integral equation of Volterra type.

29. Professor L. E. Dickson: *On the theory of numbers and generalized quaternions.*

This paper opens with a new simple proof by the method of descent of Bachet's theorem that every positive integer is a sum of four integral squares, and the new generalization that the roots of the four squares may be chosen to satisfy four assigned linear congruences.

Next there is given a simpler proof of the author's formulas for all integral solutions of $x^2+y^2+z^2+w^2=uv$. These formulas and generalized quaternions occur in the author's *Algebras and Their Arithmetics*, University of Chicago Press, 1923, pp. 187–198. The present paper discusses an arithmetic of generalized quaternions for which there exists a greatest common left divisor, although it cannot be found by a process of division with remainders of decreasing norms. The new method is first applied to the arithmetic of quaternions and shows the difference between the integral quaternions of Lipschitz and those of A. Hurwitz. Application is made to diophantine equations. The memoir will appear in the AMERICAN JOURNAL.

30. Professor L. E. Dickson: *Quadratic fields in which factorization is always unique.*

This paper appears in this number of this BULLETIN.

31. Professor A. B. Coble: *Geometric interpretation of the expression of an algebraic form as a determinant.*

Dickson and Everett have discussed the possibility of expressing a given algebraic form in k variables as a determinant whose elements are forms of the same order. It is the purpose of this paper to show that this possibility rests when $k=3$ upon the existence of coresidual point groups on the given algebraic curve; when $k=4$ upon the existence of coresidual linear systems of curves on the given algebraic surface; etc. If the determinant is to be symmetric, contact systems come into play.

32. Mr. H. S. Vandiver: *Sets of three consecutive integers which are quadratic or cubic residues of primes.*

In this paper the author proves several theorems such as the following: If p is of one of the forms $40k+1$,

$40k+11$, $40k+29$, $40k+39$, then $a+1$, a, $a-1$ are all quadratic residues of p, where $a^2 \equiv 5 \pmod{p}$ and 5 is a biquadratic residue of p.

33. Mr. H. S. Vandiver: *On Kummer's memoir of* 1857 *concerning Fermat's last theorem.*

In two previous papers under this title the author pointed out that Kummer made some errors in the proofs of certain theorems. In the first paper (PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 6 (1920), pp. 266–269) these theorems were numbered I to IV. The second paper (this BULLETIN, vol. 28 (1922), pp. 400–407) contained proofs of Theorems I and IV. The present paper is devoted to the proof of Theorem III and an extension of Theorem II is also proved which enables the author to establish the result: If the class numbers of the field $\Omega(\alpha)$, $\alpha = e^{2i\pi/p}$, is divisible by p but not by p^2, then $x^p + y^p + z^p = 0$ is not satisfied in integers in the field $\Omega(\alpha + \alpha^{-1})$, not zero, p being an odd prime. This includes Kummer's results as special cases.

34. Professor H. L. Smith: *Necessary and sufficient conditions for the existence of a class of Stieltjes integrals.*

In this paper necessary conditions are obtained for the existence of the Stieltjes integral $\int_0^1 \psi(t) d\varphi(t)$ which are also sufficient if the associated curve $x = \varphi(t)$, $y = \psi(t)$ $(0 \leqq t \leqq 1)$ is continuous and such that its multiple points are a reducible set. The conditions remain sufficient for discontinuous curves which can be embedded in continuous curves of a certain character. A corollary is that every simple closed (continuous) curve is squarable if the line integral $\int y dx$ exists over it. An example is given of a simple closed squarable curve over which the line integral $\int y dx$ does not exist.

35. Professor S. Lefschetz: *On polyhedra in euclidean n-space.*

In this paper are first developed certain properties of covering manifolds, which are then applied to the proof of this theorem: A bounded, non-singular, $(n-1)$-dimensional polyhedron in euclidean n-space bounds a finite region which together with the polyhedron is homeomorphic to an n-cell plus its boundary.

36. Professor R. L. Moore: *An extension of the theorem that no perfect set is countable.*

Sierpinski has shown that if the closed and bounded point set M is the sum of a countable number of mutually exclusive closed point sets $M_1, M_2, M_3, \ldots$ then M is not connected. In the present paper it is shown that, if the additional requirement is imposed that the point sets $M_1, M_2, M_3, \ldots$ should be connected, then not every one of them contains a limit point of the sum of all the others. It is shown that this latter conclusion does not hold if either this additional requirement or the requirement that M be bounded be omitted.

37. Professor Louis Brand: *Note on the integral theorems of vector analysis.*

The integral theorems of vector analysis fall into two principal groups. In the first group the volume integral of $\nabla\omega$ where ω denotes a scalar, vector, or dyadic point function is transformed into an integral of $\mathbf{u}\omega$ over the bounding surface, $\mathbf{u}$ being the unit surface normal. In the present note it is shown that the second class of theorems, which transform surface to circuit integrals, may be deduced from the first class. The proof hinges on the fact that $\mathbf{u}$ may be regarded as the gradient of a scalar point function whose level surfaces are parallel to the original surface.

38. Miss I. M. Schottenfels: *The Kurschak field of complex numbers.*

The author extends to the complex number field the results obtained by J. Kürschak in his paper in Crelle's JOURNAL, vol. 142 (1913). The work of Ostrowski (ACTA MATHEMATICA, vol. 41 (1918)) is also reorganized and extended.

39. Miss I. M. Schottenfels: *The error in Hartog's proof of the Zermelo theorem.*

The author points out an incompleteness in Hartog's proof of Zermelo's theorem which appeared in MATHEMATISCHE ANNALEN, vol. 76 (1915) and furnishes a complete proof, far simpler and more concise than those published by Kuratowski in volume 2 of the FUNDAMENTA MATHEMATICAE.

ARNOLD DRESDEN,
Assistant Secretary.

A CHARACTERIZATION OF SURFACES OF TRANSLATION*

BY E P. LANE

Let the non-homogeneous cartesian coordinates x, y, z of an arbitrary point on a surface be given as analytic functions of two independent variables u, v by equations of the form

$$x = U_1 + V_1, \quad y = U_2 + V_2, \quad z = U_3 + V_3,$$

where U_1, U_2, U_3 are functions of u alone and V_1, V_2, V_3 are functions of v alone. Such a surface is called a surface of translation because it can be regarded in two ways as generated by the motion of a curve which is translated so that its various points describe congruent curves. It is the purpose of this note to give another characterization of surfaces of translation, based on some notions which have arisen in the study of projective differential geometry.

Since, at each surface point P, the tangents to the curves $u =$ const. and $v =$ const. separate the two asymptotic tangents harmonically,† it follows that the parametric net is a conjugate net. Therefore the tangents to the curves $u =$ const., constructed at the points of a fixed curve $v =$ const., form a developable surface. The point where the tangent to the curve $u =$ const. through P touches the edge of regression of this developable is one of the two *ray points* of P, with respect to the parametric conjugate net. The other ray point is similarly defined on the other parametric tangent. The line joining these two ray points is called the *ray* of P. The totality of rays of all the points on the surface constitutes a congruence called the *ray congruence* of the fundamental conjugate net. And the curves on the surface which correspond to the developables of the ray congruence are the *ray curves.* These definitions serve for any conjugate net on an arbitrary surface, and are due to Wilczynski.

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* Presented to the Society, April 25, 1924.

† Lie, MATHEMATISCHE ANNALEN, vol. 14 (1879), pp. 332–337.

But since our surface is a surface of translation and the parametric conjugate net is the net of generating curves, it follows that the developable circumscribing the surface along a curve $u = \text{const.}$, or a curve $v = \text{const.}$, is a cylinder. Therefore the ray points of every surface point are both at infinity, and the ray of every surface point is entirely at infinity. The ray congruence consists therefore of lines all lying in one plane, namely, the plane at infinity, and the ray curves are indeterminate.

One of my students, Mr. M. L. MacQueen, in his master's thesis, University of Wisconsin, 1923, has studied the class of surfaces on each of which there exists a conjugate net with indeterminate ray curves. He shows that the rays of such a conjugate net all lie in one plane, which we may call the *ray plane* of the net. He shows, furthermore, that every surface on which there exists a conjugate net with indeterminate ray curves can be projected into a surface of translation by precisely the projective transformation which carries the ray plane into the plane at infinity.

We may therefore characterize surfaces of translation by saying that *surfaces of translation are the only surfaces that have the property that on each of them there exists a conjugate net with indeterminate ray curves and with ray plane at infinity.*

THE UNIVERSITY OF CHICAGO

CONCERNING A SUGGESTED AND DISCARDED GENERALIZATION OF THE WEIERSTRASS FACTORIZATION THEOREM*

BY L L. DINES

1. *Introduction.* A basic theorem in the theory of analytic implicit functions, proven by Weierstrass, may for our purpose be stated as follows:

Let (1°), $f(y; x_1, \ldots, x_n)$ *be analytic at the origin and vanish there*; *and* (2°), $f(y; 0, \ldots, 0)$ *be not identically zero. Then, throughout a certain neighborhood of the origin, there holds an identity of the form*†

$$f(y; x_1, \ldots, x_n) = (P_0 y^m + P_1 y^{m-1} + \cdots + P_m)\, g(y; x_1, \ldots, x_n) \tag{1}$$

where $g(y; x_1, \ldots, x_n)$ *is analytic and does not vanish at the origin*; *and where* $P_j, j = 0, 1, \ldots, m$, *is an analytic function of* $x_1, x_2, \ldots, x_n$ *and for* $j > 0$ *vanishes when* $x_1 = x_2 = \cdots = x_n = 0$.

In his *Madison Colloquium Lectures,* Osgood called attention to the fact that the hypothesis (2°) may be omitted in the case of a function $f(y; x)$ of only two variables, without disturbing the validity of the conclusion; and suggested tentatively but without proof that the theorem in this stronger form might be true for a function $f(y; x_1 \ldots, x_n)$ of $n+1$ variables. In a later paper‡ he showed very definitely that the theorem is *not* true in general, with the omission of the hypothesis (2°). His proof of this fact consisted in the exhibition of a function of the form

$$f(y; x_1, x_2) \equiv x_1 - x_2 F(y) \tag{2}$$

which is not factorable in the form (1).

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*Presented to the Society, September 7, 1923.

† The theorem of Weierstrass states, in addition, that $P_0 = 1$, and that m is equal to the degree of the term of lowest degree in the series $f(y; 0, \ldots, 0)$.

‡ TRANSACTIONS OF THIS SOCIETY, vol. 17, page 4.

The object of the present note is to consider further the possibility of an identity of form (1) in case (2°) is not satisfied. In the next article, we shall derive a simple necessary condition (Theorem I) for the existence of such an identity, and we shall show in particular (Corollary II) that a function of the form (2) is not factorable in the manner under consideration unless $F(y)$ is a *rational* function of y.

2. *A Necessary Condition.* If an identity of form (1) holds, then there is determined a second identity

$$\Phi\cdot f = P \tag{3}$$

where $\Phi = 1/g$, and $P = P_0 y^m + P_1 y^{m-1} + \cdots + P_m$. Let us suppose the power series expansions of the functions Φ, f, and P to be written in the following forms:

$$\begin{aligned} \Phi(y; x_1, \ldots, x_n) &= \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots \\ f(y; x_1, \ldots, x_n) &= f^{(k)} + f^{(k+1)} + f^{(k+2)} + \cdots \\ P(y; x_1, \ldots, x_n) &= P^{(k)} + P^{(k+1)} + P^{(k+2)} + \cdots \end{aligned} \tag{4}$$

where $\Phi^{(i)}, f^{(i)}$, and $P^{(i)}$ are homogeneous polynomials of degree i in $x_1, \ldots, x_n$. The polynomials $\Phi^{(i)}$ and $f^{(i)}$ have coefficients which are power series in y, while the coefficients of $P^{(i)}$ contain y to no higher power than the m th. Since g is analytic and does not vanish at the origin, it follows that $\Phi^{(0)}$ has a constant term different from zero. From the identity (3) it follows that f and P must begin with terms of the same degree in $x_1, \ldots, x_n$, and since we are considering the case in which (2) is not satisfied, i. e., in which $f(y; 0, \ldots, 0) \equiv 0$, we shall assume that this degree k is greater than zero.

Substituting the series (4) in (3), and equating polynomials of lowest degree on the two sides we obtain the identity

$$\Phi^{(0)} f^{(k)} = P^{(k)}. \tag{5}$$

A homogeneous polynomial of degree k contains power products of the form $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ where $j_1 + j_2 + \cdots + j_n = k$. Let the sequence of all such power products in any definite order be denoted by $X_1, X_2, \ldots, X_N$. Then we have

$$\begin{aligned} f^{(k)} &= f_1^{(k)} X_1 + f_2^{(k)} X_2 + \cdots + f_N^{(k)} X_N, \\ P^{(k)} &= P_1^{(k)} X_1 + P_2^{(k)} X_2 + \cdots + P_N^{(k)} X_N, \end{aligned}$$

where $f_i^{(k)}$ is, for every i, a power series in y, while $P_i^{(k)}$ is, for every i, a polynomial in y. Substituting these expressions for $f^{(k)}$ and $P^{(k)}$ in (5) and equating coefficients of like power products, we find that

$$\Phi^{(0)} f_i^{(k)} = P_i^{(k)}, \quad i = 1, 2, \ldots, N.$$

Therefore

$$\Phi^{(0)} = \frac{P_1^{(k)}}{f_1^{(k)}} = \frac{P_2^{(k)}}{f_2^{(k)}} = \cdots = \frac{P_n^{(k)}}{f_N^{(k)}}; \tag{6}$$

or otherwise expressed, for every i and j for which $f_i^{(k)}$ and $f_j^{(k)}$ are different from zero,

$$\frac{f_i^{(k)}}{f_j^{(k)}} = \frac{P_i^{(k)}}{P_j^{(k)}} = \textit{a rational function of } y.$$

Hence we may state the following theorem.

THEOREM I. *If the leading homogeneous polynomial in* $x_1, x_2, \ldots, x_n$ *of* $f(y; x_1, \ldots, x_n)$ *be*

$$f^{(k)} = f_1^{(k)} X_1 + f_2^{(k)} X_2 + \cdots + f_N^{(k)} X_N$$

where $X_1, X_2, \ldots, X_N$ *are the various power products of degree* k *in* $x_1, x_2, \ldots, x_n$ *and* $f_1^{(k)}, f_2^{(k)}, \ldots, f_N^{(k)}$ *are power series in* y, *then a necessary condition that* f *admit factorization in the form* (1) *is that the ratio formed from each pair of non-vanishing coefficients* $f_i^{(k)}, f_j^{(k)}$ *be a rational function of* y.

COROLLARY I. *If any one of the sequence of coefficients* $f_1^{(k)}, f_2^{(k)}, \ldots, f_N^{(k)}$ *is a rational function, then all of these coefficients must be rational if* $f(y; x_1, \ldots, x_n)$ *is to admit factorization in form* (1).

COROLLARY II. *A necessary condition that the function*

$$f(y; x_1, x_2) \equiv x_1 - x_2 F(y)$$

admit factorization in the form (1) *is that* $F(y)$ *be a rational function of* y.

3. *Sufficient Conditions.* If the function $f(y; x_1, \ldots, x_n)$ is homogeneous in $x_1, \ldots, x_n$, that is if $f = f^{(k)}$, the condition of our theorem is *sufficient*. For if there exist poly-

nomials in y, viz. $P_1^{(k)}, P_2^{(k)}, \ldots, P_N^{(k)}$ such that $f_1^{(k)} : f_2^{(k)} : \cdots : f_N^{(k)} = P_1^{(k)} : P_2^{(k)} : \cdots : P_N^{(k)}$, there will exist such polynomials in which the lowest powers of y are respectively equal to the lowest powers of y in the series $f_1^{(k)}, f_2^{(k)}, \ldots, f_N^{(k)}$. Hence there will exist a series $\Phi^{(0)}(y)$ with non-vanishing constant term satisfying (6). It is then easy to construct the polynomial $P^{(k)}$ satisfying the relation $\Phi^{(0)} f^{(k)} = P^{(k)}$. In view of the homogeneity of f this relation may be written in the form $\Phi f = P$. From this may be obtained the equivalent form (1').

If $f(y; x_1, \ldots, x_n)$ is not homogeneous in $x_1, x_2, \ldots, x_n$ the condition of the theorem is *not sufficient.*

THE UNIVERSITY OF SASKATCHEWAN

THE CLASS NUMBER RELATIONS IMPLICIT IN THE DISQUISITIONES ARITHMETICAE*

BY E. T BELL

1. *Introduction.* The point of this note is its moral, which is to the effect that in arithmetic attention to trifles sometimes leads to beautiful and recondite truths. In particular, certain important expansions of Kronecker and Hermite relating to the number $F(n)$ of uneven classes of binary quadratic forms of negative determinant $-n$ are implicit in § 292 of the *Disquisitiones Arithmeticae* of Gauss, and might have been read off from there at a glance by anyone familiar with the *Fundamenta Nova* of Jacobi, thirty years before Kronecker first came upon them by the devious route of complex multiplication. The relevant trifle in this instance is changing the sign of an arbitrary constant throughout an algebraic identity.

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* Presented to the Society, April 5, 1924.

It is well known that the tenth of Kronecker's relations* is equivalent to his expansion

$$\text{(XI)}\quad 12\sum_{0}^{\infty} E(n)\, q^{n} = \left[\sum_{-\infty}^{\infty} q^{n^{2}}\right]^{3} = \vartheta_{3}^{3}(q),\ E(n) = F(n) - F_{1}(n),$$

where $F_1(n)$ is the number of even classes; which is merely the algebraic expression of the theorem of Gauss (loc. cit.) on the number of representations of n as a sum of three squares; and it has recently been remarked by Mordell† that Hermite's important formula

$$(1)\quad \sum_{0}^{\infty} F(8n+3)\, q^{8n+3} = (q + q^{9} + q^{25} + \cdots)^{3} = \tfrac{1}{8}\vartheta_{2}^{3}(q^{4}),$$

and those of Kronecker (changed from k, K to ϑ notation),

$$(2)\quad 4\sum_{0}^{\infty} F(4n+2)\, q^{4n+2} = \vartheta_{3}(q^{4})\,\vartheta_{2}^{2}(q^{4}),$$

$$(3)\quad 4\sum_{0}^{\infty} F(4n+1)\, q^{4n+1} = \vartheta_{2}(q^{4})\,\vartheta_{3}^{2}(q^{4}),$$

are implicit in (XI) and can be obtained from it by elementary algebra. Note in passing that (1) is simply the algebraic translation of § 293 of the *Disquisitiones Arithmeticae.* Both Kronecker and Hermite made much use of (1), (2), (3) in their successive simplification of the classical relations and in further deductions, and these identities reappear repeatedly in works of later writers. It is interesting therefore to see that (1), (2), (3) follow from (XI) even more simply than as shown by Mordell; in fact they result from (XI) by changing the sign of q. That is, (XI), (1), (2), (3) are immediate consequences of Gauss' theorem. The deduction of class number relations from these four identities by combining them with the expansions of Jacobi is not only classical but obvious, and it seems surprising that the immediate step of translating the application which Gauss made of his theorem, to representations as sums of four squares, into the equivalent class number relation was not taken much earlier than it was.

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* Cf. Dickson's *History,* vol. 3, pp. 108, 109, 113.

† MESSENGER OF MATHEMATICS, vol. 45 (1915) pp. 78, 79.

2. *Consequences of Gauss' Theorem.* In (XI) change q into $-q$:

$$\text{(XI}_1\text{)} \qquad 12\sum_0^{\infty}(-1)^n E(n)q^n = \left[\sum_{-\infty}^{\infty}(-1)^n q^{n^2}\right]^3 = \vartheta_0^3(q).$$

Let $n_1, n_2, \ldots$ denote arbitrary integers $\gtreqless 0$, $m_1, m_2, \ldots$ odd integers $\gtrless 0$, and write $N(n = x^2 + y^2 + z^2)$ for the number of integer solutions $x, y, z \gtreqless 0$ of $n = x^2 + y^2 + z^2$. Equate coefficients of q^n in (XI$_1$):

$$\begin{aligned} 12(-1)^n E(n) & \\ = N(n &= 4n_1^2 + 4n_2^2 + 4n_3^2) + N(n = 4n_1^2 + m_2^2 + m_3^2) \\ + N(n &= m_1^2 + 4n_2^2 + m_3^2) + N(n = m_1^2 + m_2^2 + 4n_3^2) \\ - N(n &= m_1^2 + m_2^2 + m_3^2) - N(n = m_1^2 + 4n_2^2 + 4n_3^2) \\ - N(n &= 4n_1^2 + m_2^2 + 4n_3^2) - N(n = 4n_1^2 + 4n_2^2 + m_3^2), \\ = N(n &= 4n_1^2 + 4n_2^2 + 4n_3^2) + 3N(n = 4n_1^2 + m_2^2 + m_3^2) \\ - N(n &= m_1^2 + m_2^2 + m_3^2) - 3N(n = m_1^2 + 4n_2^2 + 4n_3^2). \end{aligned}$$

Since

$$\begin{aligned} 4n_1^2 + 4n_2^2 + 4n_3^2 &\equiv 0 \mod 4, \\ 4n_1^2 + m_2^2 + m_3^2 &\equiv 2 \mod 4, \\ m_1^2 + m_2^2 + m_3^2 &\equiv 3 \mod 8, \\ m_1^2 + 4n_2^2 + 4n_3^2 &\equiv 1 \mod 4, \end{aligned}$$

we have $12E(n) = N(4n = 4n_1^2 + 4n_2^2 + 4n_3^2)$, which is merely (XI) again, and

$$\begin{aligned} N(8n+3 = m_1^2 + m_2^2 + m_3^2) &= 12E(8n+3), \\ N(4n+2 = 4n_1^2 + m_2^2 + m_3^2) &= 4E(4n+2), \\ N(4n+1 = m_1^2 + 4n_2^2 + 4n_3^2) &= 4E(4n+1); \end{aligned}$$

which are equivalent respectively to

$$\begin{aligned} (\textstyle\sum q^{m_1^2})^3 &= 12\textstyle\sum q^{8n+3}E(8n+3), \\ \textstyle\sum q^{4n_1^2} \times (\textstyle\sum q^{m_1^2})^2 &= 4\textstyle\sum q^{4n+2}E(4n+2), \\ \textstyle\sum q^{m_1^2} \times (\textstyle\sum q^{4n_1^2})^2 &= 4\textstyle\sum q^{4n+1}E(4n+1). \end{aligned}$$

But these are identical respectively with (1), (2), (3), since $E(8n+3) = \tfrac{2}{8}F(8n+3)$, $E(4n+2) = F(4n+2)$,

$$E(4n+1) = F(4n+1).$$

THE UNIVERSITY OF WASHINGTON

NUMBER OF CYCLES OF THE SAME ORDER IN ANY GIVEN SUBSTITUTION GROUP*

BY G. A. MILLER

1. *Introduction.* If G is a transitive substitution group and if the subgroup composed of all the substitutions of G which omit a given letter is of degree $n-\alpha$, then there are exactly α substitutions involving no letters except possibly those of G which are commutative with every substitution of G. These α substitutions include the identity. If $\alpha > 1$ the remaining $\alpha-1$ substitutions may or may not appear in G. From this well known theorem, it follows directly that G involves exactly $\alpha-1$ sets of conjugate cycles which are such that no two distinct cycles of the set have a common letter. Each of these cycles appears in g/n different substitutions of G, where g denotes the order of G. A necessary and sufficient condition that a transitive substitution group be regular is that no two of its sets of conjugate cycles have a common letter.

When no two conjugate cycles of G have a letter in common it is evident that every pair of cycles in a set of conjugates must be commutative, but these cycles may also be commutative when G is non-regular. When G involves at least one set of conjugate cycles which has the property that every pair of cycles in the set is composed of commutative cycles, G must be imprimitive unless all these cycles involve the same letters and are also of prime order. In this special case, G is evidently always primitive. From the fact that a cycle of prime degree p is transformed into each of its various powers which are incongruent to 1 (mod p) only by substitutions of degree $p-1$ on the letters of this cycle it results directly that such substitutions involve cycles that are not commutative for every such power. Hence we have the following theorem.

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* Presented to the Society, December 29, 1923.

THEOREM I. *If every set of conjugate cycles of a transitive substitution group is composed of relatively commutative cycles, then the subgroups composed separately of all the substitutions of the group which omit a letter must omit a number of letters which is divisible by the product of the prime factors of the order of the group.*

In particular, it results from this theorem that a transitive group which has the property that all of its sets of conjugate cycles are composed of cycles that are relatively commutative must also involve at least one set of conjugate cycles that is composed of cycles such that no two of them have a common letter.

Suppose that G is intransitive and involves λ transitive constituents of orders $g_1, g_2, \ldots, g_\lambda$, respectively. Let k represent the order of an arbitary cycle of some substitution of G. From the fact that the number of the letters contained in every set of conjugate cycles of any substitution group is equal to the order of the group, whenever each cycle is counted for every substitution in which it appears, it results directly that the numbers of the cycles of order k in the transitive constituents of G are, respectively,

$$\frac{m_1 g_1}{k}, \quad \frac{m_2 g_2}{k}, \quad \ldots, \quad \frac{m_\lambda g_\lambda}{k},$$

where $m_1, m_2, \ldots, m_\lambda$ are positive integers or zero, and that the number of cycles of order k contained in G is mg/k, where g is the order of G and $m = m_1 + m_2 + \cdots + m_\lambda$. It should be noted that this theorem is independent of the way in which G is constructed by means of its transitive constituents, and that there are an infinite number of possible groups for every arbitrary pair of values of m and k, since all of the cycles of the same order contained in the symmetric group of degree n are conjugate under this group.

In particular, it results from the theorem noted above that if at least two of the transitive constituents of G involve cycles of the same order, the number of letters found in all of the cycles of this order contained in G must always exceed the order of G, each cycle being counted for every substitution in which it appears. It is evident that the numbers $m_1, m_2, \ldots, m_\lambda$, cannot exceed the degrees of the corresponding transitive constituents diminished by one, and when they attain these maximal values k must be a prime number p and all the cycles of the corresponding constituents must be of order p. In particular, the order of such a constituent must be of the form p^m. It should also be noted that all the substitutions of a given order k may be conjugate under the group while the cycles of this order do not constitute a single set of conjugates, and that conversely all the cycles of order k contained in a group may constitute a single set of conjugates while the substitutions of this order do not have this property.

2. *Smallest Number of Cycles of Order p in a Substitution Group of Order p^m.* If G is a substitution group of order p^m, where p is a prime number, it results directly from the theorems noted above that the number of cycles of order p contained in G cannot be less than g/p, and this minimum can be attained only when G is transitive. It is also evident that this minimum cannot be attained unless $p = 2$, since all the cycles of order p must be conjugate under G when it is attained. Hence we shall assume in the rest of this section, unless the contrary is stated, that $p = 2$, and that the number of the transpositions contained in the substitutions of G is exactly $g/2$.

Since all of these transpositions are conjugate under G, they must appear in an invariant substitution of G. In fact, in every substitution group of order p^m, where p is any prime number, all the cycles of order p that appear in an invariant substitution of order p constitute a complete set of conjugates and must therefore appear in more than one substitution whenever the group is non-regular. In the particular case under consideration, G involves only one set of conjugate cycles of order 2, and hence the substitutions of order 2 contained in G, if there is more than one such substitution, generate an abelian invariant sub-group.

Except in the trivial case when $g=2$ there must be a substitution s of order 4 in G which permutes all the transpositions of the invariant substitution of order 2 contained in G, since the transitive group according to which these transpositions are transformed contains an invariant substitution of order 2. The group generated by s and the substitutions contained in G whose orders divide 2, is composed of substitutions of order 4 and of degree n, where n is the degree of G, in addition to the subgroup composed of the substitutions whose orders divide 2. Hence all of the substitutions of order 4 have a common square and the group generated by them is abelian.

If G involves no substitutions whose orders exceed 4, the cycles of the invariant substitution of order 2 contained in the transitive group G must be transformed under G according to a regular group, since this group is transitive and involves only substitutions of order 2 besides the identity. Hence all of the substitutions of order 4 must have a common square and G can involve only one substitution of order 2, as otherwise all of the subgroups corresponding to subgroups of order 2 in the regular group according to which the cycles of order 2 are permuted could not be abelian. The order of this regular group cannot exceed 4, since a substitution of order 4 must transform into its inverse every substitution of this order which is not generated by it. This proves the following theorem.

THEOREM II. *If a transitive substitution group of order* p^m, *where* p *is a prime number, involves no substitution of order* p^3, *and if the number of its cycles of order* p *is* p^{m-1}, *then it must be regular and simply isomorphic with one of the following three groups: the quaternion group, the cyclic group of order* 4, *or the group of order* 2.

When $m=n-2$ for the cycles of a given order k found in the transitive group G of degree n, m must be unity for the cycles of the other possible order and the latter cycles must constitute a single set of conjugates under G. It was noted above that k is either 2 or 4 when the order of G is a power of a prime number. From the theorem at the end of § 1, it results that when $k = 2$, G must be one of three groups. The fact that G must be the octic group when $k = 4$ will be proved in the following section.

3. *Minimum Number of Cycles of Order* 4. When the cycles of order 4 in the transitive group G constitute a single set of conjugates, the subgroup composed of all the substitutions of G which omit a letter must involve at least one substitution besides the identity which omits at least two letters, since G cannot be regular and each of the cycles of the invariant substitution of order 2 contained in G is of order 2. It will first be proved that the cycles of order 2 found in the squares of the cycles of order 4 contained in G may be assumed to appear in the invariant substitution of order 2.

If this were not the case none of these squares could be commutative with all the cycles of this invariant substitution, and hence these cycles would be transformed according to cycles of order 4 by the cycles of order 4 contained in G. The transitive group according to which these cycles are transformed has a degree which is one-half the degree of G, and hence two cycles of order 4 in G would correspond to one cycle of this order in this transitive group. The cycles of order 4 in G would therefore be transformed under G according to an imprimitive group having two letters in each of these systems of imprimitivity. In the transitive group according to which these cycles would be transformed, there would therefore be again a single set of conjugate cycles of order 4, while each of the remaining cycles would be of order 2. Since this transitive group would have a lower order than the original group, we may assume for the time being that G is a transitive group of lowest order having the properties in question, and hence the squares of its cycles of order 4 are found in the invariant substitution of order 2.

Since the substitutions of G transform the cycles of the invariant substitution of order 2 according to a group which contains only operators of order 2, this group must be abelian; and since it is transitive, it must also be regular. If it is of order 2, G must be the octic group. If its order exceeds 2, the cycles of order 4 can appear in only one of the co-sets of G which correspond to the various substitutions of order 2 according to which the cycles in the invariant substitutions are transformed under G, since all of these cycles of order 4 would have to appear in such a co-set if one such cycle appears there, and all the substitutions of these co-sets must involve all the letters found in G.

Therefore the order of the regular group in question cannot exceed 4, since at least one-fourth of the substitutions of G must have orders which exceed 2. Hence each substitution of order 4 contained in G, even when the squares of the cycles of order 4 contained in G are not found in the invariant substitution of order 2, is regular, and hence only one-fourth of the substitutions of G have an order greater than 2. It follows that G is either the octic group or the direct product of this group and an abelian group of type (1, 1, 1, . . .). Hence the degree of G cannot be less than half its order, and G must be the octic group if all its cycles of order 4 are conjugate.

4. *Non-Prime Power Transitive Groups in which* $m = n - 2$. When the order of the transitive group G is not a power of a prime number and the number of the cycles of order k contained in G is $(n-2)g/k$, where n is the degree of G, it results directly that all the cycles of G are of prime order. When $n = 3$, G is the symmetric group of order 6; and when $n = 4$, G is the alternating group of order 12. We shall therefore assume in what follows that $n > 4$ unless the contrary is stated.

It will be convenient to use the following general theorem.

THEOREM III. *If a transitive substitution group which contains only one set of conjugate cycles of order k involves an invariant subgroup H such that H and each of the corresponding co-sets involves all these conjugate cycles, then H must be transitive and must contain only one set of conjugate cycles of order k.*

Since each of these cycles appears in H and in every co-set of G with respect to H, it results directly that it has as many distinct conjugates under H as it has under G. That is, all these cycles form a single set of conjugates under H. If H were intransitive, all of these cycles would therefore appear in one and in only one of its transitive constituents. This is impossible since the transitive constituents of H are transformed transitively under G.

It is now easy to prove that cycles of G which constitute a single set of conjugates must be of order 2. In fact, if these cycles were of prime order $p > 3$, the substitutions which would transform one such cycle into all its different powers less than p would involve cycles of composite order. It remains therefore only to consider the case when these cycles are assumed to be of order 3. In this case there would also be cycles of order 2 in G, and hence the order of G would be of the form $2^{\alpha}3^{\beta}$. In particular, G would be solvable.

If G contains an invariant subgroup H of index 3 and all the cycles of order 3 are conjugate, each of the co-sets corresponding to the operators of order 3 in this quotient group involves all the different cycles of order 3 contained in G, and these cycles are also found in H. Hence it follows from Theorem II that H would be transitive and would involve only one set of conjugate cycles of order 3, while all of its other cycles would be of order 2. This subgroup being of the same degree as G could therefore be used in place of G.

By repeating this process we finally would arrive at a transitive group which would be of the same degree as G and could be used for G, and which would have no invariant subgroup of index 3. This group would therefore contain a subgroup of index 2 involving all of its cycles of order 3. If not all the remaining substitutions were of order 2, all the cycles of order 3 would again be conjugate under this invariant subgroup, and hence this subgroup would again be transitive and involve cycles of order 2. Since it may be assumed that all of the remaining substitutions would be of order 2, this subgroup would have the property that each of its substitutions could correspond to its inverse in an automorphism, and hence it would be abelian. Since it would also be transitive, it would be regular. This is clearly impossible since it has been assumed that $n > 4$. We have therefore proved the following theorem.

THEOREM IV. *The symmetric group of degree* 3 *is the only non-prime power transitive group which has the properties that it involves cycles of only two different orders and that all the different cycles of the larger order are conjugate.*

When all of the different cycles of order 2 contained in G form a single set of conjugates, it may be assumed that G involves no subgroups of index 2. In fact, if there were such a subgroup each of the remaining substitutions would have to be of even order, and hence each of them would involve one and only one cycle of order 2, while the subgroup of index 2 would involve no cycle of order 2, and hence it would be of odd order. Since this is impossible, and since G is solvable, it follows that G involves an invariant subgroup of index p, where p is an odd prime number. The order of G is therefore $2^\alpha p^\beta$.

THE UNIVERSITY OF ILLINOIS

ALGEBRAS AND THEIR ARITHMETICS*

BY L. E. DICKSON

1. *Introduction.* Beginning with Hamilton's discovery of quaternions eighty years ago, there has been a widespread interest in linear associative algebras, a subject known also under the name of hypercomplex numbers. The list of investigators in this field includes the following well known names: Hamilton, Cayley, Clifford, and Sylvester in England; Poincaré and Cartan in France; Weierstrass, Frobenius, Lipschitz, Molien, Scheffers, and Study in Germany; A. Hurwitz and Du Pasquier in Switzerland; Benjamin Peirce, C. S. Peirce, Taber, Wedderburn, Hazlett, and others in America.

Needed guides to the extensive literature on this subject are furnished by the recent book by Scorza and the two books by the writer. Many of the papers, especially the older ones, contain serious errors and obscurities. Again, a large proportion of the papers are now obsolete, since they either treat only special algebras or fail in an attempt to give a general theory, and especially since they deal only with algebras over the field of all complex numbers. But the results obtained for this very special case have since been extended to algebras over any field, and it is the latter general subject which is the really important one both for algebra and for the theory of numbers.

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* A Report presented to the Society, by invitation of the program committee, at Cincinnati, December 28, 1923. This Report, the work of the author reported in it, and two other papers presented by him to the Society at the Cincinnati meetings (see this BULLETIN, page 280 of this issue) formed the basis for the award to the author of the seventy-fifth anniversary prize of the American Association for the Advancement of Science for the most notable contribution to the advancement of science presented at the meetings of that Association and its affiliated societies at Cincinnati on December 27, 1923 to January 2, 1924. (See this BULLETIN, vol. 30, Nos. 1–2, p. 90.) THE EDITORS.

We shall not attempt to give a complete survey of the entire subject of algebras, but shall restrict attention to a few results on the algebraic side which will be required in our account of the chief results in the recent remarkable theory of the arithmetics of algebras.

Instead of presenting the formal definition of a general algebra by postulates, we shall employ typical illustrations.

2. *Algebra of Complex Numbers.* A complex number $a \cdot 1 + bi$ is said to have the real *coordinates* a and b and the *basal units* 1 and i. All complex numbers form an algebra of *order* 2 over the field of all real numbers. We obtain another algebra by restricting the coordinates a and b to rational values; it is an algebra of order 2 over the field composed of all rational numbers. In general, a set of real or complex numbers is called a *field* if the sum, difference, product, and quotient (except by zero) of any two numbers of the set are also numbers of the set.

3. *Algebra of Matrices.* We shall now define a more typical algebra which plays an important rôle in our further discussion. Consider 2-rowed square matrices

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \mu = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

whose elements a, b, c, d, etc., are numbers of any chosen field F. We define the sum and the product of these matrices to be

$$m + \mu = \begin{pmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{pmatrix}, \qquad m\mu = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}.$$

If k is any number of the field F, we call

$$km = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

the *scalar product* of the number k and the matrix m. Consider the four special matrices

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the foregoing matrix m can be expressed in the form

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = ae_{11} + be_{12} + ce_{21} + de_{22}.$$

Hence the algebra formed of all 2-rowed matrices with elements in the field F has the four basal units $e_{11}, e_{12}, e_{21}, e_{22}$ and is of order 4.

Similarly, all n-rowed square matrices with elements in a field F form an algebra of order n^2, called a *simple matric algebra.*

In the definition of any algebra over any field F, we employ three operations called addition, multiplication, and scalar multiplication, which are assumed to have properties entirely analogous to those holding for the foregoing three operations on matrices.

4. *Quaternions.* The four special matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; i = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \; j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$k = ij = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

satisfy the relations

$$i^2 = j^2 = k^2 = -I, \; ij = k = -ji, \; ki = j = -ik, \; jk = i = -kj,$$

and are the basal units of quaternions

$$xI + yi + zj + wk.$$

Since matrix I plays the rôle of unity, it is usually denoted by 1. Quaternions may therefore be obtained very simply from matrices. Only when the field F contains $\sqrt{-1}$ is the present algebra of quaternions over the field F the same as the foregoing simple matric algebra of order 4.

5. *Definitious.* Let F be a field all of whose numbers are real. Consider any quaternion $q = x + yi + zj + wk$ whose four coordinates belong to F and are not all zero. It is readily verified that the product of q by its conjugate

$q' = x - yi - zj - wk$ is $N = x^2 + y^2 + z^2 + w^2$. Since $x, \ldots, w$ are not all zero, we have $N \neq 0$. Evidently q has the inverse $q^{-1} = (1/N)q'$, which is a quaternion with coordinates in F. Then the equation $xq = r$ has the solution $x = rq^{-1}$, while $qy = r$ has the solution $y = q^{-1}r$. Hence our algebra of all quaternions over the real field F is an example of a *division* algebra, in which each of the two kinds of division (except by zero) can always be performed uniquely.

The special quaternions $x + yi$ form an algebra of order 2 called a *sub-algebra* of the algebra of all quaternions.

A sub-algebra I of an algebra A is called *invariant* in A if the product of every element of I by every element of A is an element of I, and if likewise the product of every element of A by every element of I is in I.

In case A has no invariant sub-algebra other than itself, A is called a *simple* algebra. It is a fundamental theorem that every simple algebra A is a direct product of a simple matric algebra and a division algebra D; this may be understood to mean that all elements of A can be expressed as matrices whose elements belong to D.

An element is called *nilpotent* if some power of it is zero. For example, the square of the foregoing matrix

$$e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is the matrix zero all four of whose elements are zero, whence e_{12} is nilpotent.

An algebra is called nilpotent if all of its elements are nilpotent. A *semi-simple* algebra is one which has no nilpotent invariant sub-algebra.

An algebra A is said to be the *sum* of two sub-algebras B and C if every element of A can be expressed as a sum of an element of B and an element of C. If also the product of every element of B and every element of C is zero, and vice versa, and if B and C have in common only the element zero, then A is called the *direct sum* of B and C.

Every semi-simple algebra is either simple or is a direct sum of simple algebras, and conversely.

The principal theorem on algebras states that every algebra which is neither nilpotent nor semi-simple is the sum of its unique maximal nilpotent invariant sub-algebra and a semi-simple sub-algebra.

6. *Evolution of the Arithmetic of Algebras.* We have now defined those terms and stated those theorems concerning algebras which are needed in our account of the arithmetic of algebras. That subject has been surprisingly slow in its evolution. Quite naturally the arithmetic of quaternions received attention next to the arithmetic of complex integers.

Lipschitz, in his book of 1886, called a quaternion integral if and only if its four coordinates are all ordinary integers. But his theory was extremely complicated; it was not a success since such integral quaternions do not obey the essential laws of divisibility of ordinary integers. For example, there does not exist a greatest common left divisor of 2 and $q=1+i+j+k$. For, the only factors of 2 are the products of 2, 1, $1+i$, $1+j$ or $1+k$ by the various *units* ± 1, $\pm i$, $\pm j$, $\pm k$ (i. e., divisors of unity); while the only factors of q are the products of q, 1, $1+i$, $1+j$ or $1+k$ by the units. But no one of the four common factors listed is divisible by all of the others since $1+i$, $1+j$ and $1+k$ are indecomposable.

But A. Hurwitz in his memoir of 1896 and book of 1919 obtained a wholly satisfactory arithmetic of quaternions. He called a quaternion *integral* if its coordinates are either all integers or all halves of odd integers, and proved that the essential laws of divisibility of ordinary integers hold also for integral quaternions. In particular, there exists a greatest common left (or right) divisor; that of 2 and q is 2 since q is the product of 2 by the integral quaternion $\frac{1}{2}q$.

7. *Postulational Formulation.* Although Hurwitz stated his definition only for the case of quaternions, it may be formulated for any associative algebra A over the field of rational numbers. The integral elements of A are defined to be the elements which belong to a set of elements having the following four properties:

C (closure): *The sum, difference, and product of any two elements of the set are also elements of the set.*

B (basis): *The set has a finite basis* (i. e., *contains elements* $b_1, \ldots, b_k$ *such that every element of the set is a linear combination of the b's with ordinary integral coefficients*).

U': *The set contains the basal units of* A.

M (maximal): *The set is a maximal set* (i. e., *is not contained in a larger set having properties* C, B, U').

Note that Lipschitz's integral quaternions with integral coordinates have the properties C, B, U', while Hurwitz's integral quaternions have also property M. That a maximal set is superior to other sets is in accord with the history of the evolution of our number system and our experience in various branches of mathematics.

Du Pasquier, a pupil of Hurwitz, published during the past 15 years many papers in which he modified Hurwitz's definition by replacing U' by the milder assumption U that the set contains the modulus 1 which plays the rôle of unity in multiplication.

8. *Former Definitions Unsatisfactory.* But the definitions by Hurwitz and Du Pasquier are both unsatisfactory in general. This fact will be illustrated for the special algebra having two basal units 1 and e, where $e^2 = 0$.

Under Du Pasquier's definition, any set of elements with properties B and U has a basis of the form 1, $q = r + se$, where r and s are rational numbers and $s \neq 0$. Since q^2 belongs to the set by property C, we must have $q^2 = a + bq$, where a and b are ordinary integers. Thus

$$r^2 + 2rse = a + b(r + se), \quad r^2 = a + br, \quad 2rs = bs,$$

whence $2r = b$, $r^2 = a + 2r \cdot r$, $r^2 = -a$. Thus r is an integer. We may therefore replace the initial basis $1, q$ by $1, q-r = se$. Our set, designated by $(1, se)$, is evidently contained in the larger set $(1, \frac{1}{2}se)$, which in turn is contained in the still larger set $(1, \frac{1}{4}se)$, etc., where each such set has properties C, B, U. In other words, there does not exist a maximal set, so that the algebra does not possess integral elements, and the definition of integral elements is unsatisfactory. In such a case, Du Pasquier suggested that we omit the desirable requirement M and define the integral elements to be those of an arbitrarily chosen one of the infinitude of sets $(1, se)$. But it has been definitely proved by the writer* that factorization into indecomposable integral elements is then not unique and cannot be made unique by the introduction of ideals however defined.

These insurmountable difficulties arise also under the definition by Hurwitz, which imposes on the foregoing sets $(1, se)$ the condition that s be the reciprocal of an integer, so that the basal unit e shall belong to the set.

9. *Final Theory. Rank-Equation Postulate.* The writer has recently published a satisfactory theory of the integral elements of any rational algebra in his book *Algebras and Their Arithmetics* (University of Chicago Press). He employs postulates C, U, M and (in place of B) the following assumption.

R (rank equation): *For every element of the set, the coefficients of the rank equation are all ordinary integers.*

If $\xi_1, \ldots, \xi_n$ are independent variables in the field of rational numbers, the element $x = \xi_1 u_1 + \cdots + \xi_n u_n$ of a rational algebra A having the basal units $u_1, \ldots, u_n$ is a root of a uniquely determined *rank equation*

$$\omega^r + c_1 \omega^{r-1} + \cdots + c_r = 0$$

in which $c_1, \ldots, c_r$ are polynomials in $\xi_1, \ldots, \xi_n$ with rational coefficients, while x is not a root of an equation of degree less than r all of whose coefficients are such polynomials. For example, the quaternion $\alpha+\beta i+\gamma j+\delta k$ and its conjugate $\alpha-\beta i-\gamma j-\delta k$ are roots of

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* BULLETIN, vol. 28 (1922), pp. 438–442; JOURNAL DE MATHÉMATIQUES, (9), vol. 2 (1923), pp. 281–326.

$$\omega^2-2\alpha\omega+(\alpha^2+\beta^2+\gamma^2+\delta^2)=0,$$

which is the rank equation if α, β, γ, δ are independent variables in the field of rational numbers.

As a first justification of our new definition of the integral elements of any algebra A, note that for the case in which A is any algebraic field, there is a unique set of its elements which have properties C, U, R and M, and this set coincides with the totality of integral algebraic numbers of the field. In other words, the new theory is a direct generalization of the classic theory of algebraic numbers.

Next, the serious difficulties observed in the foregoing algebra with the basal units 1 and e entirely disappear under the new definition. For $x=a+be$, we evidently have $(x-a)^2=0$, which is the rank equation if a and b are independent variables in the field of rational numbers. Its coefficients are integers if and only if a is an integer. Evidently the unique maximal set of elements having properties C, U, R is composed of all the $x=a+be$ in which a is an integer and b is rational. These elements x are therefore the integral elements of the algebra. For any rational number k, the product of the integral elements $u=1+ke$ and $1-ke$ is 1, whence each is called a *unit*. Let $a\neq 0$ and choose $k=-b/a$. Then $xu=a$. The product of x by any unit u is said to be *associated* with x. Hence the integral elements are here associated with the ordinary integers a, so that the arithmetic of our algebra is associated with (and reduces to) the arithmetic of ordinary integers.

Note that our set of integral elements is the aggregate of the infinitude of non-maximal sets $(1, se)$ of Du Pasquier. Our satisfactory set may therefore by derived by a suitable enlargement of any one of Du Pasquier's unsatisfactory sets. Similarly, Hurwitz obtained his satisfactory set of integral quaternions by a suitable enlargement of Lipschitz's unsatisfactory set. There are many instances in the history of mathematics where success has been achieved by the principle of enlargement; examples are the growth of our number system and the introduction of ideals in the theory of algebraic numbers.

Note also that our integral elements $a+be$ do not have a finite basis, since b ranges over all rational numbers, and hence do not form a set of integral elements according to the definition either of Hurwitz or Du Pasquier. Thus the writer's conception of integral elements is entirely different from the conceptions of Hurwitz and Du Pasquier. It was only after long experimentation and tests of various kinds that the writer became fully convinced that his conception of the proper subject matter of arithmetics of algebras is from every standpoint wholly satisfactory and in particular far more desirable than all earlier conceptions. Moreover, the new conception greatly facilitated the development of a rich array of fundamental theorems, wholly lacking under former conceptions. The resulting remarkable science of the arithmetics of algebras furnishes the final justification of the new conception of the proper subject matter. It is obviously more difficult to justify a new determination of the proper subject matter of an embryo science than to compare different foundations of an established science.

10. *Theorems on Arithmetics.* According to the principal theorem on algebras stated above, any rational algebra A is a sum of its maximal nilpotent invariant sub-algebra N and a semi-simple sub-algebra S. The fundamental theorem on arithmetics states that the arithmetic of A is associated with that of S in the sense that every integral element (whose determinant is not zero) of A is the product of an integral element of S by a unit. For, all integral elements x of A are obtained by assigning values to the coordinates of those basal units which belong to S, such that the

S-component of x is an integral element of S, and assigning arbitrary rational values to the coordinates of the basal units which belong to N. Furthermore we can choose an element ν of N such that the product of x by the unit $1+\nu$ reduces to the component of x which belongs to S. In other words, the effect of multiplying x by a suitably chosen unit is to suppress the nilpotent component belonging to N (viz., be in our above example).

Next, the problem of the arithmetic of a semi-simple algebra S reduces to that of the arithmetics of simple algebras. For, we saw that S is a direct sum of simple algebras S_1, S_2, ..., so that each element σ of S is a sum of components σ_1, σ_2, ... belonging to S_1, S_2, ... respectively. It is an important theorem that when σ is an integral element of S, each σ_i is an integral element of S_i, and conversely. Moreover, the divisibility properties for S follow at once from those of the component algebras S_i.

We saw that the elements of any simple algebra Σ can be expressed as matrices whose elements range over the same division algebra D. It can be proved that the integral elements of Σ are those matrices whose elements range over the integral numbers of D, and conversely.

Let D be such that its integral numbers possess a division process yielding always a remainder whose norm is numerically less than the norm of the divisor. We may then establish a theory of reduction and equivalence of matrices whose elements are integral numbers of D. The resulting theory is a direct generalization of the classic theory of matrices whose elements are ordinary integers. In that case factorization into prime matrices is unique apart from unit factors. In our more general case, each matrix is the product of units and a diagonal matrix having exclusively zeros outside the diagonal, so that the arithmetic is associated with the simpler arithmetic of diagonal matrices.

We have therefore reduced the study of arithmetics of all rational algebras to the study of arithmetics of simple algebras, i. e., of matric algebras over a division algebra D, and we have completed the latter study when D is of a certain type.

11. *Conclusion.* Under the new definition, any set of integral elements of the same order as the order of the rational algebra A has a basis if and only if A is semi-simple. Hence the new definition is in complete accord with the older definitions by Hurwitz and Du Pasquier only in the important case of semi-simple algebras. For the remaining algebras, the older definitions led to insurmountable difficulties, whereas under the new definition the arithmetic of such an algebra is associated with that of its semi-simple sub-algebra S, since we may suppress the components belonging to its maximal nilpotent invariant sub-algebra N. It is fortunate that we can get rid of these bizarre nilpotent components since they would interfere seriously in applications. Their elimination also greatly simplifies the theory.

The theory of algebraic numbers finds applications only to problems involving forms which contain only two variables homogeneously and hence can be factored into linear forms. This serious limitation may often be removed by employing hypercomplex numbers. For example, $x^2+y^2+z^2+w^2$ has as factors the quaternion $x+yi+zj+wk$ and its conjugate. Since the new theory of arithmetics of algebras finds applications to problems involving forms in any number of variables it furnishes us with an effective new tool for problems in algebra and the theory of numbers.

THE UNIVERSITY OF CHICAGO

FOUR BOOKS ON SPACE

Der Raum: Ein Beitrag zur Wissenschaftslehre. By Dr. Rudolf Carnap. Berlin, Reuter und Reichard, 1922. 87 pp.

Mathematik und Physik: Eine erkenntnistheoretische Untersuchung. By E. Study. Braunschweig, Friedrich Vieweg und Sohn, 1923. 31 pp.

Die realistische Weltansicht und die Lehre vom Raume: Zweite Auflage; Erster Teil. By E. Study. Braunschweig, Friedrich Vieweg und Sohn, 1923. x + 83 pp.

Mathematische Analyse des Raumproblems. Vorlesungen gehalten in Barcelona und Madrid. By Dr. Hermann Weyl. Berlin, Julius Springer, 1923. vii + 117 pp.

Kant described our knowledge of space and time as synthetic and a priori. By *synthetic,* he distinguished it from the analytic, more or less tautological judgments of abstract logic, while by *a priori,* he signified that it is independent of the concrete content of our senses. His account of the extensional properties of the universe was an attempt to bridge the gap between the purely abstract, ratiocinative science of geometry, and the obviously empirical nature of the space-world to which it applies.

While Kant's problem still exists, the last century has seen a tremendous change both in our notions of geometry and in our notions of the spatial world. Geometry no longer means Euclid, for since the days of Bolyai and Lobachevski we have become aware that there are other possible systems which yield no whit to the traditional geometry in the matter of logical rigor. The axioms of geometry signify no longer self-evident indubitable truths, but arbitrarily set assumptions. In short, from the mathematical standpoint, geometry is but a branch of logic, and like the rest of logic, is concerned with the consistency, the deductive sequence of its theorems, not with their truth. On the other hand, the universe is no longer treated as fitting primarily into the euclidean scheme, but into the more complicated schemes of the special or the general Einstein space-time system. For all this, the problem still remains as to how we can associate with our empirically known world of sense a mathematical structure which in at least its analysis situs properties is essentially that of Euclid, and in particular, how we can perform this association in a preliminary fashion, not merely as the final result of a long chain of careful experiments, but automatically, almost at first glance. This problem, as to the nature of our knowledge of space and time, has indeed become far more acute because of the demands made by the Einstein theory on our imagination. It has led to a large and important literature, of which the books and pamphlets here reviewed form a significant part.

That of Carnap is written most distinctly from the standpoint of the professional philosopher, but shows a far deeper acquaintance than common with the mathematical and mathematico-logical advances of recent years. It is clearly written, and contains a good description of the various types of spaces and geometries recognized by the mathematicians, together with a valuable bibliography. Carnap tries to retain the spirit of the Kantian treatment of space, while discarding those details which the progress of the last century has shown to be indefensible. He distinguishes three levels of spatial knowledge, and three types of space to which they respectively pertain. The formal space of the mathematician is on the same epistemological basis as pure logic. It is known a priori, and there is nothing in its theorems that has not been put in by free assumption in its postulates. On the other hand, the space of the physicist is susceptible to an experimental investigation differing in no fundamental way from that by which we study the phenomena of heat or sound or electricity. Our knowledge of this space is inductive and a posteriori. Mediating between these two spaces is the realm where Carnap hopes to preserve what is of permanent value in the Kantian philosophy. This is the so called space of intuition, which is known as a condition of physical experience, independently of the amount of physical experience which we possess. The developments of the last century have made it impossible for us to attribute to metrical or even to projective geometry the importance of intrinsic properties of intuitional space, but (so Carnap maintains) the analysis situs properties of the universe are known as Kant considered all space to be known: a priori, but synthetically, and with more than purely logical content.

It is hard to believe that the analysis situs properties of the universe, even *im Kleinen,* are forever to be immune to a criticism of the type which has led to the theory of relativity. Relativity itself, indeed, questions our conventional notions of the *im Grossen* connectivity of the world. There are signs that the time may not be far distant when the atomicity which the quantum theory recognizes as a basal characteristic of the universe shall be referred to a fundamentally atomic conception of its space-time framework. Veblen indeed has put forward the suggestion that physics may come to describe the world by a modular space. The analysis situs of such a space—if it may properly be said to have any analysis situs—is immeasurably different from that with which we are familiar.

Like Carnap, Study attacks the problem of mediating between the deductive science of mathematics, which includes such parts of mathematical physics as concern themselves with the formal development of the conclusions of assumptions not called into question in the course of the investigation, and the inductive science of experimental physics. Except that he is concerned with mathematics and experimental physics more broadly than as to their spatial aspects, Study's notion of the objects and methods of these disciplines does not depart widely from that of Carnap. The characteristic tool of the intermediate discipline, however, he conceives to be idealization. This idealization is of the nature of a *fiction*, a schematization, a diagram, which replaces the unmanageably complex properties of the empirical world by a working model of similar but simpler nature, satisfying the postulates of some known mathematical system. It is not of purely logical character, but involves a judgement of value, for similarity *in et per se* is an empty notion. It consists, as Study says, in dissecting out a single component of a complicated situation, and analyzing it apart from disturbing factors.

The theory of idealization certainly comes very close to the facts of our knowledge of space and of the external world. A not dissimilar view is held by Whitehead and by Russell. These authors have tried to develop the machinery by which we schematize the space and time of our experience—the space of bodies and the time of events—into the point-space and instant-time of theoretical physics. Study is unable to accept their treatment of this matter because of a fundamental epistemological difference.

Study is a realist. That is, he maintains that things have a reality entirely apart from their being perceived. Now, Russell is a realist also, but the realities of Russell are of the nature of sense-data, at least in large measure, and while they exist independently of being known, and may conceivably be shared by several observers, they are primarily given as objects of the experience of a single observer. In order to build out of these data the external world of physics, Russell makes the working hypothesis of the existence of other observers. Study points out that this hypothesis is of a precisely similar character to that of the existence of the external world of physics *tout simple.* Study accordingly starts his realism with this hypothesis. Like all hypotheses, he holds that it is capable of being confirmed or refuted by its consequences, whether it can be tested directly or not. It is clearly to be distinguished from a fiction, which is an assumption made for the purpose of conceptually simplifying a complicated situation, and with a full consciousness of its falsity.

Study, however, is fully appreciative of the value of Russell's analytical work. With all its incompleteness, its logical props and supports, it does represent a definite contribution to our understanding of what really is contained in our hypotheses concerning the universe. One can only regret that Study has made no similar attempt to dissect out the precise content of the hypothesis of the existence of the external world. One thing at least is certain: that the hypothesis of the external world, taken as a simple and indivisible supposition, has no explanatory force whatever. It is only the hypothesis of a very special sort of external world which fills the need which Study finds. The nature of this external world needs elucidation and analysis of the Russellian type.

Study's *Die realistische Weltansicht und die Lehre vom Raume,* in addition to a clear and forceful exposition of his realism, contains much of a polemic nature. He combats the Neokantians, the Pragmatists, the Conventionalists such as Poincaré, and many schools besides. All of his discussion is valuable, for like Poincaré and unlike the generality of philosophers, Study has the firm basis of a real acquaintance *in concreto* with the universe which he analyses *in abstracto,* combined with a generous measure of dialectic skill. There are, however, places where one feels that the vehemence of his wrath might have been tempered with a little more regard for the amenities of scientific discussion. Nothing is gained by speaking in a contemptuous tone of the "*merkwürdige Psyche*" of William James.

Weyl's book brings us back again to the geometry of the mathematician. While he recognizes in his introduction the trichotomy of geometrical space, the world of bodies, and some third intermediate space of physics, he makes no attempt to elucidate the nature of this transitional realm. His purpose is purely to develop the structural characteristics of the space of mathematics, more especially in its differential aspects. While he avoids an explicit employment of the postulational method, his work is postulational in spirit. It consists in showing the conditions imposed on the metrical nature of space by certain suppositions of a very abstract and general character. His fundamental notion is that of *affiner Zusammenhang.* He conceives of space as consisting at every point of a sheaf of infinitesimal vectors. Given a system of coordinates, it is possible to relate the sheafs at neighboring points by means of the "parallel translation" of Levi-Civita. A method which enables us to select from all the possible parallel translations of P_1 to the neighboring point P_2 a particular "genuine" one, whatever P_1 and P_2 may be, determines what he calls the *affiner Zusammenhang* of space. In the case of the geometry of Riemann, where each linear element has once for all a length which is a quadratic form in its components, the *affiner Zusammenhang* of space is determined by the condition that genuine parallel translations preserve length *im Kleinen.* In Weyl's generalized differential geometry, where the lengths of two vectors at different points can only be compared by translating one to the other along some path, and in which the result of comparison depends in general on the path, the *affiner Zusammenhang* is still completely determined by the system of measurement.

Weyl takes up at some length his geometry, in which the lengths of the different vectors at each point are still given relatively by the square root of a quadratic form in their components, but the relative change in the length of a vector under a small displacement of its origin, its components remaining the same, is given as a linear form in the components of displacement of the origin.

Weyl exhibits differential expressions, respectively known as the *Streckenwirbel* and the *Vektorwirbel,* the vanishing of which gives the condition that space be Riemannian or euclidean, as the case may be. He also shows that the familiar non-euclidean geometries of Lobachevski and of Riemann are the only homogeneous metrical geometries: the only geometries, that is, in which there is no intrinsic metrical characteristic by which any two points may be distinguished. To this homogeneity he assigns a certain epistemological significance; namely, that if any particular metrical structure of space be given *a priori,* it must be a homogeneous one.

Up to this point Weyl has been recognizing the quadratic form as fundamental for the determination of a system of measurement. The question of course arises, "Why not a quartic or sextic form?" In the case of a homogeneous space, following Helmholtz, Weyl shows that the condition that free mobility about a point be possible limits our differential form of distance to the square root of a quadratic form. In the more general case of an arbitrary metrical space, he here proves for the first time the exceedingly important theorem that the only possible measure of the length of all infinitesimal vectors which determines invariantly the volume of an infinitesimal parallelopiped with a point P as vertex and which permits the unique determination of an *affiner Zusammenhang* by the condition that distances are preserved, is the square root of a non-degenerate quadratic form in the components of the vectors.

It will be seen that, like the postulationalists, Weyl is interested in penetrating to certain very general propositions which lie at the basis of geometry. He is fortunate, however, in not being bound in the pedantic straight-jacket of independence-proofs and postulate-counting which has strangled a most promising young science almost in its cradle. Weyl is not behind the postulationists in rigor, but he is far ahead of them in imagination, and he relegates his meticulous dissection of logical minutiae to the place where it belongs,—the back of the book.

NORBERT WIENER

TWO RECENT BOOKS ON ALGEBRAS

Algebras and their Arithmetics. By Leonard Eugene Dickson. University of Chicago Press, 1923. ix + 241 pp.

Corpi Numerici e Algebre. By Gaetano Scorza. Messina, Giuseppe Principato, 1921. ix + 462 pp.

On first picking up Professor Dickson's book, the reader's attention is caught by the fact that the author assumes no knowledge beyond the most elementary parts of a first course in the theory of equations. As the reviewer turned from one page to another, there was a certain ease and simplicity in the style which called to mind some of those science primers of the days of our grandfathers. One felt once more the half-forgotten glamor of the rainy day long ago when one discovered several of these "primers", during one's secret investigation of a pompous old book-case on the landing of the stair.

In the first chapter, he begins with the definition of a field of numbers and the elementary notions of a linear transformation and then easily makes the transition to matrices. Having pointed out that the set of all complex numbers, $a + bi$, is an algebra over the field of all real numbers and that the set of all p-rowed square matrices with elements in any field, F, is an algebra over F, in which multiplication is usually not commutative and in which division may fail, the author defines the general algebra over any field by five postulates which differ from those used in Scorza's book in one respect. After definitions of a few elementary notions in connection with algebras, he returns to matrices and shows that the matric algebra consisting of all two-rowed square matrices whose elements are complex numbers is carried by a linear transformation of the units into the familiar quaternions. Thus, at the very beginning, he gives an illustration of one of the most interesting and important results in the general theory of algebras, to wit, that any associative division algebra (an algebra in which division is always uniquely possible when the divisor is not zero) is equivalent, in a suitably enlarged field, to a matric algebra.

The second chapter is a brief one giving several elementary but useful theorems about linear sets, which are the same as algebras except that they are not necessarily closed under multiplication. Although the third chapter is also rather brief, it presents, in that simple, direct manner so characteristic of most of this author's work, about a dozen theorems and corollaries grouped around the fundamental notions of subalgebra, invariant subalgebra (analogous to self-conjugate subgroup in the theory of groups), reducible algebras and simple algebras. Of these the most significant is the one which asserts that any reducible algebra with a modulus (i. e., a number which behaves under multiplication like unity) can be expressed as a direct sum of irreducible algebras each with a modulus in one and essentially only one way. Together, these two chapters contain many of the theorems used in the later proof of the fundamental theorem of linear algebras due to Wedderburn.

Chapter 4 is concerned primarily with nilpotent algebras and idempotent elements. A nilpotent element is one such that some power of it is zero, and an analogous definition applies to nilpotent algebras. Although such algebras may seem scarcely more than figments of the imagination, yet it is very easy to exhibit one such, to wit, the algebra consisting of a single unit, e_{12}, a square matrix of order two having unity in the first row and second column and zeros elsewhere, for $(e_{12})^2 = 0$. In fact, one wishes that the author had taken a few lines to give a simple example of this type to remove the scruples of the materialistic soul who insists on having everything in mathematics securely anchored to something tangible. After defining idempotent elements (i. e., numbers u such that $u^2 = u$) and properly nilpotent elements (Scorza's *elementi eccezionali),* he proves Peirce's theorem about the decomposition of an algebra relative to an idempotent element. The last part of the chapter is concerned with semi-simple algebras—those containing no nilpotent invariant subalgebra. In addition to proving theorems which show that, under certain conditions, a subalgebra of a given algebra, A, is necessarily semi-simple, he shows that any semi-simple algebra which is not simple is the direct sum of simple algebras, and conversely. Although the results are necessarily technical, as are all results in mathematics, yet the author manages to keep the number of technical terms down to a minimum, so that, if the reader is willing to do any thinking, he will find that the text is most disarming in its simplicity.

One of the most fascinating chapters in the book is Chapter 5. This is devoted entirely to division algebras, which might be described as fields that are not necessarily commutative, if we use the word "field" (Korper) in Hurwitz' more general sense. In view of Wedderburn's theorem (proved in the next chapter) that every simple algebra over a field F can be expressed as the direct product of a division algebra over F and a simple matric algebra over F, a study of division algebras is exceedingly important. After eight elementary theorems and corollaries, each of which is proved in a very few lines, the author then devotes the remainder of the chapter to some important results on the determination of all division algebras. He first gives a simplification of his own elementary and neat proof of the famous theorem due to Frobenius and C. S. Peirce, to wit, that the only division algebras over the field of reals are (1) the field of reals, (2) the field of complex numbers and (3) the algebra of real quaternions. The text then defines the type of division algebra D of order n^2 over a field F which were discovered by the author and called Dickson algebras by Wedderburn. For the purpose of this review it will suffice to say that they are natural generalizations of real quaternions and are intimately connected with the theory of abelian equations. If the reader is curious, we can but refer him to the text, where the author gives in simplified form the most important results of his very interesting memoir, *Linear associative algebras and abelian equations,* in the TRANSACTIONS for 1914, together with other results. At the end of the chapter is a brief summary in which he tantalizingly refers the reader to Appendices I and II for further results in the general theory of division algebras. These results were proved by Wedderburn in two articles in the TRANSACTIONS for 1914 and 1921 and demonstrate still further the importance of Dickson algebras.

While Chapter 6 may not appeal to the casual reader quite so much as the one on division algebras, yet to any one who has read the preceding chapters thoughtfully, the present chapter on the structure of the general algebra will appear as a thing of beauty. Here will be found Wedderburn's important theorem that essentially every simple algebra can be expressed as the direct product of a division algebra and a simple matric algebra; and conversely. By this theorem are determined all simple algebras over the field of reals. Finally, he proves a vital theorem which, when shorn of technicalities, essentially reduces the problem of the determination of all algebras to the determination of certain special types.

In the next chapter, the author views algebras from an entirely different standpoint. Hitherto, he has proved every theorem by the aid of Wedderburn's calculus of linear sets; but now he recurs to the older method by which an algebra is thought of as given by a multiplication-table. Here he proves Peirce's theorem that every associative algebra is equivalent to a subalgebra of a matric algebra. This is followed by the orthodox set of theorems on characteristic matrices, determinants and equations, and the rank equation.

Although the final chapter of the first part of the book is brief, it is of vital importance. The author picks up the thread which he momentarily dropped at the end of Chapter 6 and then, by using certain principles of Chapter 7, he proceeds to prove the principal theorem which asserts that any associative algebra over a non-modular field, F, can be expressed as the sum of its maximal nilpotent invariant subalgebra and a semi-simple algebra. Thus, in one hundred and twenty-seven pages, he gives the essence (the triple extract, as it were) of all that is interesting and important in the general theory of algebras.

While the first part of the book is largely Wedderburn's work recast by Dickson, the second part (on the theory of arithmetics) is entirely due to Dickson. Just as the theory of complex integers, due to Gauss, is in some respects equivalent to the representation of all substitutions, with rational coefficients, which leave unaltered a sum of two squares, so similarly the transformations which leave unaltered a sum of three squares lead to quaternions. Lipschitz, by the consideration of such substitutions having rational coefficients, was lead to consider integral quaternions and then irreducible or prime quaternions.

In his book,* he defines integral quaternions as those having all coordinates rational integers and then develops a theory of such quaternions by using a rather complicated theory of congruences. But, although his definition of an integral quaternion was in some ways a natural one, it led to a theory with strange contradictions. From this definition he proves that an integral quaternion has, in general, a set of irreducible factors which is, in a certain sense, essentially unique; but if the norm of the quaternion is divisible by 4, then the quaternion has not merely one but twenty-four sets of irreducible factors which are essentially distinct in any understanding of the term.

To Hurwitz belongs the credit of the discovery that the reason for this and other anomalies lay in the definition of integral quaternion used by Lipschitz. In his memoir in the GOTTINGER NACHRICHTEN for 1896, Hurwitz gave a new definition which enlarged the set of integral quaternions and thus removed the above contradictions. But Du Pasquier, in a series of papers beginning in 1909, noticed that Hurwitz's definition is not satisfactory for every linear algebra and suggested other definitions. But these, in turn, are not suitable for every linear algebra, as they may either be vacuous or lead to insurmountable difficulties.

During the last few years, Professor Dickson has been interested in this problem, and his more important results are given in the second part of this book. In Chapter 9 he gives, in a simple and brief manner, the barest essentials of the theory of integral algebraic numbers; similarly, in Chapter 11, he gives the elements of the theory of fields, so that the reader who is unacquainted with these branches of mathematics can read the long and important Chapter 10, which gives Dickson's new work, with both understanding and appreciation.

In this chapter he defines an integer for any (associative) algebra having a modulus, over the field of rational numbers, in such a way that, when applied to the special case of quaternions, the new definition yields Hurwitz' set of integral quaternions and yet, when applied to the general algebra, it leads to no contradiction or difficulty. He thus proves that the arithmetic of an algebra A is known as soon as we know that of a certain semi-simple subalgebra, S. Similarly, the arithmetic of a semi-simple algebra, S, is known when we know the arithmetic of each of the simple algebras, S_i, of which it is the direct sum. In turn, the integers of a simple algebra are known when we know the integers of a certain division algebra, D. Thus, under the new definition, the problem of arithmetics of all algebras reduces to the case of simple algebras and finally, in large measure, to the case of division algebras. Moreover, the arithmetic of certain simple algebras is treated by generalizing the classic theory of matrices whose elements are integers. As a direct application, he obtains all integral solutions of some Diophantine equations which had not been solved completely previous to Dickson's solution.*

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* *Untersuchungen über die Summen von Quadraten,* Bonn, 1886; French translation in JOURNAL DE MATHÉMATIQUES, (4), vol. 2 (1886), pp. 393–439.

Taking it all in all, this most recent book from the hand of this indefatigable writer will be a source of keen joy to any one who feels that a composition in mathematics, like one in poetry, should have a fine balance between nicety of detail and a broad sweep that gives inspiration. It approaches very close to the French dictum, "Nothing can be added and nothing removed to improve the effect".

Professor Scorza's book is of a very different type. It seems a fair philosophy, both for life in general and for book-reviewing in particular, to judge the attainment of an individual by his purpose. Since the author says, in his preface, that this book is intended for young students in the university, one is inclined to expect a little more organization and conscious orientation than in a volume written for specialists. But the phrase that seems best to describe this book is "an encyclopedic treatise on the theory of algebras"; using the term "algebra" in the broad sense as including theory of numbers, matrices, and hypercomplex numbers. The book was to have consisted of three parts: (1) theory of number fields, (2) general theory of hypercomplex numbers or linear algebras, (3) application of (2). But, he confesses, the second part grew so large that he was forced to limit the third part to an appendix of fifty pages. It is to be regretted that he did not include more of the applications, especially those that would show to the mathematician who is inclined to be skeptical of the virtues of any abstract theory, the relation of hypercomplex numbers to quadric surfaces, abelian integrals, etc. Moreover, as will be seen below, the book would have gained both in power and charm if the second part had been compressed.

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* The only typographical errors noticed were the following:—

p. 53, label at top—Replace "§ 37" by "§ 39".

p. 60, l. 10—Replace "primitive element" by "primitive idempotent element".

p. 87, l. 5—Replace "$A-N$ is simple, also A is simple" by "A is simple, also $A-N$ is simple".

p. 178, equation (18)—Replace "$e_i = e_i$" by "$e_i^2 = e_i$".

Part I, on the general theory of number fields (130 pages) is divided into three chapters. Of these, the first gives the definition of a number field via four postulates, the rudiments of the theory and the basic differences between finite and infinite fields. In this chapter he also introduces the notation for n-tuples and gives the elements of the theory of matrices and determinants. The second chapter is the longest of the three and is devoted to polynomials. After the usual definitions for polynomials in several variables, he gives the orthodox work on divisibility, rational fractions, the zeros of a polynomial and the solution of a system of linear equations together with Leibniz's formula for a general field (modular or otherwise), Galois' generalization of Fermat's theorem and Euclid's algorithm for the highest common factor. In the final chapter, he gives the fundamental properties of Galois fields.

The reader is soon struck by the fact that the author seems to want every definition and theorem expressed with a meticulous care that is rare. All through the text he makes precise statements similar to this: "Polynomials in the same field and in the same indeterminates that differ at most by null monomials are regarded as not distinct". At first, such precise care will please the cautious mathematician; but, after page upon page of such detailed precision, the reader begins to weary. In the case of some theorems, this results in his spreading over a page or more a proof that could be given in a few lines with complete rigor and greater simplicity. In fact, one is inclined to wonder if this style would not tend to dull the interest of the young student for whom the text is written.

Part II, on the general theory of linear algebras (273 pages), is divided into six chapters, of which the first gives in forty-seven pages not only the orthodox elementary work with matrices, but also much interesting material not usually found in a text-book (such as Kowalewski's *Determinantentheorie*) of which the most important are Cayley's theorem that every square matrix satisfies its characteristic equation, Frobenius' beautiful theorem that the equation of lowest degree satisfied by a square matrix is a factor of the characteristic determinant obtained in a certain definite manner and Hadamard's well known result for the maximum value of a determinant.

The remaining chapters of this part are devoted to the general theory of algebras and look much like a revision of the first part of the author's memoir *Le algebre di ordine qualunque e le matrici di Riemann* which appeared in the RENDICONTI DI PALERMO for 1921. In many sections, the point of view and the method are those of Wedderburn's beautiful memoir in the PROCEEDINGS OF THE LONDON SOCIETY for 1908, though Professor Scorza has polished a number of the proofs—some to a slight degree and some to a greater degree, notably the proof of the theorem on the structure of simple algebras.

Although Chapter 2 is, with one exception, the longest in the book, it contains scarcely more than a long array of definitions and minor results in the general theory of linear algebras. To one already familiar with the essentials of the theory, the significance of this chapter will, of course, be clear; but the reviewer wonders if, when these multitudinous minor results are presented at the beginning of the theory, in a terminology unnecessarily complicated, they may not confuse or discourage some students. Chapters 3 and 4 are brief and discuss properties of an algebra connected with its invariant subalgebras and idempotent elements, respectively. Of these, the most important is the theorem that any algebra is the sum of its maximum nilpotent subalgebra and a semi-simple algebra (page 255). Chapter 5 is also comparatively brief and is concerned with the coordinates of an algebra, characteristic determinants, etc. The last chapter is the longest, occupying nearly one fifth of the entire book. In this he proves the set of theorems on the structure of the different types of algebras: simple algebras, matric algebras, division algebras; but ignores the very interesting and important work on division algebras by Dickson and Wedderburn referred to above in connection with Dickson's book.

Although the subject-matter of these five chapters is essentially the same as that of the first half of Dickson's book, it would be scarcely possible to imagine two mathematical books more different in their treatment. This difference is partly due to the unnecessary detail in Scorza's presentation, and partly due to the unfortunate fact that Scorza uses the older terminology which has been discarded. Also, it sometimes seems to the reader that he delights in multiplying the number of technical terms. For example, since his *algebra eccezionale* of an algebra A is simply the maximum nilpotent invariant subalgebra of A, why not call it such? The latter term is not only more descriptive, but is also the usual one.

At the end he has placed a bibliography of fifty-three books and memoirs on number fields, matrices and linear algebras. Although this is surely far better than none, it is not all that might be desired, since there are several errors, both of omission and commission. For example, why does it omit

Steinitz, Ernst, *Algebraische Theorie der Körper*, JOURNAL FÜR MATHEMATIK, vol. 137 (1909), pp. 167–309;

Dickson, L. E., *Linear associative algebras and abelian equations*, TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), pp. 31–46;

Wedderburn, J. H. M., *A type of primitive algebra*, ibid., pp. 162–166?

The last two articles (together with a brief but suggestive paper by Wedderburn in the TRANSACTIONS for 1921 which, presumably, appeared too late to be included in the bibliography) are, without doubt, the most interesting and the only important articles on the general theory of division algebras, and contain all that has been discovered about this difficult and extremely fascinating branch of linear algebras since Wedderburn's memoir in the PROCEEDINGS OF THE LONDON SOCIETY mentioned elsewhere in this review.

One notes with dismay that there is no index. This lack is an inconvenience in any book of this size and is only slightly ameliorated by putting in bold-face type the caption of every section and of every definition. In fact, one might almost say that the presence or lack of an index is a characteristic invariant which distinguishes Anglo-American texts from Continental ones.*

But altogether, the book contains a great deal of information not previously available outside of technical periodicals; and Professor Scorza is to be congratulated on the courage with which he attempted and the care with which he finished the task that he had set himself. As one reads, one can not but feel that the task has been to him a pleasant one and that when it was completed, he left it with a caress.

OLIVE C. HAZLETT

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* In addition to the errors listed in the *errata* at the end of the volume, the reviewer noted only the following non-trivial mistakes:
p. 118, l. 20—Interchange "Γ_1" and "Γ".
p. 167, l. 20—Replace "$\| \xi_{j,1} - \delta_{j,1}\, \sigma \| = 0$" by "$| \xi_{j,1} - \delta_{j,1}\, \sigma | = 0$".
p. 374, l. 4,6—Place square brackets around each "u".

A CORRECTION

In line 16, page 4, volume 29 of this BULLETIN (Jan., 1923), insert the words *and contain only decomposable continua* between the word *points* and the word *then*. The desirability of this correction was called to my attention by Professor J. R. Kline.

G. A. PFEIFFER

SHORTER NOTICES

Prolegomena to Analytical Geometry in Anisotropic Euclidean Space of Three Dimensions. By Eric Harold Neville. Cambridge, University Press, 1922. 22 + 368 pp.

The author's prolegomena are inspired by his feeling that to "let" the coordinates be complex instead of real is an indefensible absurdity. In order to justify the use of complex space he states and develops definitions, principles, and especially a vocabulary appropriate to real analytic geometry, builds a complex space, proves it to be unique, and then invests this ideal space with a geometry whose vocabulary he has previously developed. A word as to the title is necessary. In a euclidean four-space there are *isotropic* three-spaces, isotropic in the sense of the word as introduced by Laguerre. The study of the geometry of these isotropic three-spaces forms no part of the discussion.

The treatise is divided into five books. Books I–III deal with the avoidance of ambiguity in the measurement of angles and with projection, with vector analysis, and with Cartesian and vector frames. Book IV is devoted to the construction of algebraic space, the fundamental axioms involving vectors. The book concludes with a proof that complex space is unique. Especial attention is given here and later to the isotropic plane; a table of paragraphs relating to the isotropic plane is a welcome aid. Book V is devoted to a discussion of the geometry of this ideal space with special chapters on curves and surfaces especially conics and conicoids, on circles, and on spheres.

By the exercise of great care and ingenuity in his choice of definitions the author is enabled to state many important theorems without exceptions. Everywhere exceptional or degenerate cases are adequately covered. The treatment is rigorous, thorough, and careful. This care extends to the proof-reading. Only two slips of any kind were noted: (*a*) in 221.41, (*b*) in a contradiction between 553.49 and 562.53. In paragraph 445 the significance of geometric theorems relative to complex space is summarised in the clearest, most incisive manner that the reviewer has ever seen.

On the other hand, many readers will find that the subject-matter generally is, frankly, not inspiring. The chapters on vectors will attract some, the chapters on isotropic planes others, the frivolous (and others) will rejoice over the humor of the preface and foot-notes; but despite the elegance of this book, it is doubtful if many readers will be attracted to a serious perusal.

B. H. Brown

Die Mathematischen Hilfsmittel des Physikers. By Dr. Erwin Madelung. Berlin, Julius Springer, 1922. 247 pp.

This book gives a tabular list of the mathematical results needed in theoretical physics. It is very comprehensive, and covers everything that is not of a distinctly special character. The book also contains a second part devoted to formulating the fundamental laws and principles of physics, and discussing their formal aspects very briefly. Originally, this latter was intended merely for the purpose of illustrating the use of some of the mathematics given in the first part, but the author developed it so that it has a certain completeness. In our opinion, this part is coordinate in importance with the first, so that the book is decidedly more than its name implies.

We give a very reduced extract from the table of contents:

Algebra: Lineare Gleichungssysteme. Matrizes und Determinanten. Kombinatorik (8 pp.). *Funktionen*: Allgemeine Funktionentheorie. Spezielle Funktionen (35 pp.). *Reihen* (9 pp.). *Differential- und Integralrechnung* (8 pp.). *Differentialgleichungen*: Allgemeines über Differentialgleichungen. Gewöhnliche Differentialgleichungen. Partielle Differentialgleichungen (37 pp.). *Lineare Integralgleichungen* (3 pp.). *Variationsrechnung* (5 pp.). *Transformationen*: Allgemeines uber Transformationen. Koordinaten-Systeme und Koordinaten-Transformation. Berührungstransformation (22 pp.). *Vektoranalysis*: Koordinatenfreie Formulierung der Vektoranalysis. Koordinatenmassige Formulierung der Vektoranalysis in n-dimensionalen Räumen (28 pp.). *Wahrscheinlichkeitsrechnung und einige Physikalische Anwendungen* (10 pp.) *Mechanik*: Prinzipien der Mechanik. Einzelne Massenpunkte. Systeme von Massenpunkten. Starrer Korper. Mechanik der Kontinua (22 pp.). *Elektrizitätslehre* (16 pp.). *Relativitätstheorie*: Lorentztransformation. Elektrodynamik. Dynamik der Masse. Allgemeine Relativitätstheorie (14 pp.). *Thermodynamik*: (including the three laws, "phases", "mass-action", etc. 24 pp.).

The treatment of elliptic functions and integrals is characteristic of the book: periodic properties, Liouville's theorem, etc., are stated without proof; while all the steps of the reduction of the general integral

$$\int F(x, \sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4})\, dx, \quad F \text{ rational}$$

to the three standard forms are given. Apparently the great concentration of the material has made it impossible to state all the restrictions of continuity, etc., in every case (notably in the definition of *analytic function,* which is not satisfactory).

This book will exert a broadening influence on both mathematician and physicist. Although it is intended as a table rather than a textbook, it will lead to a desire for breadth of knowledge.

B. O. KOOPMAN

Grundzüge der Mehrdimensionalen Differentialgeometrie in direkter Darstellung. By D. J. Struik. Berlin, Springer, 1922. 198 pages.

In addition to the research papers which are published in any subject, there comes a time when one feels the need of a book which unifies the whole. The differential geometry of hyperspace has reached this point. Practically the only books on the subject are Killing's *Nichteuklidische Raumformen* and chapters in Bianchi's *Geometria Differenziale* so that the appearance of a book devoted entirely to differential geometry of n-dimensions is most welcome. This subject can well be said to have started with Riemann, followed by Christoffel, Beltrami, Lipschitz, and others who devoted a great deal of time to the study of the quadratic differential forms in n variables. These works were published between 1860 and 1880. In the latter eighties, Ricci and Bianchi began their work on n-dimensional geometry, and they and their followers have continued the work to the present time.

Ricci began his work in the absolute calculus in 1887, and he and Levi-Civita developed the subject almost to its present completeness many years ago, but it attracted but little attention. In fact it might be said that their work was practically unknown prior to 1913. Bianchi does use the covariant derivative as a notation but gives no indication of the powerful tool which Ricci made of it. However, when Einstein wrote his theory of gravitation and used the absolute calculus in its development, the subject became a live one and since that time there has been a whole army of mathematicians, scattered all over the world, working on it. The name which Ricci used was changed by Einstein to tensor calculus, and many people today are almost in total ignorance of what Ricci and Levi-Civita have done. It was, therefore, pleasant to find that Struik has dedicated his book to Ricci.

One of the outstanding features of the book is the "direct development". By introducing a sort of vector notation his work is freed from the dependence on the particular coordinate system used. The multiple algebra of the subject is developed to some extent but when one finds so many products used, it is quite a task on the memory of the reader to keep them all in mind. The notation, however, does allow one to sidestep the great mass of summation signs used by Ricci. The Clebsch-Aronhold notation is used to write forms of higher degree than the first as symbolic products of forms of the first order. Ricci's term "system" is replaced by "Affinor", and tensor is defined as a symmetric affinor. The first chapter is devoted to the multiplication of "affinors" both covariant and contravariant and mixed. In the second chapter the covariant derivative is developed and applied. The notion of geodesic parallelism introduced by Levi-Civita is made the basis of the work of this chapter. A differential is defined which bears the same relation to an ordinary differential that geodesic parallelism bears to ordinary parallelism and from this the operator ∇ is defined similar to the usual definition. The absolute derivative is then derived as ∇V where V is a vector or system of the first order. The various products of ∇ with the various affinors give the formulas needed in hyperspace. Chapter three deals with the curvature properties of a curved space of m dimensions contained in a curved space of n dimensions which do not depend on the Riemann-Christoffel tensor. The whole development is similar to the ordinary treatment of an m-spread in a euclidean space of n dimensions. There is a remarkable similarity in the formulas in the two cases.

The fourth chapter treats those curvature properties which depend on the Riemann-Christoffel tensor.

The text is preceded by an introduction of twelve pages which it is worth anyone's while to read. The last twenty pages are devoted to a bibliography which contains nearly four hundred titles. This gives one a good notion as to the historic development of the subject. The author has been very careful to give credit to the proper author for all ideas and formulas. The reference one meets most frequently is to Schouten, for the notation used is largely due to him.

The book is not easy reading, but as one becomes more familiar with the notation, he will find that the difficulty decreases, and he will feel amply repaid for his trouble.

C. L. E. Moore

A First Course in Nomography. By S. Brodetsky. London, G. Bell and Sons, Ltd., 1920. 135 pp.

Brodetsky has supplied us with the sort of book the subject has needed: a brief, readable exposition of elementary character. It will well serve two purposes. It is an excellent introduction to D'Ocagne, and it also enables its reader to gain a practical working knowledge of the nature and uses of nomograms with a minimum expenditure of time.

The author states in his preface that "it is the object of this First Course to offer a clear and elementary account of the construction and use of such (nomographic) charts" and that "it is a treatment that should be found useful by the reader who desires to become acquainted both with the theory of nomography and with its practical use."

On the whole the author has carried out his intentions fairly well, but, as a text for college students, the book is open to criticism. While a considerable amount of knowledge of algebra, trigonometry, and analytic geometry is presupposed, the author's treatment is very uneven in its demands upon the reader's knowledge and intelligence.

While simple matters of mechanical detail are carefully explained, the theoretical discussions contain many serious gaps and fail to make skilful use of the knowledge the reader is supposed to possess.

Moreover, the reader's confidence in the author is severely shaken by a piece of grossly illogical reasoning on pages 37 and 38. The author asks: "Is it possible to subtract in such a way that the result is given on a scale graduated with the same unit as the one begun with?". His answer is: "No! If we examine Fig. 4 and Fig. 14 (the only two nomograms which give the result of subtraction, the former by $b=x-a$, the latter by $x=a-b$) we see that in each case the final unit is different from the first unit." Not only is the logic faulty, but the conclusion itself is incorrect. Fig. 19 on page 35, although designed to be used for the addition $x=a+b$, obviously is equally useful for the subtraction $a=x-b$, and the unit for a is the same as the unit for x. Hence the book, although very useful for some purposes, cannot be regarded as wholly satisfactory.

R. D. BEETLE

Un Théorème de Géométrie et ses Applications. By Georges Cuny. Paris, Vuibert, 1923. 6+102 pp.

Let us denote the cross ratio of the four lines OX, OY, OM, ON by R_M. Let OX and OY meet an algebraic curve C_n in $A_1, \dots, A_n$, and $B_1, \dots, B_n$, respectively. Let C_n meet an algebraic curve C_p in $\alpha_1, \dots, \alpha_{np}$. Let C_p meet A_iB_i in $\beta_1, \dots, \beta_{np}$. The theorem is that the product of the R_α's is equal to the product of the R_β's. As a special case, if C_n has a multiple point of order $(n-1)$ at 0, this product is a function of p alone.

No one can fail to be impressed by the bewildering array of well known theorems which appear as special cases of this remarkably general theorem. As examples we note two general theorems on algebraic curves due to Newton, Pascal's hexagon theorem, Pappus' theorem, Carnot's theorem, Poncelet's theorem on the intersections of a cubic and conic, Frégier's theorem, Cazamian's theorem; examples selected from among scores of familiar projective and metric theorems. The methods of polar reciprocation, of inversion, and Laguerre's projective definition of angle are most happily employed. Extensions to three dimensions are briefly indicated.

The book should be of great value, especially to the younger graduate student. The reviewer believes that the possibilities of the theorem are by no means exhausted. Further investigations might well be incorporated in masters' theses. The subject matter is admirably adapted for student lectures in a course on modern geometry.

B. H. BROWN

Vector Analysis and the Theory of Relativity. By Francis D. Murnaghan. Baltimore, The Johns Hopkins Press, 1922. x + 122 pp.

This book is a brief exposition of what is commonly called the mathematics of relativity. The term vector analysis is used in a broad sense as equivalent to tensor analysis. Addressing himself primarily to the physicist the author connects up the new concept with those parts of mathematics with which the physicist may be assumed to be familiar. Thus after defining non-metrical n-dimensional space and spreads therein, the subject of tensors is approached through the consideration of integrals over such spreads which are independent of the parameters in terms of which the spread is given. After tensor algebra has been developed, it is applied to the proof of a generalized form of Stokes's lemma.

Metrical space and its attendant tensors are next considered and the relation between tensor analysis and the ordinary vector analysis noted. The resolved part of a tensor in an arbitrary direction is found and application made to a proof of a generalized form of Green's Theorem and to a discussion of Maxwell's equations.

The fifth chapter is a digression which the author advises be omitted in a first reading. It deals with the connection of tensor algebra with integral invariants and application to the statement of Faraday's law of moving circuits. It serves as another example of the fact emphasized in the book, that the methods of tensor analysis are by no means limited in their physical application to the relativity theory.

In the next chapter come covariant differentiation, the Riemann four index symbol, and Einstein's gravitational tensor. And finally the last chapter leads from the general tensor equations for the gravitational tensor to their solution in terms of a particular coordinate system for the field of a single particle and thence to the motion of the perihelion of Mercury and the bending of a light ray which grazes the sun. This jump from the general to the specific, which in so many of the works on relativity seems fraught with danger to life and limb, is here presented with a clarity and definiteness that delights the heart. Frequent examples involving the transformation from rectangular cartesian to space polar coordinates are used to illustrate the tensor analysis.

To the reviewer the approach to the tensor through integrals over a spread which are invariant under all changes of the parameters giving the spread is not as simple and direct as that through the differential properties of the spread. But the book is not an introduction to the relativity theory, and if we assume the reader already introduced to the theory we probably may assume a bowing acquaintanceship with its mathematics, and if this is the case the new point of view here developed will give him added insight into tensor analysis.

CHARLES E. WILDER

Dimensional Analysis. By P. W. Bridgman, New Haven, Yale University Press, 1922. ii 112 pp.

The substance of this book was given as a series of five lectures delivered at Harvard University in the spring of 1920. It treats of the *principle of similarity* used so effectively by the late Lord Rayleigh. A physical equation, according to the usual view, must express an equality between magnitudes of the same character, i. e., of the same dimensions. Nevertheless the author writes on page 42 the equation $v+s=gt+\frac{1}{2}gt^2$ where s is a length and v a velocity. But a complex equation of this type requires a special definition of the +sign and we cannot agree with the statement that "it is in itself a refutation of the intuitional method of proof of the principle of similarity." There is a useful chapter on the theory of model experiments and a collection of thirty-two examples. We feel that the book would benefit by expansion. Thus brief statements such as "Freedom from viscosity and complete turbulence of motion are seen by the analysis to be the same thing" (p. 85) are difficult to understand.

F. D. MURNAGHAN

Denken und Darstellung; *Logik und Werte*; *Dingliches und Menschliches in Mathematik und Naturwissenschaften.* Von E. Study. Braunschweig, Friedrich Vieweg und Sohn, 1921. 43 pp.

This pamphlet is interesting and valuable as expressing the views of a leading mathematical investigator on the ideals of mathematical teaching, study, and research. It is a polemic against the too formal notions of M. Pasch, who apparently regards the logical minuteness which is the final achievement of the analyst as the chief and almost the sole criterion of the value of a piece of mathematical work and as the goal to be striven for in mathematical education. Study rightly emphasises that this painstaking logical detail is merely a means for the avoidance of error, not an engine for the discovery of truth, that imagination is the first virtue of the mathematician, and that logical consistency alone is but sterile. He strongly dissents from Pasch when the latter separates a finer mathematics in which the utmost of logical precision is requisite from a coarser mathematics in which a moderate amount of fallacy is permissible. He demands strict logic everywhere, but a logic combined with imagination, not bound in the straight jacket of a dead syllogistic form. He says much that is worth reading about the bases of an evaluation of pieces of research, together with the ideals which should govern mathematical publication. The book stands beside those of Poincaré as one of the best expositions of how mathematical work is actually done, by a man whose great services to mathematics enable him to speak with authority.

NORBERT WIENER

Relativity, A systematic Treatment of Einstein's Theory. By J. Rice. London, Longmans, Green and Co., 1923. xvi+397 pp. $6.00.

The Mathematical Theory of Relativity. By A. Kopff. Translated by H. Levy. New York, E. P. Dutton (printed, however, in Aberdeen)., viii+212 pp. $3.20.

These two books illustrate very well indeed the two tendencies of the flood of books on relativity that is now flowing from the presses of British publishers. On the one hand, we have a few books of great value, such as Eddington's *Mathematical Theory of Relativity.* On the other hand, we have innumerable translations of German expositions of "*Einstein*" that pay no attention to current progress in this field.

Mr. Rice's book is of the first type. In spite of a few instances of poor proof-reading, it contains a somewhat novel treatment of the restricted theory, an excellent development of the general theory, and a most valuable comparative study of the work of Einstein, Weyl, and Eddington. In an addendum, it contains a brief but interesting account of Einstein's recent amendment to his general theory, which appeared in the Berlin SITZUNGSBERICHTE of March, 1923. In short, this volume would be a valuable addition to the library of anyone interested in the problems on which Einstein and others have made noteworthy progress.

The volume by Kopff is an English translation of one of a multitude of German works on "*Einstein*" rather than on *relativity.* Kopff has presented the now classical theory in a scholarly manner. The book needed, however, an editor as well as a translator, the English edition being full of footnotes referring exclusively to German sources. For those to whom this work would be of value, the German edition would be a less expensive volume of equal value.

C. N. REYNOLDS, JR.

Die Buntordnung. By Arnold Kowalewski. Heft 1. *Entstehung und mathematischer Ausbau der Buntordnungslehre.* Leipzig, Wilhelm Engelmann, 1922. 53 pp.

The subtitle of this work describes it as "mathematical, philosophical, and technical considerations concerning a new tactical idea". The new idea is as follows: To arrange the $\left|\begin{smallmatrix} n \\ p \end{smallmatrix}\right|$ combinations of n things (elements) taken p at a time in a row in such a way that every set of k successive p-ads in the row shall contain no common element. Such an arrangement the author calls a diversified row (Buntreihe) of degree $k-1$. If such a row of degree $k-1$ is possible, while one of degree k is not possible, the row is called the "most diversified" for the given n and p. For given n and p the problem is naturally to find the most diversified rows. The solution of this problem is said to be of importance in experimental psychology (where the problem was indeed suggested) and in many other domains, including analysis situs.

The idea seems to the reviewer not without interest. Unfortunately the present booklet leaves the reader in doubt as to how far a general theory has or can be developed. The author confines himself here to a popular exposition of the more elementary aspects of the problem for small values of n and p and gives numerous special examples of diversified rows, diversified rings, etc. For a more complete account of his investigations he refers to a series of seven articles published in the SITZUNGSBERICHTE DER WIENER AKADEMIE DER WISSENSCHAFTEN beginning in 1915. The author, who is apparently a psychologist, should not be confused with the mathematician Gerhard Kowalewski.

J. W. YOUNG

Lehrbuch der Analytischen Geometrie. Zweiter Band: *Geometrie im Bündel und im Raum.* By Lothar Heffter. Leipzig and Berlin, B. G. Teubner, 1923. xii + 423 pp.

The first volume of this text on analytic geometry appeared in 1905, with L. Heffter and C. Koehler as joint authors. (See this BULLETIN, vol. 13, pp. 247–249.) In the preface to the second volume, the author expresses regret that the original plan of the first volume was modified and suggests that the first twenty articles be supplemented by his pamphlet, *Die Grundlagen der Geometrie als Unterbau für die analytische Geometrie* (1921).* He also states that in the present volume he has been especially concerned in trying to clear up the ambiguity that is often associated with the word "metric".

This volume opens with the geometry of the totality of lines and planes through a point. This completes Part II, *Geometry of two dimensions,* which was begun in volume I. Geometry in space of three dimensions occupies the remaining five-sixths of the book. The first seven chapters (125 pages) deal with projective geometry. Projective point and plane coordinates in space, projective theorems concerning points and planes, and projective coordinates of the straight line in space precede the general and special projective properties of the surfaces of the second order and second class and polarity. "Affine" geometry is disposed of in four chapters (75 pages) before the treatment of "Aquiform" geometry (seven chapters, 144 pages). This includes the principal axes and principal planes of surfaces of the second order, focal properties of central surfaces of the second order and of the paraboloid, systems of these surfaces, and biquadratic space curves.

Groups of exercises are inserted at intervals throughout the book. There are some references and there is a good index. The many figures are well drawn, and the subject matter is clearly presented.

E. B. COWLEY

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* See this BULLETIN, vol. 28, p. 224.

NOTES

The third number of volume 24, series 2, of the ANNALS OF MATHEMATICS contains: *On the potential of a homogeneous spherical cap, of a magnetic shell, and of a steady current,* by C. De Jans; *On cyclic-harmonic curves,* by Harold Hilton; *Multiple integrals in n-space,* by Philip Franklin; *On symmetric forms in n variables,* by Arnold Dresden; *Algebraic fields,* by J. H. M. Wedderburn; *A theorem concerning certain unit matrices with integer elements,* by H. R. Brahana.

At the annual meeting of the American Association for the Advancement of Science, held at Cincinnati December 27, 1923, to January 2, 1924, Dr. J. McKeen Cattell, editor of SCIENCE, was elected president. Vice-presidents were elected as follows: mathematics, Professor J. C. Fields; physics, Dr. E. F. Nichols; astronomy, Professor J. A. Miller; historical and philological sciences, Professor L. C. Karpinski. Professor T. H. Hildebrandt was elected as member of the Committee for Section A (mathematics). Professor G. A. Miller delivered his address as retiring vice-president of Section A before the joint session of that section with the American Mathematical Society and the Mathematical Association of America; the address was on *American mathematics during three-quarters of a century.* At this session Professor L. E. Dickson delivered, at the request of the program committee of the American Mathematical Society, an address on *Algebras and their arithmetics,* for which (together with his two other papers read before sessions of this Society) he was awarded the Association's prize for the most valuable contribution to science presented during the meeting (see this BULLETIN, vol. 30, Nos. 1-2, p. 90. Professor Dickson's address appears in full in the present issue of this BULLETIN on pages 247-257. A review of his book of the same title appears on pages 263-270.

Professor H. L. Rietz has been elected president of the Mathematical Association of America, and Professors J. L. Coolidge and Dunham Jackson vice-presidents.

Professor W. H. Young has been elected president of the London Mathematical Society, Professors L. N. G. Filon, Harold Hilton, and A. E. Joliffe vice-presidents, Dr. A. E. Western treasurer, and Professors G. H. Hardy and G. N. Watson secretaries.

The fiftieth anniversary of the founding of the Société Française de Physique was celebrated at Paris in December, 1923. On this occasion a general meeting of the International Union of Physics was held, on December 10.

The International Mathematical Congress to be held in Toronto on August 11–16, 1924 has been organized into the sections listed below. An Editorial Committee has been appointed, of which Professor J. C. Fields in the general chairman, and of which the American members for the several sections appear under the section heading below.

SECTION I: *Algebra, Theory of Numbers, Analysis.*
  Professors D. R. Curtiss and A. J. Kempner.
SECTION II: *Geometry.*
  Professors A. B. Coble and L. P. Eisenhart.
SECTION III: (a) *Mechanics, Mathematical Physics.*
  (b) *Astronomy, Geophysics.*
  Professors G. D. Birkhoff and F. R. Moulton.
SECTION IV: (a) *Electrical, Mechanical, Civil, and Mining Engineering.*
  (b) *Aeronautics, Naval Architecture, Ballistics, Radiotelegraphy.*
  Professors E. R. Hedrick and A. E. Kennelly.
SECTION V: *Statistics, Actuarial Science, Economics.*
  Professors T. M. Putnam and H. L. Rietz.
SECTION VI: *History, Philosophy, Didactics.*
  Professors R. C. Archibald and L. C. Karpinski.

The general Organizing Committee (Professor J. C. Fields, *Chairman*; Professor J. C. McLennan, *Acting chairman*; Professor J. L. Synge, *Secretary*) will welcome abstracts of papers offerred for presentation before the Congress; these should be sent to Professor Synge in care of the Royal Canadian Institute, Toronto. Among the noted Europeans who have indicated an intention of attending the Congress are Borel, de la Vallée Poussin, Eddington, Enriques, Fréchet, Hadamard, Koenigs, Marconi, Parsons, Pincherle, Sierpinski, Villat, Volterra, Young, Yule.

The Paris Academy of Sciences announces the award of the following prizes for 1922: the Bordin prize to Professor Emile Gau, of the University of Grenoble, for a memoir on partial differential equations of the second order integrable by Darboux's method (see this BULLETIN, vol. 27, p. 239); the Francœur prize to the Abbé Gaston Bertrand, for his work in integral equations and the theory of tides; the Montyon prize in mechanics to Henri Chipart; the Poncelet prize in mechanics to the late Auguste Boulanger; prizes in navigation to Commandant Charles Lafon for his work *Etudes sur le Ballon captif et les Aéronefs marins,* and to Captain Stanislas Millot for his memoir on *Amarrage, remorquage et mouillage des navires*; the Hebert prize in physics to Professor Edmond Bauer, of the University of Strasbourg, for his work entitled *La Theorie de Bohr*; the Montyon prize in statistics to Alfred Barriol for his statistical work, including his *Théorie et Pratique des*

Opérations financières, and an honorable mention to H. E. Soper for his work entitled *Frequency Arrays*; the Binoux prize for history and philosophy of sciences to Dr. Robert Bouvier, for his work entitled *La Pensée d'Ernst Mach*; the Petit d'Ormoy prize to Professor Elie Cartan, for his mathematical work; the Estrade-Delcros prize to Professor René Baire, for his mathematical work; the Gustave Roux prize to Professor Georges Giraud for his work in automorphic functions; a prize from the Gegner foundation to Professor Augustin Boutaric for his work entitled *La Vie des Atomes*; a prize from the Hirn foundation to Gustave Eiffel for his work in aerodynamics. The Damoiseau prize was not awarded, and the subject, on developments of the work of Poincaré and Liapounoff on figures of equilibrium of a rotating fluid mass (see this BULLETIN, vol. 27, p. 239) was withdrawn.

The scientific prize of the Lasserre Foundation has been awarded to Professor Gaston Julia, of the University of Paris.

The Royal Academy of Madrid announces the following subject for a prize, for which competing memoirs should be presented by October 31, 1924: *A study of the complex collineation in the plane, and its real representation.* The competition is open only to Spanish, Portuguese, and Spanish-American mathematicians.

The Royal Society of London has received a gift from Sir Alfred Yarrow which will be used to endow research professorships, to be awarded only to men of proved ability for independent research. The initial awards of these professorships were to Professor A. Fowler, professor of astrophysics at the Imperial College of Science and Technology, and to Mr. G. I. Taylor, fellow and lecturer in mathematics at Trinity College, Cambridge.

The Cambridge Philosophical Society announces that it will publish shortly, as separate supplements to its PROCEEDINGS, the series of papers by Dr. Niels Bohr, *On the Application of the Quantum Theory to Atomic Structure,* Part I of which has already appeared in volume 13 of the ZEITSCHRIFT FÜR PHYSIK. The translation of Part I will be closely followed by a similar translation af Part II, which it is hoped will appear simultaneously with the German version.

Professor Aimé Cotton, of the Sorbonne, has been elected a member of the Paris Academy of Sciences in the section of general physics, as successor to the late Professor Jules Violle.

Professors Einstein and von Laue have been elected honorary members of the Royal Institution, London.

Professor W. F. Osgood, of Harvard University, has been elected a member of the Leopoldinisch-Carolinische Deutsche Akademie der Naturforscher in Halle.

Dr. Horace Lamb, of Trinity College, Cambridge, has been appointed to give the first Rouse Ball lecture on some subject related to mathematical science (see this BULLETIN, vol. 28, p. 320).

Mr. R. H. Fowler, of Trinity College, Cambridge, has been appointed university lecturer in mathematics.

Dr. C. E. Weatherburn, of Ormond College, Melbourne, has been appointed professor of mathematics at Canterbury University College, Christchurch, New Zealand.

Dr. Ganesh Prasad, dean of the Faculty of Science in the Benares Hindu University, has been appointed Hardinge professor of higher mathematics in the Calcutta University, in succession to Professor C. E. Cullis.

Professors G. D. Birkhoff, of Harvard University, and R. C. Archibald, of Brown University, will lecture at the University of California in the summer of 1924.

Professor H. F. Blichfeldt, of Stanford University, will lecture at Columbia University during the summer session.

Professor Earl Church, of the Pennsylvania Military College, has resigned to take charge of the computation and least squares adjustment of the geodetic survey of the Hawaiian Islands.

Dean James N. Hart, of the University of Maine, who has been head of the department of mathematics for the past thirty years, has been granted a year's leave of absence.

Professor F. W. Owens, of Cornell University, has been granted leave of absence for the second semester of the year 1923–24, and will spend the time in Europe.

Professor C. W. Strom, of Luther College, Decorah, Iowa, has been granted leave of absence for the current year and is studying at the University of Iowa.

Mr. C. C. Wagner, of Allegheny College, has been appointed assistant professor of mathematics at Pennsylvania State College, and is now acting head of the department.

Mr. H. A. Simmons, instructor in mathematics in the University of Michigan, has been appointed assistant professor of mathematics at the University of Pittsburgh.

Associate Professor H. H. Dalaker, of the University of Minnesota, has been promoted to a full professorship of mathematics.

At Fairmount College, Wichita, Kansas, Professor A. J. Hoare has resumed his work as head of the department, having become adjusted to a serious impairment of vision which he suffered four years ago.

Mr. C. S. Whitney has been appointed professor of mathematics and head of the department at the State School of Mines, Miami, Oklahoma.

Professor J. A. Whitted has resigned his position at Hedding College and has been appointed professor of mathematics at Ohio Northern University.

At Simmons College, Abilene, Texas, Professor A. E. Chandler has been made bursar, and Associate Professor J. E. Burnam has been promoted to a full professorship.

Professor Emma K. Whiton, of the University of Redlands, has been appointed professor of mathematics at Mills College.

Professor Auguste Boulanger, directeur des études at the Ecole Polytechnique, died in 1923, at the age of fifty-seven.

Dr. Alexander Gleichen, known for his work in optics, died October 21, 1923, at the age of sixty-one.

The death is announced of Gustaf Enestrom, of Stockholm, founder and for eighteen years editor of the BIBLIOTHECA MATHEMATICA.

Professor C. B. Williams, of Kalamazoo College, and Mrs. Williams were guests at the Grand Hotel in Yokohama on the day of the great earthquake. As nothing has been heard from them since that day, the American Embassy reports that they are believed to have lost their lives in the collapse of that hotel. Professor Williams had been a member of the American Mathematical Society since 1895.

Professor J. M. Willard, of Pennsylvania State College, died December 10, 1923, at the age of fifty-eight. Professor Willard had been a member of the American Mathematical Society since 1892.

Professor James Harkness, of McGill University, acting dean of the faculty of arts of that university, died December 7, 1923, at the age of fifty-nine years. Professor Harkness had been a member of the American Mathematical Society since 1891 and had held office as vice-president.

Professor Joseph Lipka, of the Massachusetts Institute of Technology, died January 15, 1924, at the age of forty. Professor Lipka, who received his doctorate from Columbia in 1912, had been on the mathematical staff of the Institute since 1908, with the rank of assistant professor since 1917. He had been requested by the program committee to give a talk at the May meeting of this Society.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BACHMANN (P.). Zahlentheorie. Versuch einer Gesamtdarstellung dieser Wissenschaft in ihren Hauptteilen. Teil 4: Die Arithmetik der quadratischen Formen. 2te Abteilung. Leipzig, Teubner, 1923. 22 + 537 pp.

BAKER (H. F.). Principles of geometry. Volume III: Solid geometry: quadrics, cubic curves in space, cubic surfaces. Cambridge, University Press, 1923. 20 + 228 pp.

BRENKE (W. C.). See DAVIS (E. W.).

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THE FIRST JOSIAH WILLARD GIBBS LECTURE

The first Josiah Willard Gibbs Lecture was delivered under the auspices of this Society on February 29, 1924, by Professor M. I. Pupin, of Columbia University, in the auditorium of the Engineering Societies' Building, New York City. A large and distinguished audience was present, including, besides members of the Society, many physicists, chemists, and engineers who had been invited to attend.

In introducing the speaker, President Veblen spoke as follows:

"In instituting the Willard Gibbs Lectures, the American Mathematical Society has recognized the dual character of mathematics. On the one hand, mathematics is one of the essential emanations of the human spirit,—a thing to be valued in and for itself, like art or poetry. Gibbs made notable contributions to this side of mathematics in his work on vector analysis and multiple algebras.

"On the other hand, mathematics is the handmaiden and helper of the other sciences, both in their most abstract generalizations and in their most concrete applications to industry. In this field Gibbs may be justly described as transcendental,—even if we think only of his work in thermodynamics. For his paper *On the equilibrium of heterogeneous substances* is one of the foundation stones of physical chemistry.

"It is hoped that the Willard Gibbs Lectures will remind the mathematicians of something that we fear they sometimes forget,—the existence of an outside world. It is equally hoped that they will remind the outside world that mathematics is a going concern,—not a pedantic exercise for the torment of school boys, but a living organism growing larger and stronger each year. Also it is intended that these should be lectures which we can all understand. We hope not to have an experience like one I heard of last summer. A friend of mine, an astronomer, was describing to me his attempt to read a book by Einstein. At a certain point Einstein says 'Until you have grasped this point, dear reader, read no further.' 'At this point,' said my friend, 'I closed the book and have never opened it again.'

"In asking Professor Pupin to give the first Willard Gibbs Lecture we have made sure that we shall not be tempted to close the book before we finish it. For we all know him not only as an eminent physicist and electrical engineer, but also as a rich and vivid personality and a teacher who has inspired more than one generation of students. He is well known to you as a man who has made brilliant applications of mathematics to electrical problems. But on this occasion we recall with special pleasure his connection with the American Mathematical Society. He is one of its charter members. He contributed a very famous paper (*Wave-propagations over non-uniform electrical conductors*) to the first volume of its TRANSACTIONS, and since the recent incorporation of the Society he has been one of its trustees. May I introduce our fellow mathematician, Professor Pupin."

The very illuminating lecture that followed marked the inauguration of what is confidently expected to be a notable series contributing to an appreciation on the part of the educated public of the role that mathematics plays in modern thought. Under the title *Coordination,* the lecturer divided all physical phenomena into two groups: first, coordinated phenomena, illustrated by planetary motions, and second, non-coordinated phenomena, illustrated by molecular motions. Newton, Faraday, and Maxwell laid the foundations for the dynamics of coordinated phenomena, and prepared the way for the study of non-coordinated phenomena. Sadi Carnot, with his second law of thermodynamics, led the way into the study of the dynamics of non-coordination. The discovery of the fundamental laws of radiation by Kirchhoff, and the development of the kinetic theory of gases by Maxwell, Boltzmann, and Clausius, gradually disclosed the physical fact that all radiation from hot bodies is a non-coordinated process, and that by far the greatest quantities of energy in the universe are in a non-coordinated form. The lecturer showed that the work of Gibbs in thermodynamics, in particular his generalization of Carnot's law, entitles him to the position of one of the founders of the science of non-coordinated motions.

R. G. D. Richardson,
Secretary.

THE MARCH MEETING OF THE SOCIETY

The two hundred thirty-fourth regular meeting of the Society was held at Columbia University on Saturday, March 1, and was preceded, on February 29, by the first Josiah Willard Gibbs Lecture (see this Bulletin, p. 289). The attendance at this meeting included the following forty-seven members of the Society:

Alexander, Babb, Barnum, Bernstein, W. M. Bond, E. W. Brown, R. W. Burgess, Coolidge, Cowley, Eisenhart, Fields, Fite, Gafafer, Gehman, Gilman, Glenn, Grove, Guggenbühl, Hausle, Hill, Hille, Himwich, Huntington, Joffe, Kellogg, Kline, Langman, Harry Levy, Mirick, H. H. Mitchell, C. L. E. Moore, Mullins, Pell, Post, Rainich, Reddick, R. G. D. Richardson, Ritt, Seely, H. D. Thompson, Veblen, H. E. Webb, Wedderburn, Weiss, H. S. White, Whittemore, J. W. Young.

The Secretary announced the election of the following persons to membership in the Society:

Dr. Maria Castellani, Bryn Mawr College;
Miss Laura Guggenbuhl, Bryn Mawr College;
Mr. Henry Alfred Hoover, Washington University;
Professor Benjamin Hoffman Kerstein, State Teachers' College Silver City, N. M.;
Professor James Shannon Miller, Emory and Henry College.

Thirty-seven applications for membership were received.

On recommendation of the Council, the following amendment to the By-Laws was presented to the Society: to insert in Section 1 of Article III the words "of ex-presidents, for a period of six years after the expiration of their respective presidential terms," so that the section shall read:

"Section 1. The Council shall consist of the officers specified in Article I, of ex-secretaries who have served as Secretary for ten years or more, of ex-presidents for a period of six years after the expiration of their respective presidential terms, and of fifteen additional elected members chosen in accordance with Article IV, Section 2."

The Society voted to adopt this amendment.

At the meeting of the Council, the appointment of the following committees was announced: On arrangements for the Annual Meeting of 1924, to be held at Washington: Professors H. L. Hodgkins (chairman), J. T. Erwin; Archibald Henderson, A. E. Landry, F. D. Murnaghan, and J. M. Page, Miss Elizabeth W. Wilson, and the Secretary; on the Second Award of the Bôcher Prize: Professors G. D. Birkhoff (chairman), L. E. Dickson, and H. S. White.

The report of the Auditing Committee, appointed to examine the accounts of the Treasurer, was presented to the Council for its information; this report shows a balance for the general funds of the Society, as of December 15, 1923, of $9,638.78, of which $5,726.91 is held as reserve to protect life memberships and $2,500 is estimated as required for printing the outstanding numbers of the BULLETIN and TRANSACTIONS for 1922 and 1923.

The Committee on the next Colloquium reported a recommendation that it be held at Cornell University in the summer of 1925; this recommandation was adopted.

President Veblen presided at the scientific sessions, relieved by Professors E. W. Brown, H. S. White, and Vice-President J. H. M. Wedderburn. Titles and abstracts of the papers read at this meeting follow below. Mr. Van de Walle was introduced be Professor Huntington. Professor Murnaghan's paper was read by Mr. Rainich, and Mr. Van de Walle's by Professor Huntington. The papers of Alexander, Garabedian, Michal, Moore, Murray, Rainich (third paper), and Stone, were read by title.

1. Professor F. D. Murnaghan: *The outer multiplication of integral forms.*

The author gives a short proof by means of tensor algebra of a known theorem on the composition of integral forms. In any space S_n, not necessarily metrical, of n dimensions, the generalised Kronecker symbol $\delta^{r_1 \cdots r_p}_{s_1 \cdots s_p}$ $(p \leqq n)$ may be defined by the following properties. It is alternating in both the upper labels $r_1 \cdots r_p$ and the lower labels $s_1 \cdots s_p$. It has the value zero if the group $r_1 \cdots r_p$ is different from the group $s_1 \cdots s_p$ and has the value ± 1 according as the arrangement $r_1 \cdots r_p$ is of the same class as the arrangement $s_1 \cdots s_p$ or not. The symbol $\delta^{r_1 \cdot r_p}_{s_1 \cdot s_p}$ defines a mixed tensor of rank $2p$. If we have two alternating covariant tensors $A_{l_1 \cdots l_i}$ and $B_{m_1 \quad m_j}$ of ranks i and j respectively $(i+j \leqq n)$ then $\delta^{l_1 \cdots l_i m_1 \cdots m_j}_{\alpha_1 \cdots \alpha_i \beta_1 \cdots \beta_j} A_{l_1 \cdot \; l_i} B_{m_1 \cdots m_j}$ is an alternating covariant tensor of rank $i+j$. If we construct integral forms with the components of the alternating covariant tensors as coefficients, the form of rank $i+j$ is the outer product of the two forms of rank i and j respectively. It may be expressly mentioned that the tensor $\delta^{r_1 \cdots r_p}_{s_1 \cdots s_p}$ is non metrical, i.e., the S_n need not be Riemannian nor need it have a geometry of paths. Thus the usual presentation, in the special case $p = 1$, of $\delta^{r_1}_{s_1}$ as equal to $g^{r_1}_{s_1} = g^{r_1 k} g_{s_1 k}$ is unfortunate.

2. Mr. G. Y. Rainich: *A modification of tensor analysis.*

In this paper, vectors are represented by their components with respect to a special system of rectangular axes at each point, and points, as usual by n coordinates. The derivative of a tensor φ of rank zero is given by $(\partial \varphi / \partial x_\rho) \cdot b_{\rho i}$, where $b_{ij} = b_{ji}$, and the derivative of a tensor ψ_i of the first rank by $(\partial \psi_i / \partial x_\rho) \cdot b_{\rho j} + \psi_\rho T^{\rho}_{ij}$, etc. If we require that (1) the derivatives of the Galilean tensor δ^i_j be zero and (2) the second derivatives of a tensor of rank zero be independent of the order of differentiation, we find that T^i_{jk} is one-half the determinant made up of three rows of three terms each: $\partial a_{i\rho} / \partial x_\sigma, \cdots;$ $b_{i\rho}, \cdots;$ $b_{i\sigma}, \cdots;$ where the a_{ij} are the coefficients of the inverse transformation with respect to the b_{ij}. There is no distinction between covariant and contravariant quantities. This representation and the usual one are connected by the formula $g_{ij} = a_{i\rho} \cdot a_{\rho j}$. Greek letters indicate a summation.

3. Mr. G. Y. Rainich: *On the electromagnetic field in general relativity theory.*

If at each point two perpendicular planes are given, we say we have a skeleton. A skeleton defines at each point two vectors p and q. The principal planes of the tensors of an anti-symmetrical field constitute a skeleton; in order that an anti-symmetrical field satisfy the Maxwell equations it is necessary and sufficient that the vectors p and q be gradients. If we have a Maxwellian field a necessary and sufficient condition that the planes of its skeleton may be considered as tangent planes to two families of surfaces is that the vector q be zero. The known energy relation which connects the contracted curvature tensor F_{ij} with the electromagnetic tensor imposes on F_{ij} the condition that the secular equation formed with the F_{ij} have two positive and two negative roots all of the same absolute value. Geometrically it means that F_{ij} defines a skeleton. The vector p of this skeleton is a gradient as a consequence of the general properties of curved space, but the condition that q be a gradient seems to impose a new condition on the space.

4. Professor R. E. Gilman: *The remainder in Charlier's integral formulas.*

The formula

$$\int_a^b \lambda(x)dx = \omega\sum_{r=1}^{s-1}\lambda(a+r\omega)+\tfrac{1}{2}\omega[\lambda(b)-\lambda(a)]$$
$$-\frac{\omega}{12}[\Delta\lambda(b)-\Delta\lambda(a)]+\frac{\omega}{24}[\Delta^2\lambda(b)-\Delta^2\lambda(a)]+\cdots$$
$$\cdots+C_n\omega[\Delta^n\lambda(b)-\Delta^n\lambda(a)]+R^n$$

is typical of those considered. In this formula, $\lambda(x)$ is a single-valued function with a continuous nth derivative in the interval $a \leqq x \leqq b + n\omega$, s is an integer, $\omega = (b-a)/s$, and $\Delta\lambda(a)$ has the customary meaning defined by the relations $\Delta\lambda(a) = \lambda(a+\omega) - \lambda(a)$ and $\Delta^{k+1}\lambda(a) = \Delta(\Delta^k\lambda(a))$. An investigation of the remainder R_n is carried out from two different points of view. The principal result of the paper is that

$$R_n = \omega^{n+1}(b-a)\lambda^{(n+1)}(\xi)\int_0^1 \frac{t(t-1)\dots(t-n)}{(n+1)!}dt, \qquad a < \xi < b+n\omega.$$

5. Professor Einar Hille: *An existence theorem.*

The integral equation of the Volterra type

$$w(z) = w_0(z) + \int_z^\infty \sin(t-z)\Phi(t)w(t)dt$$

is studied, in this paper, for complex values of z. The functions $w_0(z)$ and $\Phi(z)$ are supposed to be single-valued and analytic in the strip $A < x$, $B_1 < y < B_2$ $(z = x+iy)$ and to be bounded in absolute value; $\Phi(z)$ in addition satisfies $|\Phi(z)| < M/|z|^{1+\nu}$, $\nu > 0$. The path of integration is a straight line parallel to the real axis. The existence is shown of a unique solution which is single-valued and analytic in the strip. The continuation of the solution outside of the strip in a sector is studied in the case in which $w_0(z) = c_1e^{iz} + c_2e^{-iz}$.

6. Professor E. W. Brown: *A proof of the instability of the motions of asteroids whose periods are half that of Jupiter.*

This paper consists mainly of an announcement of a proof of the instability of the motions of asteroids which have half the period of Jupiter. The well known gap in the distribution corresponding to this period has long been a puzzle. It appears that the instability is mainly one which causes eccentricities to increase so much that the configurations cannot be maintained.

7. Dr. C. A. Garabedian: *Thick rectangular plates.*

The author has recently presented a method of series in elasticity* in connection with its applications to *Circular plates of constant or variable thickness*† and to *Rods of constant or variable cross-section.*‡ When the method is applied to rectangular plates the differential equations are partial instead of total, and the difficulties of integration are such that it is natural in a first presentation to avoid the complicated case of variable thickness. In the present note the method of attack is illustrated by a computation of the exact solution for a rectangular plate of constant thickness, supported at the edges and under uniform pressure. The results will be found in the COMPTES RENDUS in a note entitled *Plaques rectangulaires épaisses.* The author will show later how the method of series applied twice (secondly to the auxiliary partial differential equations which present themselves in the plate problem) leads to total differential equations and furnishes new solutions for beams of rectangular section.

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* This Bulletin, April and May, 1923; Comptes Rendus, vol. 177 (1923), p. 942.

† Transactions of this Society, July, 1923.

‡ This paper will appear in an early number of the American Journal.

8. Dr. F. H. Murray: *Generalization of certain theorems of Bohl.*

The object of this paper is the generalization of certain existence theorems obtained by P. Bohl, concerning trajectories which remain in the neighborhood of a point of equilibrium for a one-sided variation of the time, or for all values of the time. When the equations in generalized coordinates of a dynamical problem are satisfied by a set of constant values of the variables, the theorems of Bohl cannot always be applied; in certain cases, however, similar theorems can be obtained by similar methods, and in this article a number of these are derived. They include the corresponding theorems of Bohl as special cases. (See P. Bohl, *Über die Bewegung eines mechanischen Systems in der Nähe einer Gleichgewichtslage,* CRELLE'S JOURNAL, vol. 127, pp. 179–276.)

9. Mr. A. D. Michal: *Functionals of closed plane curves, invariant under one-parameter groups of transformations of the plane, and generalizations.*

In a previous communication presented to the Society (Annual Meeting, 1923) the author commenced the study of functionals of closed plane curves invariant under a given arbitrary continuous one-parameter group of transformations of the plane. In the present paper the writer continues his discussion of such invariant functionals of closed plane curves. Furthermore the theory is extended to the case of functionals of closed plane curves depending in particular on a point on the curve. Generalizations of this invariant theory are effected in the case of functionals of closed $n-1$ spreads in n dimensions.

10. Mr. M. H. Stone: *An unusual type of expansion problem.*

In the present paper, the boundary value and expansion problems arising from the differential systems $u' + \lambda pu = 0$, $u(0) - u(1) = 0$, $0 \leqq x \leqq 1$, and $u' + \lambda pu = 0$, $u(0) + u(1) = 0$, $0 \leqq x \leqq 1$, are discussed. The function $p(x)$ is bounded, and is so restricted that the Lebesgue integral $P(x) = \int_0^x p dx$ has only a finite number of oscillations. The special case when p is unity is taken up first. The general case is then considered. When P is not monotone, the formal expansions of an arbitrary function represent, not the function itself, but a related function which may, in special cases, coincide with it. Infinitely many functions have the same formal expansions in terms of solutions of the two differential systems.

11. Mr. M. H. Stone: *On the order of an analytic function at a singular point.*

Hadamard in his Thèse defined the order of an analytic function at a singular point (JOURNAL DE MATHÉMATIQUES, ser. 4, vol. 8 (1892), pp. 101–186, part III). He did not prove the existence of the order. Later writers on the subject have neglected to give proofs, or have given incorrect proofs. In the present paper a demonstration of the existence of the order is given.

12. Professor J. R. Kline: *Theorems concerning the division of a plane by continua.*

Suppose the bounded decomposable continuum M, lying in the plane S, is such that if g is any closed connected subset of M, then $M - g$ is connected. Under these conditions, the author proves that M must divide the plane. He further proves that unless M is a simple closed curve it cannot be a continuous curve.

13. Professor L. P. Eisenhart: *Geometries of paths for which the equations of the paths admit a quadratic first integral.*

A geometry of paths for a general space of n dimensions is based upon the conception that the paths are fundamental entities. They are the integral curves of a system of differential equations of the form

$$\frac{d^2x^i}{ds^2}+\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0,$$

where s is a special parameter for each path and the $\Gamma^i_{\alpha\beta}(=\Gamma^i_{\beta\alpha})$ are functions of the x's. The present paper determines the functions $\Gamma^i_{\alpha\beta}$ in order that the above equations may admit a given quadratic first integral $g_{ij}(dx^i/ds)(dx^j/ds) = \text{const.}$ Veblen and Thomas have solved the converse problem: Given the functions Γ, to determine whether the equations admit a quadratic first integral.

14. Mr. W. E. Van de Walle: *On the complete independence of the postulates for betweenness.*

Of the eleven sets of postulates for betweenness published by Huntington and Kline in 1917 (TRANSACTIONS OF THIS SOCIETY, vol. 18, pp. 301–325), it is shown that sets 1–10 are completely independent, while set 11 is not. In the case of sets 1, 2, 3, 5, 6, 7, the necessary examples are given in terms of a class containing only four elements. In the case of sets 4, 8, 9, 10, 11, some of the examples require the use of a class containing five elements. The failure of set 11 to be completely independent is due to the nonexistence of an example having the following "record": $A+$, $B+$, $C+$, $3-$, $4-$, $8+$.

15. Professor J. F. Ritt: *Equivalent rational substitutions.*

This paper presents several results relative to systems of three rational functions $\varphi(z)$, $\alpha(z)$, $\beta(z)$, for which $\varphi[\alpha(z)] = \varphi[\beta(z)]$.

16. Professor J. F. Ritt: *New proofs of two well known theorems on quadratic forms.*

Let N represent a prime positive integer. It is well known that if $N\equiv 1$, mod 4, it has the form $\alpha^2+\beta^2$, where α and β are integers, and that if $N\equiv 1$, mod 6, it is of the form $\alpha^2+\alpha\beta+\beta^2$. The present note proves these results by an analytic method, which involves the construction of a Riemann surface, and its uniformization with elliptic functions. It is not believed that such a method has been used before in the theory of numbers.

17. Professor R. L. Moore: *A characterization of a bounded continuum which forms the common boundary of two domains in the plane.*

It is shown that if, in a plane S, M is a bounded continuum which has more than one prime part and no one of its prime parts separates S then in order that $S-M$ should be the sum of two mutually exclusive domains such that every point of M is a limit point of both of them it is necessary and sufficient that M should contain no subcontinuum whose omission disconnects M.

18. Professor R. L. Moore: *On the relation of a bounded continuum to its complement in the plane.*

In a plane $\bar{S}$ let $\bar{M}$ denote a bounded continuum no one of whose prime parts separates $\bar{S}$. Let the term element denote any prime part of $\bar{M}$ or any point which does not belong to $\bar{M}$. Let S denote the set of all elements and let N denote the set of all prime parts of $\bar{M}$. An element p is said to be a limit element of a set of elements K if for every positive ε there exists an element belonging to K but distinct from p and containing a point at a distance less than ε from some point of p. It is shown that every "simple closed curve" of elements separates S into two "domains," of which a particular one is called its interior. It is shown that if the interior of every simple closed curve of elements is called a region then all the axioms of the author's *Foundations of plane analysis situs** are satisfied. It follows† that between the elements of S and the points of $\bar{S}$ there exists a one to one correspondence which preserves limits. The image $(\bar{N})$ of N under this correspondence is a continuous curve. Thus, from the stand-point of analysis situs, not only is N a continuous curve (of elements) but its relation to the remainder of the plane is that of a continuous curve to its complement.

19. Professor J. W. Alexander: *Note on Brouwer's fixed point theorem.*

Consider a continuous transformation T_1 of an n-sphere into itself such that no point is carried into its opposite.

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* TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), pp. 131–164.

† Ibid., vol. 20 (1919), pp. 169–178.

Each point P may then be joined to its image P' by an arc of a great circle shorter than half a great circle; consequently, the transformation may be generated by a (singular) deformation and must be of index unity. Now, consider a continuous transformation T_2 with no fixed point. If T_2 be followed by a reflection, we obtain a transformation of the type T_1 and index one. The reflection is of index ± 1 depending on whether the sphere is a manifold of an odd or even number of dimensions. Therefore, the index of T_2 must be ± 1 depending on the dimensionality of the sphere.

20. Mr. G. Y. Rainich: *On the dependence of the curvature upon the electromagnetic field.*

Einstein introduces in his paper *Bietet die Feldtheorie Möglichkeiten für die Lösung des Quantumproblems?* (BERLIN SITZUNGSBERICHTE, December 13, 1923) a certain expression $\Psi_{ik,lm}$ which has the algebraic properties of the Riemann tensor and reduces to the energy tensor after contraction. If we designate by $\Omega_{ik,lm}$ the tensor which depends in the same way on the dual (magnetoelectric) tensor as $\Psi_{ik,lm}$ depends on the electromagnetic tensor, we have in $\Omega_{ik,lm}$ another expression with the same properties. We can form an infinity of linear combinations of these two tensors having the same properties. The expression proposed by the author in the December number of the PROCEEDINGS OF THE NATIONAL ACADEMY is a particular case of this, being equal to half of the difference of these two tensors.

R. G. D. RICHARDSON,
Secretary.

THE EQUATION OF THE EIGHTH DEGREE*

BY A B. COBLE

1. *Introduction.* The progress of mathematics as a whole is occasionally brought to our attention by the appearance of some notable book or memoir in which the resources of the subject are brought from various fields to bear upon a central problem. An early instance of this is the *Ikosaeder*[1] of Klein—the forerunner of the Klein-Fricke series. The first two chapters of this book furnish an introduction to group theory which is as yet unsurpassed. A later example is the book of Hudson on *Kummer's Quartic Surface.*[2] This surface, remarkable in itself, is more remarkable for the breadth of the theories which attach to it. Central problems in these two books are respectively the quintic and the sextic equation. It is true that Hudson assumes the solution of the sextic when needful, but his book furnishes the geometric background for this solution.

Around the sextic equation there cluster the geometric theory of the Weddle and Kummer surfaces and of certain modular three-ways in S_4 as well as the analytic theory of the hyperelliptic integrals and functions of genus two, a complex in which the methods of algebra and group theory have constant play. For the most part the allied geometry is within the spaces of our experience, and with the correspondingly small number of variables the methods of ordinary analytic geometry are effective. The analytic function theory is also fairly manageable by direct methods. When, however, we seek to extend this field of ideas to equations of higher even degree complications develop rapidly. The surfaces or manifolds involved lie in spaces of high dimension and are defined by many rather than one equation. The analytic theory is correspondingly difficult. Some solid ground there is, particularly in the theta functions, but not enough to form a basis for a closely knit theory.

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* Address as retiring chairman of the Chicago Section of this Society, presented to the Society at the meeting in Cincinnati, December 28, 1923. This is an outline of an investigation pursued under the auspices of the Carnegie Institution of Washington, D. C. References to a bibliography at the close of this paper are indicated by arabic numerals.

It is the purpose of this paper to exhibit certain points of view, certain modes of approach to the general problem, and certain analytic methods which can be applied successfully to the octavic and in most cases to equations of even degree, $2p+2$. An indication there will be of a general theory which is fairly coherent and perhaps not more vague than is inevitable with generalizations of this character.

It should be remarked that the solutions here sought for these equations are solutions in terms of hyperelliptic modular functions and not the solutions of Klein in terms of the form problems of linear groups, which perhaps are more desirable but certainly are less accessible.

2. *Two Linear Systems of Irrational Invariants.* (*a*). The given equation defines three sets of points each set on a rational norm-curve in its space. Thus the sextic defines the set Q_6^1 of points $q_1, \ldots, q_6$ on a line S_1, the set R_6^2 of points $r_1, \ldots, r_6$ on a conic N^2 in S_2, and the set S_6^3 of points $s_1, \ldots, s_6$ on a space cubic curve N^3 in S_3. Similarly the octavic defines sets of eight points, Q_8^1, R_8^3, S_8^5 on respectively the line S_1, the space cubic curve N^3 in S_3, and the rational norm quintic curve N^5 in S_5. In general the equation of even degree $2p+2$ defines the sets Q_{2p+2}^1 on S_1, R_{2p+2}^p on N^p in S_p, and S_{2p+2}^{2p-1} on N^{2p-1} in S_{2p-1}. We shall take all of these norm curves N^k in S_k in the simple parametric form, $x_i = t^i (i = 0, \ldots, k)$, and assume that the parameters $t_1, \ldots, t_{2p+2}$ of the points of each set are either the roots of the given equation or are projective to these roots. We observe that for the extreme sets Q_{2p+2}^1, S_{2p+2}^{2p-1} the norm curve requirement imposes no conditions on the points since on $k+3$ points in general position in

S_k there is a unique norm curve N^k. The intermediate set R^p_{2p+2} is special due to $(p-1)^2$ conditions that its points shall lie on N^p. The extreme sets also furnish one of the simplest examples of *associated sets* ([3]Theorem 10).

(*b*). Invariants of a set of points in S_k are formed from the determinants of the coordinates of $k+1$ points selected from the set, which are combined in such a way that each term is of the same degree in each point. This degree is the *degree* of the invariant. Such invariants are *rational* or *irrational* according as they are or are not unaltered under permutation of this points ([4]I § 3). The sets Q^1_{2p+2} and R^p_{2p+2} have linear invariants which are linear combinations with constant coefficients of products of the respective types

$$\begin{array}{lll} \text{(A):} & (12)(34)\cdots(2p+1,\ 2p+2), & (ij)=\begin{vmatrix} t_i & 1 \\ t_j & 1 \end{vmatrix}; \\ \text{(B):} & (12\cdots,p+1)(p+2,\cdots,2p+2), & \\ & (i_1 i_2 \cdots i_{p+1}) = \begin{vmatrix} t^p_{i_1} & t^{p-1}_{i_1} & \cdots & 1 \\ t^p_{i_2} & t^{p-1}_{i_2} & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ t^p_{i_{p+1}} & t^{p-1}_{i_{p+1}} & \cdots & 1 \end{vmatrix}. & \end{array} \tag{I}$$

The various products of either type are linearly related in many ways by the usual determinant identities. The set S^{2p-1}_{2p+2} contributes nothing new to this invariant theory since its determinants are proportional to the complementary determinants of its associated set Q^1_{2p+2} ([4]I (25)).

(*c*). Obviously if the roots t_i of the given equation are known the numerical values of the invariants (A) are easily found. Conversely if the numerical values of the invariants (A) are known the roots t_i of the given equation may be found by rational processes. Thus from (12) (34) (56) $\ldots$ $(2p+1,\ 2p+2)$ and (14) (32) (56) $\ldots$ $(2p+1,\ 2p+2)$ we find by division the double ratio $(13\,|\,24)$. If we take t'_1, t_2, t'_3 to be 0, 1, ∞ the values of these double ratios determine the remaining roots $t'_4, \ldots, t'_{2p+2}$ of an equation projective to the given equation. The procedure for finding the transformation which converts the known roots t' into the required roots t of the given equation is a known exercise in binary forms. We shall therefore henceforth regard the determination of the values of the invariants (A) as a "solution" of the given equation.

(d). The numerical values of the invariants (B) are in immediate relation to hyperelliptic modular functions of genus p. The basis notation[5] for these is very convenient. We divide the numbers $1, 2, \ldots, 2p+2$ into equivalent complementary sets and name the half periods by an even number of subscripts as in $P_{1,2,\ldots,2k} = P_{2k+1,\ldots,2p+2}$; the even theta functions of first order by a number of subscripts of the form $p+1 \pm 4k$; and the odd functions of the first order by a number of subscripts of the form $p+1 \pm 2(2k+1)$, $(k = 0, 1, \ldots)$. The simplest set of modular functions are the values of the even functions for the zero arguments. When, however, the functions are hyperelliptic all of these values vanish except the ones with $p+1$ subscripts ([6]pp. 456–464). The non-vanishing values are connected with the invariants (B) by the remarkable relations

$$(2) \qquad \varepsilon\cdot(1, 2, \ldots, p+1)(p+2, \ldots, 2p+2) = \vartheta^4_{1,\ldots,p+1}((0)),$$

where ε is a properly adjusted sign (for $p = 2$, cf. [4]I, p. 194; for $p = 3$, [5]p. 111).

(e). If then we find rational expressions for the invariants of type (A) in terms of those of type (B) we have a solution of the given equation in terms of hyperelliptic modular functions. I shall return in § 3 to a discussion of these relations. Meanwhile let us seek for a natural and easily extensible way of introducing the theta functions themselves into the discussion. For this purpose the set of points S^{2p-1}_{2p+2} plays the important role.

3. *The Generalized Weddle p-way and the Hyperelliptic Kummer p-way.* (a). Given a general set of $k+3$ points $s_1, s_2, \ldots, s_{k+3}$ in S_k we define, by isolating a pair of points

s_1, s_2, a Cremona involution I_{12} of order k, projective to the type $x_i' = 1/x_i$ $(i = 0, 1, \ldots, k)$ which interchanges the points s_1, s_2 and has F-points at the remaining $k+1$ points of the given set. By considering the effect of these involutions upon the directions about the points of the given set we find that they generate a group for which the following generational relations hold:

$$\begin{aligned} I_{12}I_{13} &= I_{13}I_{12} = I_{23}, \\ I_{12}I_{34} &= I_{13}I_{24} = I_{14}I_{23} \equiv I_{1234}, \\ I_{1234}I_{56} &= I_{1235}I_{46} = \cdots \equiv I_{123456}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{3}$$

Hence they generate an abelian group of order 2^{k+2}, G_2^{k+2}, all of whose elements are involutory. We observe, however, an essential difference between the odd and even values of k. When k is odd and therefore $k+3$ even, say $2p+2$, there is one element of the group, namely

$$I = I_{123\ldots, 2p+2} \tag{4}$$

which is symmetrically related to the given set of $2p+2$ points, S_{2p+2}^{2p-1} (cf. [4]II, p. 369). We define the Weddle p-way, W_p, in S_{2p-1}, to be the locus of fixed points of the involution I. That there is such a locus of fixed points and that it can be expressed parametrically by means of hyperelliptic functions we proceed to prove by a process which is new to the literature.

(*b*). The two common canonical forms of the hyperelliptic algebraic plane curve of genus p are first the form

$$y^2 = a_0(x-t_1) \ldots (x-t_{2p+2}), \tag{5}$$

commonly used for the development of the hyperelliptic integrals; and secondly the curve H_p of order $p+2$ with a p-fold point at O and $2p+2$ contacts $\tau_1, \tau_2, \ldots, \tau_{2p+2}$ of tangents from O. To the so-called "superposed points" $x, \pm y$ of (5) there correspond the pairs of points on H_p on a line through O. These pairs constitute the unique g_1^2 on H_p invariant under all birational transformations. The form (5) depends upon $2p-1$ "moduli" or "birational invariants". On the other hand the curve H_p has $3p-1$ absolute projective constants. Hence the ∞^{3p-1} projectively distinct curves H_p divide into ∞^{2p-1} birationally distinct classes, each class containing ∞^p curves. The ∞^p curves H_p in each class are birationally equivalent and it may be shown that the curves of a given class may all be obtained from a given curve of the class by selecting on this given curve a p-ad of points and transforming this p-ad into the p-fold point of the transform. This may in fact be done by Cremona transformations of the entire plane. However, each selected p-ad determines its "superposed" p-ad and either one of a pair of superposed p-ads leads to the same transformed curve. Thus the two aggregates—(1) the ∞^p projectively distinct but birationally equivalent curves H_p; and (2) one of the curves H_p and the ∞^p pairs of superposed p-ads on it—are birationally equivalent. In the next section we connect the first aggregate with the ∞^p points on the Weddle W_p; and in the following section the second aggregate with the variables of the attached theta functions and thereby obtain the desired parametric expression of W_p.

(c). Each of the ∞^p curves H_p has a p-fold point O and $(2p+2)$ branch points $\tau_1, \tau_2, \cdots, \tau_{2p+2}$ which make up a set of points, T^2_{2p+3}. The associated set lies in S_{2p-1} and by virtue of a property ([8]p. 1) of such associated sets the points in S_{2p-1} which correspond to $\tau_1, \cdots, \tau_{2p+2}$ can be identified with the fixed set S^{2p-1}_{2p+2}. Then as H_p varies in the ∞^p system the point P in S_{2p-1} which corresponds to O in the plane runs over a p-way in S_{2p-1}. In order to identify this p-way with the Weddle W_p we observe that H_p is a locus of fixed points of a Jonquière Cremona involution J_p and therefore the set T^2_{2p+3} is congruent to itself under J_p. Moreover J_p is a product of Jonquière quadratic involutions $J_0(O, \tau_1, \tau_2)$, $J_0(O, \tau_3, \tau_4), \cdots$. Hence (cf. [4]II, p. 361) the associated set in S_{2p-1} is congruent to itself under the product of the involutions $I_{12}, I_{34}, \cdots$, i. e. under I. Since the

F-points of I are all in the set S_{2p+2}^{2p-1} the point P is a fixed point of I and therefore a point of W_p. Conversely for each point P on W_p we have an associated set T_{2p+3}^{2} and a curve H_p. The known[7] relations satisfied by the set T_{2p+3}^{2} lead to geometric conditions on P. Thus for $p=2$ the six branch points τ are on a conic and therefore P is the vertex of a quadric cone on S_6^3— the usual definition of W_2. When $p=3$ the eight points τ and O are the base points of a pencil of cubics, whence P and the 8 points of S_8^5 are on a pencil of elliptic sextics in S_5 ([3]pp. 17–18).

(d). Consider now a fixed curve H_p and a p-ad of points on it which correspond to the points $(x_1), \ldots, (x_p)$ on (5). On the Riemann surface of (5) select p paths of integration from the fixed branch point t_1 to $(x_1), \ldots, (x_p)$ respectively. If v stands in succession for one of the p normal integrals of the first kind and we set

$$v_{t_1}^{(x_1)} + \cdots + v_{t_1}^{(x_p)} = u, \tag{6}$$

then to within multiples of the periods the p-ad defines a value system $u \equiv u_1, \ldots, u_p$ and conversely u defines as a solution of the inversion problem a p-ad $(x_1), \ldots, (x_p)$. Superposed p-ads give rise to $\pm u$. The particular p-ad at the point O is determined by $u=d$. In this way to a point P on W_p we attach parameters $\pm d$. We readily find (the exposition in [6]Chap. 10 is convenient here) that the involutions I_{12}, which in the planar field are represented by $J_0(O, \tau_1, \tau_2)$, transform W_p into itself and are given in terms of the parameters by the simple relations

$$d' \equiv d + \pi_{12} \tag{7}$$

where π_{12} is a particular half period. We still have to account, on the Weddle p-way, for the 2^{2p} singular points and the 2^{2p} singular tangent spaces of the Kummer p-way. It is more convenient to illustrate these by the relatively simple cases of the sextic and octavic for $p=2, 3$ respectively.

(e). When $p=2$ the six branch points $\tau_1, \ldots, \tau_6$ of H_2 with node at O are on a conic and the discriminant condition (cf. [8]§ 13) that τ_1, τ_2, τ_3 are on a line implies also that τ_4, τ_5, τ_6 are on a line. Then we find as in (d) that the even function $\Theta_{123}(d) = \Theta_{456}(d)$ vanishes. The corresponding condition on the associated set S_6^3, P of the set T_7^2 is that P is on a plane with s_1, s_2, s_3 and s_4, s_5, s_6 or that P is on a quadric section of W_2 by the pair of planes $\overline{s_1 s_2 s_3}$, $\overline{s_4 s_5 s_6}$. On transforming this condition on T_7^2 by $J_0(O, \tau_1, \tau_2)$; on S_6^3, P by I_{12}; and on $\Theta_{123}(d)$ by (7) we find a second type of discriminant condition which is satisfied when τ_3 approaches coincidence with O; when P approaches coincidence with s_3 in a direction on the quadric cone with node at s_3 and simple points at the other points of S_6^3, or P is on the quadric section of W_2 by this cone; and finally when the odd function $\Theta_3(d)$ vanishes. In this fashion we conclude that the quadric sections of W_2 by quadrics on S_6^3 are represented by the plane sections of the surface K_2 obtained by setting the coordinates equal to four linearly independent theta squares, i. e. of the Kummer surface. If four points $\tau_1, \ldots, \tau_4$ are on a line, $d \equiv \pi_{56}$, one of the fifteen proper half periods, and the point P of W_2 is on the line $\overline{s_5 s_6}$. Finally if P is on the cubic curve N^3 through S_6^3 the set T_7^2 is on a conic ([8]Theorem 10) and H_2 is this conic doubly covered while $d \equiv 0$. In fact $d \equiv 0$ implies that a pair of the g_1^2 are nodal parameters which can happen only for a doubly covered conic for which the pairs of g_1^2 are the doubly covered points and O is any of these double points. This is of course the indeterminate case for the inversion problem.

When $p = 3$ the four linearly independent quadrics on S_6^3 above are replaced by $8 = 2^p$ linearly independent cubic spreads with nodes at S_8^5 and containing the norm curve N^5 on S_8^5; and these map W_3 on the generalized Kummer K_3 of hyperelliptic type. The $2^{2p} = 64$ singular tangent spaces arise from the 8 cubic spreads (cones) with respectively a triple point at one of the eight points of S_8^5 and double curve N^5; and the 56 degenerate cubic spreads which break up into an S_4 on five of the eight points and a quadric on these five with nodes at the other three. These are connected with the 64 theta squares as above.

(*f*). Naturally we must have, in order to obtain specific analytic results such as (2), some analytic method for the hyperspace cases. Since the given equation defines in S_{2p-1} the set S_{2p+2}^{2p-1} and the norm curve N^{2p-1} on it, or if one prefers the N^{2p-1} with the set S_{2p+2}^{2p-1} on it, it is convenient to refer the points of the space to this norm curve. The case $p=2$ is perhaps too special to be fairly illustrative, but I cannot go into the higher cases here. Let the parameters t of the six points of S_6^3 on N^3 be given by the binary sextic $(\alpha t)^6 = 0$; and let the three osculating planes of N^3 which meet in a point a of S_3 be given by the binary cubic

$$(at)^3 = (a't)^3 = \cdots = 0.$$

Then, if

$$(\beta t)^2 = \beta_0 t_1^2 + 2\beta_1 t_1 t_2 + \beta_2 t_2^2$$

is an arbitrary quadratic, β_3 an arbitrary constant, we have, in

$$\beta_3(\alpha a)^3(\alpha a')^3 + (aa')^2(a\beta)(a'\beta) = 0, \tag{8}$$

for variable β_3, β_0, β_1, β_2 the web of quadrics on S_6^3 in variables a_0, a_1, a_2, a_3, the coefficients of the variable cubic $(at)^3$. The polarized form of (8) in variables a, b is

$$\beta_3(\alpha a)^3(\alpha b)^3 + (ab)^2(a\beta)(b\beta) = 0. \tag{9}$$

If we assume that the quadric (8) has a node at a, then in (9) the coefficients of b_0, b_1, b_2, b_3 vanish giving rise to four equations bilinear in a and β. If from these we eliminate the β's we have the quartic equation in variables a of the Weddle surface W_2. If, however, we eliminate the a's we have the quartic equation in variables β of the Kummer surface K_2 as an envelope. H. F. Baker has given these equations as four-row determinants ([9]p. 65 and p. 56). Baker remarks ([9]p. 60) that K is a simultaneous invariant of the quadratic $(\beta t)^2$ and the sextic $(\alpha t)^6$. Strictly speaking this is not so, because of the non-homogeneity in the coefficients. However K can be written

$$K = K_0\beta_3^4 + K_1\beta_3^3 + K_2\beta_3^2 + K_3\beta_3 + K_4, \tag{10}$$

where the K_i are such simultaneous invariants. From the degrees and to some extent from the form we infer at once that these coefficients must be numerical multiples of respectively: the catalecticant of $(\alpha t)^6$; the apolarity invariant of $(\beta t)^2$ and the quadratic covariant of $(\alpha t)^6$ of degree 3; the apolarity invariant of $[(\beta t)^2]^2$ and $(\alpha\alpha')^4(\alpha t)^2(\alpha' t)^2$, together with the product of $(\beta\beta')^2$ and $(\alpha\alpha')^6$; the apolarity invariant of $(\alpha t)^6$ and $[(\beta t)^2]^3$; and $[(\beta\beta')^2]^2$. Similarly the surface W is a simultaneous invariant of $(\alpha t)^6$ and $(at)^3$ which from the degrees must evidently be the apolarity invariant of $(\alpha t)^6$ with the product of $(at)^3$ and its cubic covariant, i. e.

$$W \equiv (aa')^2(a'a'')(a\alpha)(a''\alpha)^2(a'''\alpha)^3 = 0. \tag{11}$$

(g). The extension of this method to further cases constitutes a remarkably neat application of many phases of the theory of binary forms. Let me give one illustration. For $p = 3$ and given S_8^5, a point in S_5 is determined by a binary quintic $(at)^5 = (a't)^5 = \cdots$. The catalecticant of this quintic,

$$(aa')^2(aa'')^2(a'a'')^2(at)(a't)(a''t) = 0,$$

determines a one-parameter (t) system of cubic spreads with N^5 as a double curve and triple point at t. Thus for $t = t_1, \ldots, t_8$ we have respectively the eight cubic cones mentioned in 5. The existence of this system leads to the following theorem.

THEOREM. *The Kummer* 3-*way in* S_7 *with* 64 *singular points and* 64 *singular tangent spaces has in the hyperelliptic case an additional configuration of* 64 S_3*'s each with the following property: An* S_3 *contains* 8 *singular points and is contained in* 8 *singular tangent spaces. The eight points lie on a cubic curve in the* S_3 *with parameters* $t_1, \ldots, t_8$ *and they lie three at a time on the remaining* 56 *singular tangent spaces.*

We observe then that the generalization of the conics in the singular tangent planes of the Kummer K_2 is in part the 2-ways in the singular tangent sections of K_3, and in part the cubic curves in the 64 S_3's, each curve containing an intermediate set R_6^3.

A fuller development of these ideas leads quite naturally to the proof of the formula (2) which connects the invariants (B) and the theta fourth powers. To these modular relations I shall briefly return.

4. *The Modular Spreads Associated with the Invariants* (A) *and* (B). (*a*). The number of invariants (B) is $\frac{1}{2}\binom{2p+2}{p+1}$. I have shown ([4]I(10)) that only $\nu = \frac{1}{p+2}\binom{2p+2}{p+1}$ of these are linearly independent. Similarly the number of invariants (A) is $(2p+2)!/(p+1)!\,2^{p+1}$. I shall show in a forthcoming paper which will elaborate the indications presented here that the number of linearly independent invariants (A) is likewise ν. Thus the invariants (A) and (B) lie in linear systems of the same dimension $\nu-1$ and it is sufficient to know the values of ν independent invariants of either system in order to obtain all of them. They are of course subject also to many relations of higher degree. By means of either linear system the totality of birationally distinct binary $(2p+2)$-ics is mapped upon the points of a "modular spread" M_{2p-1} of dimension $2p-1$ in $S_{\nu-1}$.

(*b*). For the invariants (A) we secure the mapping by fixing all but one of the points of the set S_{2p+2}^{2p-1} at a convenient "base" and allowing the last one to vary over S_{2p-1}. Then the invariants (A) become spreads of order p with $(p-1)$-fold points at the fixed base points and this linear system maps the space S_{2p-1} upon M_{2p-1}. To the permutation group of the roots of the underlying binary $(2p+2)$-ic there corresponds, in $S_{\nu-1}$, a collineation group of order $(2p+2)!$, $C_{n!}$, under which M_{2p-1} is invariant; in S_{2p-1}, however, the Moore cross-ratio group which is generated by $(2p+1)!$ permutations of the fixed base points and an additional Cremona involution with $2p$ of the fixed base points as F-points and the remaining one a fixed point (cf. for the sextic [10]§ 1).

For the invariants (B) attached to the set R^p_{2p+2} we select ν independent ones as coordinates in $S_{\nu-1}$ and then find that the permutation of the roots or of the points of the set R leads likewise to a collineation group $C'_{n!}$ in $S_{\nu-1}$. It is the obvious thing to make such a selection of coordinates that the collineation groups $C_{n!}$ and $C'_{n!}$ attached respectively to the systems (A) and (B) shall coincide. Furthermore in order to distinguish between the two mappings we map by the system (A) upon the points of $S_{\nu-1}$ and by the system (B) upon the $S_{\nu-2}$'s in $S_{\nu-1}$.

(*c*). For the sextic, $\nu = 5$ and the modular spread M_3 in S_4 is a cubic three-way, the map of S_3 by quadrics on 5 points. By proper choice of the quadrics M_3 takes the form $\sum_{i=1}^{6} x_i^3 = 0$ where $\sum_{i=1}^{6} x_i = 0$. The collineation $C_{6!}$ is then merely the permutations of the six coordinates x_i. By proper choice of the invariants (B) the planar sets of 6 points on a conic, R^2_6, are mapped upon the S_3's of S_4, $u_1, \ldots, u_6$ where $\sum_{i=1}^{6} u_i = 0$ in such a way that the relation

$$\left(\sum u_1 u_2\right)^2 - 4\left(\sum u_1 u_2 u_3 u_4\right) = 0$$

is satisfied. Here then the mapping is such that the invariants (A), (B) of the same binary sextic give rise to point and tangent space at the point of the *same* modular spread ([10](33)).

For the octavic, $\nu = 14$, and the geometry is hardly suitable for hasty exposition. Thus the modular spread attached to the invariants (A) is an M_5^{40} of order 40 and dimension 5 in S_{13}. We have, however, in the formula

$$\epsilon(ij)(kl)(mn)(op) = \vartheta^2_{ijkl}\vartheta^2_{ijmn}\vartheta^2_{ijop}, \tag{12}$$

where ϵ is a properly chosen sign, the expression of the invariants (A) as uniform modular functions which are in immediate algebraic relation with the invariants (B). For detailed study the tactical relations exhibited by E. H. Moore[11] are most advantageous.

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GROUP OF A SET OF SIMULTANEOUS ALGEBRAIC EQUATIONS*

BY LOUIS WEISNER

1. *Introduction.* Consider a system of m independent and consistent algebraic equations in m variables

$$f_k(z_1, \cdots, z_m) = 0, \qquad (k = 1, \cdots, m). \tag{1}$$

Let the roots of this system of equations be $(z_{1a}, \ldots z_{ma})$ $(a = 1, \cdots, n)$. We assume that each z_{pa} is finite. Therefore by a process of elimination we obtain equations

$$F_p(z_p) = 0, \qquad (p = 1, \cdots, m), \tag{2}$$

each of which is of degree n and involves only one of the unknowns. None of these equations vanishes identically, since equations (1) are independent. We assume that none of equations (2) has a multiple root.

A rational domain R which includes the coefficients of (1) includes the coefficients of (2). It will be shown in this paper that the groups of equations (2) relative to R are identical, except for the symbols which they affect.

2. *Elimination of z_m.* On eliminating z_m from (1) we obtain $m-1$ equations involving $z_1, \cdots, z_{m-1}$. Now z_{m-1} cannot be absent from all of these equations; for in that case we would have $m-1$ equations involving fewer than $m-1$ unknowns, and equations (1) would not be consistent and independent as assumed. Repeating this argument we can find two independent and consistent equations

$$g_1(z_1, z_2) = 0, \qquad g_2(z_1, z_2) = 0, \tag{3}$$

involving the unknowns z_1, z_2 (or any other pair of the unknowns). This elimination can always be carried out in such a way that if (z_{1a}, z_{2a}) is a root of equations (3), then there is a root $(z_{1a}, z_{2a}, \cdots, z_{ma})$ of equations (1).

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3. *Relations of Rationality.* Let R' be the domain obtained by adjoining z_{2a} to R. By equations (3), $g_1(z_{1a}, z_{2a})$ and $g_2(z_{1a}, z_{2a})$ are rational functions of a root of $F_1(z_1) = 0$, with coefficients in R' which equal a number (zero) in R'; they are therefore unaltered in value by the substitutions of the group H of $F_1(z_1) = 0$ relative to R'. Suppose that a substitution of H changes z_{1a} to z_{1b}. Applying this substitution to $g_1(z_{1a}, z_{2a})$ and $g_2(z_{1a}, z_{2a})$ we obtain $g_1(z_{1b}, z_{2a}) = 0$ and $g_2(z_{1b}, z_{2a}) = 0$. By § 2, there are two roots $(z_{1a}, z_{2a}, \cdots, z_{ma})$ and $(z_{1b}, z_{2a}, \cdots, z_{mb})$ of equations (1) with the same value for z_2. Therefore $F_2(z_2) = 0$ has a multiple root, contrary to assumption. It follows that H leaves z_{1a} fixed. Hence z_{1a} is in R' and is a rational function of z_{2a} with coefficients in R.

THEOREM 1. *If $(z_{1a}, z_{2a}, \cdots, z_{ma})$ is a root of equations (1), z_{pa} equals a rational function of z_{qa} with coefficients in R $(p, q = 1, 2, \cdots, m)$**.

4. *Group Relations.* Let us regard G_p, the group of $F_p(z_p) = 0$ relative to R, as a substitution group not on the roots $z_{p1}, z_{p2}, \cdots, z_{pn}$, but on their second subscripts. Thus (12) simultaneously interchanges z_{11} and z_{12}, z_{21} and z_{22} etc. With this understanding we state our main result.

THEOREM 2. *The groups of equations* (2) *relative to R are identical.*

We shall prove G_1 identical with G_2. By Theorem 1, $z_{pa} = \varphi_{pa}(z_{1a})$ where φ_{pa} is a rational function with coefficients in R. Hence by equations (1)

$$(4) \quad f_k[z_{1a}, \varphi_{2a}(z_{1a}), \cdots, \varphi_{ma}(z_{1a})] = 0, \quad \begin{pmatrix} k = 1, 2, \cdots, m \\ a = 1, 2, \cdots, n \end{pmatrix}$$

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* The following is a proof which might be offered of this theorem: Eliminate every power of z_1 from equations (3) except the first; then z_1 is expressed as a rational function of z_2 with coefficients in R. The objection to this proof is that in the process of elimination the first power of z_1 may be incidentally eliminated. It is easy to make up examples where this actually occurs. I believe the proof given above is free from objection.

Now suppose a substitution of G_1 replaces z_{1a} by z_{1b}. Then equations (4) become

$$f_k[z_{1b},\ \varphi_{2a}(z_{1b}),\ \cdots,\ \varphi_{ma}(z_{1b})] = 0. \tag{5}$$

Therefore $[z_{1b},\ \varphi_{2a}(z_{1b}),\ \cdots,\ \varphi_{ma}(z_{1b})]$ is a root of (1). But $[z_{1b},\ \varphi_{2b}(z_{1b}),\ \cdots,\ \varphi_{mb}(z_{1b})]$ is a root of (1), and (2) has no multiple root. Hence $\varphi_{pa}(z_{1b}) = \varphi_{pb}(z_{1b})$. We may therefore omit the second subscript of φ_{pa} and write

$$z_{pa} = \varphi_p(z_{1a}). \tag{6}$$

Now let $\psi(z_{21}, \cdots, z_{2n})$ be a rational function with coefficients in R of the roots of $F_2(z_2) = 0$ which equals a number in R. Then by (6) $\psi[\varphi_2(z_{11}), \cdots, \varphi_2(z_{1n})]$ is a rational function with coefficients in R of the roots of $F_1(z_1) = 0$ which equals a number in R and hence is unaltered by G_1. But on applying a substitution of G to $\psi[\varphi_2(z_{11}), \cdots, \varphi_2(z_{1n})]$ we find that, in virtue of (6), $z_{21}, \cdots, z_{2n}$ undergo a similar substitution. Therefore $\psi(z_{21}, \cdots, z_{2n})$ is unaltered by every substitution of G_1. It follows that G_1 is a subgroup of G_2. Similarly G_2 is a subgroup of G_1. Hence G_1 and G_2 are identical.

5. *A Necessary and Sufficient Condition.* We shall call $G = G_p$ $(p = 1, \cdots, m)$ the group of equations (1) relative to R. Let $\psi(z_{11}, \cdots, z_{1n}; \cdots; z_{m1}, \cdots, z_{mn})$ be unaltered in value by all the substitutions of G. Then so is $\psi[z_{11}, \cdots, z_{1n}; \cdots; \varphi_m(z_{11}), \cdots, \varphi_m(z_{1n})]$. But the latter is a rational function with coefficients in R of the roots of $F_1(z_1) = 0$, which is unaltered by G. Hence it equals a number in R. Conversely, let $\psi(z_{11}, \cdots, z_{1n}; \cdots; z_{m1}, \cdots, z_{mn})$ equal a number in R. Then so does $\psi[z_{11}, \cdots, z_{1n}; \cdots; \varphi_m(z_{11}), \cdots, \varphi_m(z_{1n})]$. Since the latter is unaltered by G, so is the former.

THEOREM 3. *A necessary and sufficient condition that a rational function with coefficients in R, of the numbers satisfying a set of independent and consistent algebraic equations with coefficients in R, be unaltered in value by the group of the equations relative to R, is that it equal a number in R.*

THE UNIVERSITY OF ROCHESTER

ON A TYPE OF PLANE UNICURSAL CURVE

BY HAROLD HILTON

The type considered is that which meets the sides of a real triangle in points whose parameters have at most three distinct values. If we take the triangle as triangle of reference, and suitable homogeneous coordinates, we may suppose any point (x, y, z) on the curve given by the equations

$$(1)\qquad \begin{cases} x=(t-\alpha)^{p_1}(t-\beta)^{q_1}(t-\gamma)^{r_1}, \\ y=(t-\alpha)^{p_2}(t-\beta)^{q_2}(t-\gamma)^{r_2}, \\ z=(t-\alpha)^{p_3}(t-\beta)^{q_3}(t-\gamma)^{r_3}, \end{cases}$$

where

$$(2)\qquad p_1+q_1+r_1=p_2+q_2+r_2=p_3+q_3+r_3=n,$$

where n is the degree of the curve.

In this, one of the quantities p_1, p_2, p_3 is zero and the other two are zero or positive integers; and so for q_1, q_2, q_3 and r_1, r_2, r_3.

We may denote the curve (1) symbolically by the array

$$(3)\qquad \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix},$$

or, as it may be written to save space,

$$(4)\qquad (p_1\ q_1\ r_1,\ \ p_2\ q_2\ r_2,\ \ p_3\ q_3\ r_3).$$

It is evident that (*a*) each row of (3) contains at least one zero, (*b*) the sum of the elements in each column is n, (*c*) no two columns are identical, (*d*) we get no essentially distinct curve, if we interchange two rows or interchange two columns.

The first problem that suggests itself is to find the number of essentially distinct curves of the required type; i. e., to find the number of distinct arrays (3) subject to the restrictions just mentioned. We content ourselves with giving the result in the form of a table, the first column giving the array (4), and the last two giving the number of distinct curves when n is odd or even.

Note that in this table

$$(5)\qquad \begin{cases} p+P=q+Q=n, \\ \lambda+\mu+\nu=n, \\ f+F=g+G=h+H=n; \end{cases}$$

f is not less than F, g, G, h, or H.

Array		n odd	n even
$(n00,\ 0n0,\ 00n)$		1	1
$(pP0,\ 0n0,\ 00n)$		$n-1$	$n-1$
$(0pP,\ 0n0,\ 00n)$	$p \geqq P$	$\frac{1}{2}(n-1)$	$\frac{1}{2}n$
$(\lambda\mu\nu,\ 0n0,\ 00n)$	$\mu \geqq \nu$	$\frac{1}{4}(n-1)^2$	$\frac{1}{4}n(n-2)$
$(pP0,\ qQ0,\ 00n)$	$p \geqq Q;\ p>P, q$	$\frac{1}{4}(n-1)^2$	$\frac{1}{4}n(n-2)$
$(0pP,\ qQ0,\ 00n)$		$(n-1)^2$	$(n-1)^2$
$(\lambda\mu\nu,\ qQ0,\ 00n)$	$\lambda>\mu$ or $\lambda=\mu,\ q>Q$	$\frac{1}{4}(n-1)^2(n-2)$	$\frac{1}{4}n(n-2)^2$
$(fF0, f0F, 0Ff)$	$f>F$	$\frac{1}{2}(n-1)$	$\frac{1}{2}(n-2)$
$(fF0, F0f, 0fF)$	$f>F$	$\frac{1}{2}(n-1)$	$\frac{1}{2}(n-2)$
$(ff0,\ f0f,\ 0ff)$		0	1
$(fF0, f0F, 0hH)$	$f>F;\ f>h \geqq H$	$\frac{1}{8}(n-1)(n-3)$	$\frac{1}{8}n(n-2)$
$(fF0, F0f, 0hH)$		$\frac{1}{4}(n-1)(n-3)$	$\frac{1}{4}(n-2)^2$
$(fF0, g0G, 0Ff)$	$f>F;\ f>g \geqq G$	$\frac{1}{8}(n-1)(n-3)$	$\frac{1}{8}n(n-2)$
$(fF0, g0G, 0gG)$	$f>g, G;\ g \neq G$	$\frac{1}{4}(n-1)(n-3)$	$\frac{1}{4}(n-2)(n-4)$
$(fF0, g0G, 0Gg)$	$f>g, G;\ g \neq G$	$\frac{1}{4}(n-1)(n-3)$	$\frac{1}{4}(n-2)(n-4)$
$(fF0,\ g0g,\ g0g)$		0	$\frac{1}{2}(n-2)$
$(fF0, g0G, 0hH)$		$\frac{1}{6}(n-1)(n-3)(n-5)$	$\frac{1}{6}(n-2)(n-3)(n-4)$

There are in all $\frac{1}{12}n(5n^2+7)$ distinct curves, if n is odd, and $\frac{5}{12}n(n^2+2)$, if n is even. In this, however, may be included some curves of degree less than n if two rows of (3) are proportional. This can only be the case, if the curve belongs to the type $(0pP,\ 0n0,\ 00n)$ and p and n have a common factor, or to the type $(\lambda\mu\nu,\ qQ0,\ 00n)$ and q is a multiple of n', where $\nu/n = \nu'/n'$ and ν' is prime to n'.

For $n=3$, there are 13 cubics, 4 cuspidal, 4 crunodal, and 5 acnodal.

For $n = 4$, there are 29 quartics, excluding (022, 040, 004), which is a conic. Of these, 8 have a superlinear branch of order three, 3 have a cusp-linear triple point, 6 have a rhamphoid cusp and another double point, 2 have a crunode and two acnodes, 2 have a cusp and two acnodes, 2 have two cusps and an acnode, 1 has three cusps, and 5 have three acnodes.

Among them we may note the following, whose equations are given referred to the triangle with its vertices at the double points:

(211, 040, 004), $3(y^2z^2+z^2x^2+x^2y^2)+2xyz(2\sqrt{2}\,x+2\sqrt{2}\,y+z) = 0,$
(310, 130, 004), $(x^2-6xy+y^2)z^2+x^2y^2 = 0,$
(031, 220, 004), $z^2(x-y)^2-4xyz(x+y)+16x^2y^2 = 0,$
(211, 130, 004), $z\{3(x+y)+\sqrt{5}\,(x-y)\}\{(x+y)z+xy\}+2x^2y^2 = 0,$
(310, 202, 013), $y^2z^2+z^2x^2+x^2y^2+xyz(x-2y-2z) = 0,$
(310, 202, 022), $y^2z^2+z^2x^2+x^2y^2-2xyz(x+y) = 0.$

It will be noticed that the curve obtained by the quadratic transformation, which consists in replacing x, y, z by $1/x$, $1/y$, $1/z$ in (1), is a curve of the same type.

Without loss of generality we may suppose that in (1) the numerically greatest of p_2-p_3, $p_3-p_1, \ldots, r_1-r_2$ is r_3-r_1, and that $r_3 \geqq r_2 \geqq r_1 = 0$, $p_1-p_3 \geqq q_1-q_3$. We may then put (1) in the form

(6) $$x = t^a(t-1)^b,\quad y = t^c(t-1)^d,\quad z = 1,$$

on replacing t by

$$\{\gamma(\beta-\alpha)t-\alpha(\beta-\gamma)\} \div \{(\beta-\alpha)t-(\beta-\gamma)\},$$

and replacing x, y, z by suitable multiples of themselves.

In (6) we have

(7) $$a = p_1-p_3,\quad b = q_1-q_3,\quad c = p_2-p_3,\quad d = q_2-q_3,$$

(8) $$a \geqq b \geqq 0,\quad (a+b) \geqq (c+d) \geqq 0,\quad b+c \geqq 0,\quad a+d \geqq 0,$$

in virtue of the restrictions which we have supposed to hold between $p_1, p_2, \ldots, r_3$.

The sides of the triangle of reference meet the curve only where $t=0, 1,$ or ∞. We shall suppose excluded the case $ad=bc$, when the curve becomes the well known type

$$x:y:z=t^a:t^c:1.$$

The tangent to (6) is

$$(9)\quad t^{-a}(t-1)^{-b}\{(c+d)t-c\}x-t^{-c}(t-1)^{-d}\{(a+b)t-a\}y \\ +\{(a+b-c-d)t-(a-c)\}z=0.$$

The inflexions, other than $t=0, 1, \infty$, are given by

$$(10)\quad \{(a+b-c-d)t-(a-c)\}\{(a+b)t-a\}\{(c+d)t-c\} \\ +(ad-bc)t(t-1)=0.$$

We see that the degree n of (6) is $a+b$ if $c\geqq 0$, $d\geqq 0$; $a+b-d$ if $c\geqq 0$, $d\leqq 0$; $a+b-c$ if $d\geqq 0$, $c\leqq 0$.

The class is in general the greatest of

$$a+b+1,\ a+d+1,\ b+c+1.$$

It may be less than this by unity in certain cases, e. g., if $a+b=c+d$.

The knowledge of the degree and class, combined with the fact that the deficiency is zero, gives the Plücker's numbers of the curve.

If ABC is the triangle whose sides are met by (6) in points whose parameters are $0, 1, \infty$, we have from (9) and (10) the following result.

From each of A, B, C at most one tangent can be drawn touching the curve at a point not lying on a side of the triangle ABC, and the number of these tangents is equal to the number of inflexions which do not lie on the sides of the triangle.

This number is three at most, but can be two, one, or zero, if one or more of

$$a+b-c-d,\ c+d,\ b,\ c,\ d,\ a-c,\ b-d$$

is zero. For instance, the type $(ff0, f0f, 0ff)$, in which $c+d$, b, and $a-c$ are zero, has no inflexion except at the vertices of ABC. It is a curve with three superlinear branches of order f, and includes the case ot the three-cusped hypocycloid.

The only other type with no inflexion except those on a side of the triangle ABC is $(n00,\ 0n0,\ 00n)$, discussed in MESSENGER OF MATHEMATICS, vol. 49 (1920), p. 132. Another interesting type is $(fF0,\ F0f,\ 0fF)$, discussed in this BULLETIN, vol. 29 (1923), p. 303. We may also note the type $(fF0,\ Ff0,\ 00n)$ with two real inflexions if $f \geqq \frac{1}{2}(n+\sqrt{n})$, especially the case in which $f=n-1$. The number of sextactic points not on a side of the triangle ABC is twelve or less.

If a node of (6) has parameters t_1 and t_2 (not 0, 1, or ∞) and

(11) $$t_1/t_2 = \omega, \quad (t_1-1)/(t_2-1) = \Omega,$$

we have ω and $\Omega \neq 1$, $\omega \neq \Omega$, while

(12) $$\omega^a \Omega^b = 1, \ \omega^c \Omega^d = 1.$$

These give

(13) $$\omega^\varepsilon = 1, \ \Omega^\varepsilon = 1, \text{ where } \varepsilon = ad-bc,$$

whence

(14) $$t_1 = \omega(1-\Omega)/(\omega-\Omega), \ t_2 = (1-\Omega)/(\omega-\Omega).$$

It is at once seen that the values of t_1 and t_2 given by (12), (13), (14) are complex and that the corresponding values of x and y given by (6) are real. Hence we have the following result.

The only point-singularities of the curve other than those on the sides of the triangle ABC are real acnodes.

If we take

(15) $$\omega = e^{2h\pi i/\varepsilon}, \ \Omega = e^{2k\pi i/\varepsilon},$$

the equation (12) becomes

(16) $$ha+kb \equiv 0 \pmod{\varepsilon}, \ hc+kd \equiv 0 \pmod{\varepsilon}$$

and the acnodes are given on finding values of h and k between 1 and $\varepsilon-1$ satisfying (16). We may simplify (16) by putting

$$a=\lambda a_1,\ c=\lambda c_1,\ b=\mu b_1,\ d=\mu d_1,\ \varepsilon=\lambda\mu\varepsilon_1=\lambda\mu(a_1d_1-b_1c_1),$$

where a_1 is prime to c_1 and b_1 to d_1. We readily find from (16), on eliminating h or k, that $h = \mu h_1$, $k = \lambda k_1$, where h_1 and k_1 are integral, and we get

(17) $h_1 a_1 + k_1 b_1 \equiv 0 \pmod{\varepsilon_1}$, $h_1 c_1 + k_1 d_1 \equiv 0 \pmod{\varepsilon_1}$.

The nature of the singularities on the sides of the triangle ABC is readily determined. For instance, suppose in (6) $c > a > 0$. Then (6) gives an expansion for t in ascending powers of $x^{1/a}$, and thence we get for y an expansion of the form

$$y = x^{c/a}(\alpha + \beta x^{1/a} + \gamma x^{2/a} + \cdots)$$

in general, fixing the nature of the singularity for which t is zero.

BEDFORD COLLEGE, UNIVERSITY OF LONDON

SURFACES WITH ORTHOGONAL LOCI OF THE CENTERS OF GEODESIC CURVATURE OF AN ORTHOGONAL SYSTEM*

BY MALCOLM FOSTER

We consider a surface S referred to any orthogonal system. Let G_1 and G_2 be the centers of geodesic curvature of the curves $u =$ const. and $v =$ const. respectively, through any point M of S. As M is displaced over the entire surface the loci of G_1 and G_2 will in general be two surfaces S_1 and S_2, corresponding elements of which are those which result from a common displacement of M. We ask: What are the surfaces S for which the surfaces S_1 and S_2 correspond with orthogonality of linear elements?

The condition that the displacements of G_1 and G_2 be orthogonal for every displacement of M, is that the absolute displacements of these points in the directions of the axes of the moving trihedral at M satisfy the relation

$$\sum \delta x_1 \, \delta x_2 = 0, \tag{1}$$

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* Presented to the Society, April 28, 1923.

for all values of dv/du. The x-axis of the trihedral is chosen tangent to the curve $v = \text{const.}$ The radii of geodesic curvature of the curves $u = \text{const.}$ and $v = \text{const.}$ we denote by ϱ_{gv} and ϱ_{gu} respectively. The relation (1) becomes*

$$\begin{aligned}\varrho_{gu} r_1\left(-\frac{\partial \varrho_{gv}}{\partial u}du - \frac{\partial \varrho_{gv}}{\partial v}dv + \xi du\right)dv \\ + \varrho_{gv} r\left(\frac{\partial \varrho_{gu}}{\partial u}du + \frac{\partial \varrho_{gu}}{\partial v}dv + \eta_1 dv\right)du \\ - \varrho_{gv}\varrho_{gu}(p\,du + p_1 dv)(q\,du + q_1 dv) \equiv 0.\end{aligned}$$

Hence, setting the coefficients of du^2, $du dv$, and dv^2 equal to zero, we get

$$(2)\quad \left\{\begin{aligned} \varrho_{gv}\left(\varrho_{gu} p q - r\frac{\partial \varrho_{gu}}{du}\right) = 0, \\ \varrho_{gu} r_1\left(\xi - \frac{\partial \varrho_{gv}}{\partial u}\right) + \varrho_{gv} r\left(\eta_1 + \frac{\partial \varrho_{gu}}{\partial v}\right) \\ - \varrho_{gv}\varrho_{gu}(p q_1 + p_1 q) = 0, \\ \varrho_{gu}\left(\varrho_{gv} p_1 q_1 + r_1\frac{\partial \varrho_{gv}}{\partial v}\right) = 0.\end{aligned}\right.$$

Now $\varrho_{gv} = \frac{\eta_1}{r_1}$, and $\varrho_{gu} = \frac{\xi}{r}$;† using these values, and the relations between the fundamental quantities for the surface‡, the equations (2) reduce to

$$(3)\quad \left\{\begin{aligned} \varrho_{gu} p q - r\frac{\partial \varrho_{gu}}{\partial u} = 0, \\ p q_1 = 0, \\ \varrho_{gv} p_1 q_1 + r_1\frac{\partial \varrho_{gv}}{\partial v} = 0,\end{aligned}\right.$$

since $\varrho_{gv}, \varrho_{gu} \neq 0$.

Since $p = D'/\eta$, and $q_1 = -D'/\xi$,§ we see from the second member of (3) that both p and q_1, are zero, and that the parametric curves must be the lines of curvature.

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* Eisenhart, *Differential Geometry of Curves and Surfaces*, p. 170.

† Eisenhart, p. 132, formula (47), and p. 167, formulas (45).

‡ Eisenhart, p. 168 and p. 170.

§ Eisenhart, p. 174, formulas (73).

Consequently the first and third members of (3) reduce to[*]

$$\frac{\partial \varrho_{gu}}{\partial u} = 0,$$

$$\frac{\partial \varrho_{gv}}{\partial v} = 0.$$

Hence

$$(4) \qquad \varrho_{gv} = U, \qquad \varrho_{gu} = V,$$

where U and V are functions of u and v alone respectively. The parametric curves therefore have constant geodesic curvature and the system is isothermal.[†] The surface is therefore isothermal.

Making use of (4), we see that the elements ds_1^2 and ds_2^2 of the loci of G_1 and G_2 respectively, are

$$(5) \quad \begin{cases} ds_1^2 = \sum \delta x_1^2 = \dfrac{\eta_1^2}{r_1^2} \left[\dfrac{\left(\dfrac{\partial r_1}{\partial u}\right)^2}{r_1^2} + r^2 + q^2 \right] du^2, \\ ds_2^2 = \sum \delta x_2^2 = \dfrac{\xi^2}{r^2} \left[\dfrac{\left(\dfrac{\partial r}{\delta v}\right)^2}{r^2} + r_1^2 + p_1^2 \right] dv^2. \end{cases}$$

Hence the loci of G_1 and G_2 are curves and not surfaces. As the vertex of the trihedral describes a curve $u =$ const. the point G_1 remains fixed, and as the vertex of the trihedral describes a curve $v =$ const. the point G_2 remains fixed. The lines of curvature are therefore spherical in both systems; they lie on spheres whose centers lie on the loci of G_1 and G_2, and which are mutually orthogonal with S at every point.

We denote the curves which are the loci of G_1 and G_2 by Γ_1 and Γ_2 respectively. The curve Γ_1 is described by G_1 as the vertex of the trihedral describes every curve $v =$ const., and Γ_2 is described by G_2 as the vertex of the trihedral describes every curve $u = \text{const.}$ Suppose that the vertex of the trihedral describes a definite curve $u = \text{const.}$ The point G_1 remains fixed at some point on Γ_1 while G_2 describes Γ_2, and every tangent to Γ_2 will be perpendicular to the fixed tangent to Γ_1 at G_1. The curve Γ_2 is therefore either a plane curve whose plane is perpendicular to the fixed tangent to Γ_1, or a straight line perpendicular to this fixed tangent. Suppose now that the vertex of the trihedral describe a second curve $u = \text{const.}$ Then G_1 remains fixed at some second point of Γ_1 while G_2 describes Γ_2, and every tangent to Γ_2 will be perpendicular to the fixed tangent to Γ_1 at the second position of G_1. Consequently if Γ_2 be a plane curve, the locus of G_1, namely Γ_1, must be a straight line. We obtain similar conclusions if we consider the vertex of the trihedral to describe two different curves $v = \text{const.}$ Hence the locus of at least one of the points G_1 and G_2 is a straight line.

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[*] We exclude the cases where either $r = 0$, or $r_1 = 0$, since in either case the curves in one family are geodesics, and one of the points G_1 and G_2 is at infinity.

[†] Eisenhart, p. 137.

We suppose that it is the locus of G_1 which is a straight line. Consider the absolute displacements of the point G_1 in the directions of the axes of the trihedral T_u of a curve $v = \text{const.}$ We have*

$$(6) \qquad \frac{\delta x_1}{ds} = \frac{dx_1}{ds} + 1, \quad \frac{\delta y_1}{ds} = \frac{x_1}{\varrho}, \quad \frac{\delta z_1}{ds} = 0,$$

where ds is the element of arc of the curve $v = \text{const.}$, and ϱ is the radius of first curvature. From the third member of (6), we see that the line Γ_1 which is the locus of G_1, lies in the osculating plane of the curve $v = \text{const.}$ at every point. Now

$$x_1 = -\varrho_{gv} = -\frac{\eta_1}{r_1},\dagger \qquad ds = \sqrt{E}\,du = \xi du;$$

using these relations, together with the relations between the fundamental quantities for the surface,‡ equations (6) become

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* Eisenhart, p. 32.

† It is necessary tha ϱ_{gv} be measured in the opposite direction to that in which the parameter u increases. Cf. Darboux, vol. II, p. 359.

‡ Eisenhart, p. 168 and p. 170.

$$(7)\qquad \frac{\delta x_1}{ds} = \frac{\eta_1 \dfrac{\partial r_1}{\partial u}}{\xi r_1^2}, \quad \frac{\delta y_1}{ds} = -\frac{\eta_1}{\varrho r_1}, \quad \frac{\delta z_1}{ds} = 0.$$

Hence

$$(8)\qquad ds_1 = \frac{\eta_1 \left[\varrho^2 \left(\dfrac{\partial r_1}{\partial u}\right)^2 + \xi^2 r_1^2\right]^{1/2} ds}{\varrho \xi r_1^2},$$

where ds_1 is the element of the line Γ_1. The direction-cosines of Γ_1 relative to the trihedral T_u are therefore

$$(9)\ \alpha_1 = \frac{\varrho \dfrac{\partial r_1}{\partial u}}{\sqrt{\varrho^2 \left(\dfrac{\partial r_1}{\partial u}\right)^2 + \xi^2 r_1^2}}, \quad \beta_1 = \frac{-\xi r_1}{\sqrt{\varrho^2 \left(\dfrac{\partial r_1}{\partial u}\right)^2 + \xi^2 r_1^2}}, \quad \gamma_1 = 0.$$

Since Γ_1 is a straight line fixed in space, we must have

$$\frac{\delta \alpha_1}{ds} = \frac{\delta \beta_1}{ds} = \frac{\delta \gamma_1}{ds} = 0.$$

These equations become on using (9),*

$$(10)\quad \left\{ \begin{aligned}
&\frac{\delta \alpha_1}{ds} = \frac{d\alpha_1}{ds} - \frac{\beta_1}{\varrho} \\
&= \frac{r_1^2 \left[\varrho \xi \left(\varrho \dfrac{\partial^2 r_1}{\partial u^2} + \dfrac{\partial \varrho}{\partial u} \dfrac{\partial r_1}{\partial u}\right) - \varrho^2 \dfrac{\partial r_1}{\partial u} \dfrac{\partial \xi}{\partial u} + \xi^3 r_1\right]}{\varrho \left[\varrho^2 \left(\dfrac{\partial r_1}{\partial u}\right)^2 + \xi^2 r_1^2\right]^{3/2}} = 0, \\
&\frac{\delta \beta_1}{ds} = \frac{d\beta_1}{ds} + \frac{\alpha_1}{\varrho} \\
&= \frac{r_1 \dfrac{\partial r_1}{\partial u} \left[\varrho \xi \left(\varrho \dfrac{\partial^2 r_1}{\partial u^2} + \dfrac{\partial \varrho}{\partial u} \dfrac{\partial r_1}{\partial u}\right) - \varrho^2 \dfrac{\partial r_1}{\partial u} \dfrac{\partial \xi}{\partial u} + \xi^3 r_1\right]}{\xi \left[\varrho^2 \left(\dfrac{\partial r_1}{\partial u}\right)^2 + \xi^2 r_1^2\right]^{3/2}} = 0, \\
&\qquad \frac{\delta \gamma_1}{ds} = -\frac{\beta_1}{\tau} = \frac{\xi \eta_1 r_1}{\tau \left[\varrho^2 \left(\dfrac{\partial r_1}{\partial u}\right)^2 + \xi^2 r_1^2\right]^{1/2}} = 0,
\end{aligned} \right.$$

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* Eisenhart, p. 32.

where τ is the radius of second curvature of the curve $v = \text{const.}$ Hence we must have

$$(11)\qquad \left\{\begin{aligned} &\varrho\xi\left(\varrho\frac{\partial^2 r_1}{\partial u^2}+\frac{\partial\varrho}{\partial u}\frac{\partial r_1}{\partial u}\right)-\varrho^2\frac{\partial r_1}{\partial u}\frac{\partial\xi}{\partial u}+\xi^3 r_1=0,\\ &\frac{1}{\tau}=0. \end{aligned}\right.$$

The curves $v = \text{const.}$ are therefore plane; and since they are spherical, they are circles. Consequently $\varrho = \text{const.}$, and the first member of (11) reduces to

$$(12)\qquad \varrho^2\left(\xi\frac{\partial^2 r_1}{\partial u^2}-\frac{\partial r_1}{\partial u}\frac{\partial\xi}{\partial u}\right)+\xi^3 r_1=0.$$

The relation (12) may also be written in the form

$$(13)\qquad \varrho^2\frac{\partial}{\partial u}\left(\frac{\frac{\partial r_1}{\partial u}}{\xi}\right)+\frac{\partial\eta_1}{\partial u}=0.$$

The line Γ_1 therefore lies in the plane of every curve $v = \text{const.}$, and consequently the surfaces have plane lines of curvature in one system for which all the planes pass through the straight line Γ_1; such surfaces are called surfaces of Joachimsthal.*

Finally, the surfaces considered are isothermal surfaces of Joachimsthal for which the lines of curvature which lie in coaxial planes are circles, and for which either the relation (12) holds, or the corresponding relation is satisfied with reference to the curves $u = \text{const.}$

YALE UNIVERSITY

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* Eisenhart, pp. 308–310.

QUADRATIC FIELDS IN WHICH FACTORIZATION IS ALWAYS UNIQUE*

BY L. E DICKSON

1. *Definitions.* Let m be an integer, other than 0 and 1, such that m is not divisible by a perfect square exceeding unity. All numbers $r+s\sqrt{m}$ in which r and s are rational constitute a *field* $R(\sqrt{m})$. Its *algebraic integers* are known to be $x+y\theta$, where x and y are rational integers, and

$$(1) \qquad \theta=\sqrt{m} \quad \text{if } m\equiv 2 \text{ or } m\equiv 3 \pmod 4,$$

$$(2) \qquad \theta=\tfrac{1}{2}(1+\sqrt{m}),\ \theta^2=\theta-k, \quad \text{if } m\equiv 1 \pmod 4,$$

where $k=\frac{1}{4}(1-m)$. The *conjugate* of $\xi=x+y\theta$ is defined to be $\xi'=x+y\theta'$, where $\theta'=-\theta$ in case (1), and $\theta'=\frac{1}{2}(1-\sqrt{m})$ in case (2). The product $\xi\xi'$ is called the *norm* of ξ, and is denoted by $N(\xi)$. According as the case is (1) or (2), we have

$$(3) \qquad N(x+y\theta)=x^2-my^2 \quad \text{or} \quad x^2+xy+ky^2.$$

If ξ is an algebraic integer such that $N(\xi)=\pm 1$, then ξ is called a *unit.* The only units in $R(i)$ are ± 1 and $\pm i$.

2. *Object of the Paper.* It is known† that $-1, -2, -3, -7$ and -11 are the only negative values of m for which the greatest common divisor process yielding numerically decreasing norms is always applicable in $R(\sqrt{m})$, so that if a and b are any algebraic integers $(b\neq 0)$ there exist algebraic integers q and r of the field such that

$$a=bq+r, \quad |\text{norm } r|<|\text{norm } b|.$$

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* Presented to the Society, December 29, 1923. See also This Bulletin, p. 90, Jan.–Feb., 1924, and footnote, p. 247, May–June, 1924.

† For a geometric proof, see Birkhoff, American Mathematical Monthly, vol. 13 (1906), pp. 156–159.

For a few positive values of m, as 2, 3, 5, 11, such a process exists. But there are many values of m for which this process is not applicable, although there exists a greatest common divisor as shown by the theory of ideals when the number of classes of ideals is unity.

Avoiding the theory of ideals, we shall give an elementary proof of the following result.

THEOREM 1. *Let m be any integer for which there is a single* class of properly primitive binary quadratic forms capable of representing positive integers and having the discriminant $4m$ or m according as the case is* (1) *or* (2). *Then the algebraic integers of $R(\sqrt{m})$ admit unique factorization into primes apart from the association of unit factors.*†

This proof places at the disposal of students of elementary theory of numbers an effective tool utilized by Gauss and Dirichlet in the case $m=-1$. Moreover, the proof furnishes a model for the investigation‡ of the arithmetics of linear algebras for which no theory of ideals is available.

3. LEMMA 1. *If a and b are relatively prime and*

$$(4)\qquad a^2-mb^2=pq \quad \textit{or} \quad a^2+ab+kb^2=pq,\quad p>0,$$

there exist integers z and w such that

$$(5)\qquad p=z^2-mw^2 \quad \textit{or} \quad p=z^2+zw+kw^2,$$

according as case (1) *or* (2) *holds.*

For, b and q are relatively prime since otherwise a common prime divisor of them would divide a^2 and hence a. Hence there exist integers s and t such that $a = sb + tq$. Inserting this in (4), we get

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* In this BULLETIN, vol. 17 (1910–11), pp. 534–37, the writer proved that there is a single class of positive primitive quadratic forms of negative discriminant $-P$ with $P<1{,}500{,}000$ only when $P=3, 4, 7, 8, 11, 12, 16, 19, 27, 28, 43, 67, 163$. But the cases in which the discriminant is positive are very numerous.

† Theorem 1 holds also if there are only two classes and these are *opposite* classes of properly primitive forms of discriminant $4m$ or m. In the first case, every such form is equivalent to x^2-my^2 or mx^2-y^2. The proof differs from that in the text only by the occasional insertion of the double sign ±.

‡ See L. E. DICKSON, AMERICAN JOURNAL, 1924.

$$pq = Ab^2 + Bbqt + q^2t^2,$$

where $A = s^2 - m$, $B = 2s$ in case (1); while $A = s^2 + s + k$, $B = 2s + 1$ in case (2). Hence A must be divisible by q. Write $A = qe$. Then

$$p = eb^2 + Bbt + qt^2. \tag{6}$$

The discriminant of this form is $B^2 - 4eq = 4m$ or m according as the case is (1) or (2). If any odd prime divides e, B and q, its square divides $B^2 - 4eq$, whereas m has no square factor. If, in case (1), 2 divides both e and q, then $m = s^2 - eq \equiv s^2 \equiv 0$ or 1 (mod 4), contrary to (1). Hence in every case (6) is a properly primitive form representing the positive integer p and therefore, by our hypothesis of a single class, is equivalent to the respective form (5) of discriminant $4m$ or m.

4. LEMMA 2.* *If a and b are relatively prime integers and if $N(a + b\theta)$ is divisible by the rational prime p, then p decomposes and one of its integral algebraic factors divides $a + b\theta$.*

By hypothesis, we have (4) and hence (5).

(i) Let the first equations (4) and (5) hold. Then

$$mb^2 \equiv a^2, \quad z^2 \equiv mw^2 \pmod{p}.$$

By multiplication we get $bz \equiv \pm aw \pmod{p}$ if m is not divisible by p, and we may choose the upper sign after changing the sign of w if necessary. If m is divisible by p, we have $z \equiv 0$, $a \equiv 0 \pmod{p}$. Hence $bz \equiv aw \pmod{p}$ in all cases. Define rational numbers x and y by means of

$$p(x + y\theta) = (a + b\theta)(z - w\theta), \tag{7}$$

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* Another proof follows from the writer's theorem in this BULLETIN, vol. 29 (1923), pp. 464–467, that all solutions of $N(a + b\theta) = pq$ are products of the same integer ρ by the numbers obtained from (8) and $p = N(z + w\theta)$, $q = N(x + y\theta)$. Here $\rho = \pm 1$ since a and b are relatively prime. But if we omit §§ 3–4, we must replace § 5 by one of the standard proofs of unique representation of a prime as a norm.

whence

$$x=\frac{az-mbw}{p},\quad y=\frac{bz-aw}{p},$$

so that y is an integer. Since $p=(z+w\theta)(z-w\theta)$, we may cancel $z-w\theta$ from (7) and get

$$(z+w\theta)(x+y\theta)=a+b\theta. \tag{8}$$

Taking norms, we have $p(x^2-my^2)=pq$, whence x^2 is the integer my^2+q. Hence both x and y are integers in (8), which is the desired result.

(ii) Let the second equations (4) and (5) hold. Then

$$(2a+b)^2-mb^2=4pq,\quad 4p=(2z+w)^2-mw^2,$$

$$mb^2(2z+w)^2\equiv(2a+b)^2mw^2 \pmod{p}.$$

If m is not divisible by p, we get

$$b(2z+w)\equiv\pm(2a+b)w \pmod{p}. \tag{9}$$

First, let the upper sign hold in (9). Then, if $p\neq 2$, $bz\equiv aw \pmod{p}$. This follows also if m is divisible by p, whence $2a+b\equiv 0$, $2z+w\equiv 0$. Define rational numbers x and y by means of

$$p(x+y\theta)=(a+b\theta)(z+w\theta'), \tag{10}$$

whence

$$x=\frac{a(z+w)+kbw}{p},\quad y=\frac{bz-aw}{p},$$

so that y is an integer. Since $p=(z+w\theta)(z+w\theta')$, we may cancel $z+w\theta'$ from (10) and get (8). By the norm of (8), $p(x^2+xy+ky^2)=pq$. Thus x is a rational root of an equation with integral coefficients and leading coefficient unity; hence x is an integer.

Second, let the lower sign hold in (9). Then if $p\neq 2$, $b(z+w)\equiv -aw \pmod{p}$. Introduce the integers $Z=z+w$, $W=-w$. Then $bZ\equiv aW$ and

$$Z^2+ZW+kW^2=z^2+zw+kw^2=p, \tag{11}$$

and we are led to our first case with Z and W in place of z and w. Hence $a+b\theta$ has the factor $Z+W\theta=z+w\theta'$.

Finally, let $p=2$. If b is even, so that a is odd, a^2+ab

$+kb^2$ is odd, whereas it is divisible by $p=2$ by (4). If w is even, z is even in $2=z^2+zw+kw^2$, which is then divisible by 4. Hence b and w are both odd. By (11) we may add w to z and hence make z even or odd at pleasure and hence make $z\equiv a \pmod 2$. Then $bz\equiv aw \pmod 2$. We proceed as in the first case.

5. LEMMA 3. *If a rational prime p is expressible as a norm, it is the product of two algebraic integers, neither a unit, in one and only one way apart from unit factors and apart from the arrangement of the two integers.*

We have $p=(z+w\theta)(z+w\theta')$. Let also $p=\pi\varrho$, where $\pi=a+b\theta$ and ϱ are algebraic integers neither a unit. Then $p^2=N(\pi)N(\varrho)$, whence $N(\pi)=N(\varrho)=\pm p$. If a and b had a common prime factor, its square would divide $N(\pi)=\pm p$. Hence we may apply the proof of Lemma 2 with $q=\pm 1$ and obtain (8) or the similar equation with $z+w\theta$ replaced by $z+w\theta'$. Then $x+y\theta$ is a unit since its norm is $q=\pm 1$. Hence every factorization $p=\pi\varrho$ differs from the given one only by the insertion of unit factors or by the interchange of the given factors.

6. *Algebraic Primes.* An algebraic integer not a unit of $R(\theta)$ is called an *algebraic prime* if it is not a product of two algebraic integers neither a unit of $R(\theta)$.

If a rational prime p is a norm, so that $p=\pi\pi'$, then π and π' are algebraic primes. For, if $\pi=\alpha\beta$, where neither α nor β is a unit, then $p=N(\alpha)N(\beta)$, and one of the norms is ± 1, so that α or β is a unit.

7. LEMMA 4. *If $N(c+d\theta)$ is divisible by a rational prime p, then either $c+d\theta$ is divisible by p or else p decomposes and $c+d\theta$ is divisible by one of the algebraic prime factors of p.*

Let g be the greatest common divisor of $c=ga$, $d=gb$. Then

$$c+d\theta=g(a+b\theta) \tag{12}$$

is divisible by p if g is. Next, let g be not divisible by p. Since $N(c+d\theta)=q^2N(a+b\theta)$ is divisible by p, $N(a+b\theta)$ is divisible by p and Lemma 2 shows that p decomposes and that $a+b\theta$ is divisible by one of the algebraic prime factors of p. Hence (12) is divisible by that factor.

8. THEOREM 2. *If an algebraic prime divides a product AB, it divides A or B.*

(i) Suppose the algebraic prime is a rational prime p. We may write $A=q\alpha$, where q is a rational integer and α has relatively prime coordinates. Then p divides qP, where $P=\alpha B=r+s\theta$. Then p divides qr and qs. If p divides q, it divides A. In the contrary case, p divides r, s and hence also αB. Write $B=t\beta$, where β has relatively prime coordinates. As before, either p divides t and hence B, or else p divides $\alpha\beta$, so that p^2 divides $N(\alpha)N(\beta)$. We may then assume that p divides $N(\alpha)$ for example, whence, by Lemma 2, p decomposes, whereas it is an algebraic prime.

(ii) Let π be an algebraic prime not the product of an integer by a unit. Hence the coordinates of π are relatively prime. The integer $N(\pi)$ is divisible by a rational prime p. Hence, by Lemma 2, $p=\varrho\varrho'$ and ϱ is a divisor of π. Thus the prime π is the product of ϱ by a unit, whence $N(\pi)=\pm N(\varrho)=\pm p$. Suppose that π divides AB, but divides neither A nor B. Then $\pm p=N(\pi)$ divides $N(A)N(B)$. Let therefore p divide $N(A)$ for example. By Lemmas 3 and 4, A is divisible by π or π'. Hence A is divisible by π'. Write $A=\pi'\alpha$, $P=\alpha B$. Then π divides $AB=\pi'P$, so that $p=\pm\pi\pi'$ divides π'^2P. We shall prove that p divides $\pi'P$. Write

$$\pi'=a+b\theta,\ P=c+d\theta,\ \pi'^2P=r+s\theta,$$
$$r\equiv s\equiv 0\pmod{p}.$$

First, let $\theta=\sqrt{m}$. Then $\pm p=a^2-mb^2$ and

$$r=a^2c+2mabd+mb^2c,\quad s=a^2d+2abc+mb^2d.$$

Elimination of a^2 by means of $a^2\equiv mb^2\pmod{p}$ gives

$$r\equiv 2mb(ad+bc)\equiv 0,\quad s\equiv 2b(ac+mbd)\equiv 0\pmod{p}.$$

If $2mb$ is not divisible by p, the numbers in parenthesis, which are the coordinates of $\pi' P$, are divisible by p. Hence P is divisible by π. Thus π divides αB, but divides neither α nor B. A repetition of the argument leads evidently to a contradiction. Next, if b were divisible by p, a would be also in view of $\pm p = a^2 - mb^2$, which would then be divisible by p^2. If m were divisible by p, a would be also, whence π'^2 is divisible by p, and hence π' by π, so that π' is the product of π by a unit. Since A is divisible by π', it is divisible by π. If $p = 2$, $\pi'^2 \equiv a^2 + b^2 m \equiv 0 \pmod{p}$.

Second, consider case (2). Then $\pm p = a^2 + ab + kb^2$, so that b is not divisible by p. In

$$r = a^2c - 2kabd - kb^2c - kb^2d,$$
$$s = a^2d + b^2c + 2abc + 2abd + (1-k)b^2d$$

we replace a^2 by $-ab - kb^2 \pmod{p}$, cancel factors b, and get

$$\begin{aligned} &2kad + ac + 2kbc + kbd \equiv 0, \\ &bc + 2ac + ad + (1-2k)bd \equiv 0 \qquad \pmod{p}. \end{aligned} \tag{13}$$

Elimination of ac gives $(4k-1)E \equiv 0$, where $E = ad + bc + bd$. But if $4k - 1 = -m$ were divisible by p, $\pm 4p = (2a+b)^2 - mb^2$ shows that $2a + b \equiv 0$, whence

$$\pi'^2 = a^2 - kb^2 + b(b + 2a)\theta \equiv a^2(1-4k) = ma^2 \equiv 0 \pmod{p},$$

so that A is divisible by π as in the first case. In view of this contradiction, we have $E \equiv 0$. Adding $-2kE$ to the first equation (13), we get $F \equiv ac - kbd \equiv 0$. Hence

$$\pi' P = F + E\theta \equiv 0 \pmod{p}.$$

The University of Chicago

THE JACOBIAN OF A CONTACT TRANSFORMATION*

BY E F. ALLEN

The equations

$$(1)\qquad x_1 = X(x, z, p), \quad z_1 = Z(x, z, p), \quad p_1 = P(x, z, p),$$

where X, Z, and P are functions of class C'', represent a transformation of line-elements in the xz plane to line-elements in the x_1z_1 plane. With Lie we shall define every transformation in x, z, p, which leaves the Pfaff differential equation

$$(2)\qquad dz - p\,dx = 0$$

invariant, as a contact transformation of the xz plane to the x_1z_1 plane. Hence the equations (1) must satisfy an identity of the form

$$(3)\qquad dz_1 - p_1 dx_1 = \varrho(dz - p\,dx),$$

where ϱ is a function of x, z, and p alone.

The following relations connecting X, Z, P, and their partial derivatives are easily obtained:†

$$(4)\qquad \begin{cases} Z_x - PX_x = -p\varrho, \\ Z_z - PX_z = \varrho, \\ Z_p - PX_p = 0; \end{cases}$$

$$(5)\qquad \begin{cases} [XZ] = X_p(Z_x + pZ_z) - Z_p(X_x + pX_z) = 0, \\ [PX] = \varrho, \text{ and } [PZ] = \varrho P. \end{cases}$$

The jacobian of transformation (1) is

$$(6)\qquad J = \begin{vmatrix} X_x & X_z & X_p \\ Z_x & Z_z & Z_p \\ P_x & P_z & P_p \end{vmatrix}.$$

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* Presented to the Society, December 1, 1923.

† Lie und Scheffers, *Geometrie der Berührungstransformationen*, p. 68, Chap. 3.

We shall show that this jacobian is equal to ϱ^2. Let us multiply the first row by P and subtract the product from the second row; then

$$(7)\qquad J=\begin{vmatrix} X_x & X_z & X_p \\ Z_x-PX_x & Z_z-PX_z & Z_p-PX_p \\ P_x & P_z & P_p \end{vmatrix}.$$

Hence, using equations (4), we find

$$(8)\qquad J=\begin{vmatrix} X_x & X_z & X_p \\ -p\varrho & \varrho & 0 \\ P_x & P_z & P_p \end{vmatrix}.$$

This reduces to

$$(9)\qquad J=\begin{vmatrix} X_x+pX_z & X_z & X_p \\ 0 & \varrho & 0 \\ P_x+pP_z & P_z & P_p \end{vmatrix},$$

when the second column is multiplied by p and the sum is added to the first column. Evaluating this determinant, we have

$$(10)\qquad J=\varrho[P_p(X_x+pX_z)-X_p(P_x+pP_z)].$$

Therefore, by equation (5), we may write

$$(11)\qquad J=\varrho^2.$$

The equations of a contact transformation may be regarded as the equations of a point transformation, which transforms points in xzp space to points in $x_1z_1p_1$ space. In general a surface in xzp space, represented by the equation $F_1(x, z, p)=0$, will be transformed into a surface in $x_1z_1p_1$ space, represented by the equation $F_2(x_1, z_1, p_1)=0$. Or if we regard equations (1) as the equations of a transformation of line-elements, it will transform a differential equation in x, z, p, into one in x_1, z_1, p_1, and also the solutions of the first differential equation into the solutions of the second.

Now if we set ϱ equal to zero,* we will have the equation of a surface in xzp space, or we might say that we have a differential equation in xz space. Let us see into what this surface or into what this differential equation is transformed when it is subjected to the transformation (1). A few examples result in obtaining curves in $x_1z_1p_1$ space or in obtaining differential equations free from p_1. This leads to the following theorem.

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* In some cases there are no values of the variables that will make ρ equal to zero. The following theory does not apply to such cases.

THEOREM. *The surface* $\varrho = 0$ *is transformed into a curve in space by the transformation* (1).

If the partial derivative of ϱ with respect to z is not identically equal to zero, the equation $\varrho = 0$ may be solved for z.* Assuming that this is true, when the value thus obtained for z is substituted in X, Z, and P, they become functions of x and p alone. Regarding x and p as parameters, the equations (1) are the parametric equations of a surface. A necessary and sufficient condition† that

$$(12)\qquad x_1 = f(x, y),\quad y_1 = g(x, y),\quad z_1 = h(x, y)$$

define a curve in space and not a surface is that

$$(13)\qquad EG - F^2 = A^2 + B^2 + C^2 \equiv 0,$$

where

$$(14)\qquad A = \frac{\partial(y_1, z_1)}{\partial(x, y)},\quad B = \frac{\partial(z_1, x_1)}{\partial(x, y)},\quad C = \frac{\partial(x_1, y_1)}{\partial(x, y)}.$$

To prove our theorem it is necessary and sufficient to show that the A, B, and C connected with equations (1) are identically equal to zero. That is, it is sufficient to show that all the determinants of the following matrix vanish identically:

$$(15)\qquad \left|\begin{matrix} X_x + pX_z & X_p + \dfrac{\partial z}{\partial p} X_z \\ Z_x + pZ_z & Z_p + \dfrac{\partial z}{\partial p} Z_z \\ P_x + pP_z & P_p + \dfrac{\partial z}{\partial p} P_z \end{matrix}\right|.$$

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[*] If $\partial\rho/\partial z \equiv 0$ we will be able to solve for either x or p if $\rho \neq$ const.

[†] Eisenhart, *Differential Geometry*, p. 71.

Let us see what the effect will be when the value of z as obtained from $\varrho = 0$ is substituted in equations (1). Suppose that the substitution has been made in X and Z. It is easy to see that X_z and Z_z are equal to zero, and that to differentiate X completely with respect to x, it is necessary to differentiate with respect to x and then to use the function of a function rule, thus $X_x + X_z(\partial z/\partial x)$, and similarly for the other letters. Thus using the fact that $\varrho = 0$, we may write the equations (5) in the form

$$(16)\quad \left\{\begin{aligned} &\left(X_p+\frac{\partial z}{\partial p}X_z\right)(Z_x+pZ_z)-\left(Z_p+\frac{\partial z}{\partial p}Z_z\right)(X_x+pX_z)=0,\\ &\left(P_p+\frac{\partial z}{\partial p}P_z\right)(X_x+pX_z)-\left(X_p+\frac{\partial z}{\partial p}X_z\right)(P_x+pP_z)=0,\\ &\left(P_p+\frac{\partial z}{\partial p}P_z\right)(Z_x+pZ_z)-\left(Z_p+\frac{\partial z}{\partial p}Z_z\right)(P_x+pP_z)=0.\end{aligned}\right.$$

It is very easy to see that these equations are now the expanded form of the determinants of the matrix (15). Hence the theorem is proved.

THE UNIVERSITY OF MISSOURI

INTEGRO-DIFFERENTIAL INVARIANTS OF ONE-PARAMETER GROUPS OF FREDHOLM TRANSFORMATIONS[*]

BY A. D. MICHAL

1. *Statement of the Problem.* The author[†] has already considered functionals of the form $f[y(\tau_0'), y'(\tau_0')]$ (depending only on a function $y(\tau)$ and its derivative $y'(\tau)$ between 0 and 1) which are invariant under an arbitrary Volterra one-parameter group of continuous transformations. The

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[*] Presented to the Society, December 1, 1923.

[†] Cf. *Integro-differential expressions invariant under Volterra's group of transformations* in a forthcoming issue of the ANNALS OF MATHEMATICS. This paper will be referred to as "*I. D. I. V.*"

calculation of the invariants in question was effected in the case of a large class of functionals known as analytic functionals.

The purpose of this note is to consider the problem of finding analytic functionals $f[y(\tau_0'), y'(\tau_0')]$ invariant under a Fredholm group of transformations

$$y_1(x) = y(x) + \int_0^1 K(x, s \mid a) y(s) ds, \tag{1}$$

where a is the parameter of this continuous one-parameter group of transformations, and where $a = 0$ corresponds to the identical transformation.

We restrict ourselves to transformations (1) for which the $y'(\tau)$'s exist and are continuous in the interval $I: 0 \leqq \tau \leqq 1$; $K(x, s \mid a)$ and $\partial K / \partial x$ are continuous in x and s in the square $S: 0 \leqq x \leqq 1,\ 0 \leqq s \leqq 1$; and $\partial K / \partial x$ is not identically zero when $a \neq 0$.

The infinitesimal transformation corresponding to (1) will be of the form

$$\delta y(x) = \left[\int_0^1 H(x, s) y(s) ds\right] \delta a \tag{2}$$

with

$$\delta y'(x) = \left[\int_0^1 H_1(x, s) y(s) ds\right] \delta a, \tag{3}$$

where

$$H_1(x, s) \equiv \frac{\partial H(x, s)}{\partial x},$$

as the extended group of infinitesimal transformations.

Here follow the well known relations[*] between the kernel $K(x, s \mid a)$ of the Fredholm finite transformation (1) and the kernel $H(x, s)$ of the corresponding infinitesimal transformation (2):

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[*] Gerhard Kowalewski, *Über Funktionenräume,* Wiener Sitzungsberichte, 1911, vol. 120, II A.

$$K(x, s \mid a) = \sum_{i=1}^{\infty} \frac{a^i H^i}{i!}, \tag{4}$$

$$H(x, s) = \left[\frac{\partial K(x, s \mid a)}{\partial a}\right]_{a=0} = \frac{1}{a}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{K^i}{i}, \quad a \neq 0, \tag{5}$$

where H^i and K^i are to be interpreted according to Volterra's symbolic multiplication.

By methods similar to those employed in proving the lemma of Part I of *I. D. I. V.*, we can prove* without difficulty the following lemma.

LEMMA. *Necessary and sufficient conditions that $H(x, s)$ and $\partial H/\partial x$ be continuous in x and s and that $\partial H/\partial x$ be not identically zero are that $K(x, s \mid a)$ and $\partial K/\partial x$ be continuous in x and s and that $\partial K/\partial x$ be not identically zero, when $a \neq 0$.*

2. *A Sufficient Condition for Invariance.*

THEOREM 1. *Let $f[y(\tau_0'), y'(\tau_0')]$ be an analytic functional of $y(\tau)$ and $y'(\tau)$,* i. e., *developable in a Volterra expansion*†

$$f_{00} + \sum_{j=1}^{\infty}\left\{\frac{1}{j!}\int_0^1\int_{0_{(j)}}^1\cdots\int_0^1\left[\sum_{k=0}^{j}\binom{j}{k} f_{j-k,k}(t_1, \cdots, t_{j-k}; t_{j-k+1}, \cdots, t_j) \times \prod_{i=1}^{j-k} y(t_i) \prod_{i=j-k+1}^{j} y'(t_i)\right] dt_1 dt_2 \cdots dt_j\right\}. \tag{6}$$

We shall assume that $f_{j-k,k}$ is continuous in its j arguments, symmetric separately in the sets of arguments $t_1, t_2, \ldots, t_{j-k}$ and $t_{j-k+1}, \ldots, t_j$ respectively; and for convenience we assume also that

$$|f_{j-k,k}| < \gamma, \qquad |y| < \varrho_1, \qquad |y'| < \varrho_2, \tag{7}$$

where γ, ϱ_1, ϱ_2 are positive constants. Then a sufficient condition that $f[y(\tau_0'), y'(\tau^0)]$ be invariant under a given group of transformations (1) *is that it satisfy the relation*

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* The proof comes by a direct calculation of the series involved.

† This is a generalization of Taylor's series given by Volterra. See for example his *Leçons sur les Équations Intégrales*, 1913.

$$(8) \qquad \int_0^1 H_1(t,\, t_{i+k}) f_{i,\,k}(t_1,\, \ldots,\, t_i;\; t_{i+1},\, \ldots,\, t_{i+k-1},\, t)\,dt$$
$$- \equiv \int_0^1 H(t,\, t_{i+k}) f_{i+1,\,k-1}(t,\, t_1,\, \ldots,\, t_i;\; t_{i+1},\, \ldots,\, t_{i+k-1})\,dt.$$

The necessary and sufficient condition that $f[y(\tau_0'),\, y'(\tau_0)]$ be invariant under (1) is that under (2)

$$(9) \qquad \delta f[y(\tau_0'),\, y'(\tau_0')] \equiv 0 \text{ in } y \text{ and } y'.$$

Since the analyticity of our functionals insures the validity of a Volterra variation, we may use Volterra's* form of the variation of a functional. Then condition (9) becomes

$$(10) \qquad \int_0^1 f_y(t)\delta y(t)\,dt + \int_0^1 f_{y'}(t)\delta y'(t)\,dt \equiv 0$$

in y and y', where $f_y(t)$ and $f_{y'}(t)$ are the partial functional derivatives of $f[y(\tau_0'),\, y'(\tau_0')]$ with respect to $y(\tau)$ and $y'(\tau)$, respectively, both taken at the point t.

Substituting in (10) the values of $\delta y(t)$ and $\delta y'(t)$ as given by (2) and (3), respectively, rearranging and dividing through by δa, we get

$$(11) \qquad \int_0^1 y(s)\Big[\int_0^1 f_y(t)H(t,\, s)\,dt + \int_0^1 f_{y'}(t)H_1(t,\, s)\,dt\Big]ds \equiv 0$$

in y. We may now apply Lemma 2 of *I. D. I. V.*; doing so, we find

$$(12) \qquad \int_0^1 f_{y'}(t)H_1(t,\, s)\,dt = -\int_0^1 f_y(t)H(t,\, s)\,dt.$$

Such operations as functional differentiations term by term are valid since the series involved are uniformly convergent under our hypotheses.† Calculating the partial functional derivatives $f_y(t)$ and $f_{y'}(t)$, respectively, and substituting them in (12), we get by an easy reduction‡

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* A more general expression for δf would be in the form of Stieltjes integrals.

† Cf. *I. D. I. V.*, Part II, and Volterra's *Leçons sur les Équations Intégrales*, 1913, p. 18.

‡ Cf. similar reductions of *I. D. I. V.*

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\left\{\frac{1}{(j-1)!}\int_0^1\int_0^1\cdots\int_0^1\left[\sum_{l=0}^{j-1}\binom{j-1}{l}H_1(t,s)f_{j-1-l,\,l+1}(t_1,\ldots\right.\right.\\
&\ldots,t_{j-1-l};\,t_{j-l},\ldots,t_{j-1},t)\\
&\qquad\left.\left.\times\prod_{i=1}^{j-1-l}y(t_i)\prod_{i=j-l}^{j-1}y'(t_i)\right]dt_1\ldots dt_{j-1}dt\right\}\\
(13)\quad&\\
&\equiv-\sum_{j=1}^{\infty}\left\{\frac{1}{(j-1)!}\int_0^1\int_0^1\cdots\int_0^1\left[\sum_{k=0}^{j-1}\binom{j-1}{k}H(t,s)f_{j-k,k}(t,t_1,\ldots\right.\right.\\
&\ldots,t_{j-k-1};\,t_{j-k},\ldots,t_{j-1})\\
&\qquad\left.\left.\times\prod_{i=1}^{j-k-1}y(t_i)\prod_{i=j-k}^{j-1}y'(t_i)\right]dt_1\ldots dt_{j-1}dt\right\},
\end{aligned}
$$

where $k=l+1$. Equating coefficients of similar terms in y and y', we find

$$
\begin{aligned}
&\int_0^1 H_1(t,s)f_{j-1-l,\,l+1}(t_1,\ldots,t_{j-1-l};\,t_{j-l},\ldots,t_{j-1},t)dt\\
&\qquad\equiv-\int_0^1 H(t,s)f_{j-l,\,l}(t,t_1,\ldots,t_{j-1-l};\,t_{j-l},\ldots,t_{j-1})dt,
\end{aligned}
$$

which can be written in the form (8).

3. *Calculation of the Invariants* $f[y(\tau_0'),y'(\tau_0')]$. In order that $f[y(\tau_0'),y'(\tau_0')]$ be invariant under (1) it is sufficient that the following recurrence formula hold

$$
\begin{aligned}
(14)\qquad &f_{i,k}(t_1,\ldots,t_i;\,t_{i+1},\ldots,t_{i+k-1},t)\\
&\equiv-\frac{H(t,t_{i+k})}{H_1(t,t_{i+k})}f_{i+1,k-1}(t,t_1,\ldots,t_i;\,t_{i+1},\ldots,t_{i+k-1}).
\end{aligned}
$$

We shall now prove the following theorem.

THEOREM II. *A necessary and sufficient condition on* (1) *that an analytic functional* $f[y(\tau_0),y'(\tau_0')]$ *be invariant under* (1) *when* (14) *holds is that the kernel* $H(x,s)$ *of the infinitesimal transformation be of the form*

$$
(15)\qquad H(x,s)\equiv\psi(s)e^{x/c},
$$

where $\psi(s)$ *is an arbitrary function of* s, *and where* c *is a constant.**

It is evident from (14) that

$$\frac{H(t,\, t_{i+k})}{H_1(t,\, t_{i+k})}$$

must be independent of t_{i+k}, and hence it is necessary that it be a function of t alone, say $\varphi(t)$. On applying (14) until $f_{i,k}$ is written in terms of f's with second index zero, we get the recurrence formula

$$(16)\qquad \begin{aligned} &f_{i,k}(t_1, t_2, \ldots, t_i;\, t_{i+1}, \ldots, t_{i+k-1}, t) \\ &\quad = (-1)^k[\varphi(t)]^k f_{i+k,\,0}(t, t_1, \ldots, t_i, t_{i+1}, \ldots, t_{i+k-1}). \end{aligned}$$

By hypothesis $f_{i+k,\,0}$ is symmetric in all its arguments. Therefore, interchanging t_1 and t_{i+1} leaves the right-hand side of (16) unchanged. Hence if (16) is to hold, $f_{i,\,k}$ must be symmetric with respect to t_1 and t_{i+1}, and therefore it must be symmetric in all its arguments. On interchanging t and any t_j in (16), we see at once that $\varphi(t)$ must be a constant, say c; i. e., $H(x, s)$ must satisfy the equation

$$(17)\qquad c\frac{\partial H(x,s)}{\partial x} - H(x,s) = 0,$$

whose most general solution is (15).

We may now remark that the arbitrariness of the coefficients $f_{i+k,0}$, in terms of which all the other $f_{i,k}$'s can be evaluated, on making use of the recurrence formula, enables us to state immediately the following theorem.

THEOREM III. *Let the kernel* $H(x, s)$ *of the infinitesimal transformation* (2) *be of the form* $\psi(s)e^{x/c}$, *and let us take an analytic functional* $f[y(\tau_0'), y'(\tau_0')]$, *all of whose* $f_{i,k}$*'s are symmetric in all their arguments, and assign arbitrarily for initial conditions the coefficients* $f_{i+k,\,0}$ *in its Volterra expansion; that is, take an arbitrary* $F[y(\tau_0')]$ *such that* $F[y(\tau_0')] \equiv f[y(\tau_0'), y_0'(\tau_0')]$, *and for convenience take* $y_0'(\tau) \equiv 0$. *Then, if the* $f_{i,k}$*'s are calculated by the recurrence formula*

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* That is, if $H(x,s) \equiv \psi(s)e^{x/c}$, we may assert that invariant analytic functionals $f[y(\tau_0')\ y'(\tau_0')]$ always exist.

$$(18) \quad f_{i,k}(t_1, \ldots, t_{i+k}) = (-1)^k c^k f_{i+k,0}(t_1, \ldots, t_{i+k}),$$

we shall have an analytic functional $f[y(\tau_0'), y'(\tau_0)]$ *which will be invariant under a transformation* (1) *whose kernel* $K(x, s \mid a)$ *is given by*

$$K(x, s \mid a)$$

$$(19) \quad = \sum_{i=1}^{\infty} \frac{a^i}{i!} \psi(s) e^{x/c} \int_0^1 \int_{0_{(i-1)}}^1 \cdots \int_0^1 e^{\frac{1}{c}\sum_{j=1}^{i-1} t_j} \prod_{l=1}^{i-1} \psi(t_l) dt_1 \ldots dt_{i-1}.$$

4. *Example.* We here give an easy example in which the direct verification by means of the finite transformation is very simple. Let us suppose that

$$\delta y(x) = \left[e^x \int_0^1 s y(s) ds\right] \delta a$$

is the given infinitesimal transformation, i. e., that $H(x, s) = se^x$. By means of an easy calculation, the finite transformation may be written in the form

$$(20) \qquad y_1(x) = y(x) + (e^a - 1) e^x \int_0^1 s y(s) ds,$$

i. e., $K(x, s \mid a) = (e^a - 1) e^x s$. Let us take for initial condition

$$f[y(\tau_0'), 0] \equiv F[y(\tau_0')]$$

$$\equiv f_{00} + \int_0^1 f_{10}(t_1) y(t_1) dt_1 + \frac{1}{2!} \int_0^1 dt_2 \int_0^1 f_{20}(t_1, t_2) y(t_1) y(t_2) dt_1.$$

Then the functional $f[y(\tau_0'), y'(\tau_0')]$ given by

$$f[y(\tau_0'), y'(\tau_0')] = f_{00} + \int_0^1 f_{10}(t_1)[y(t_1) - y'(t_1)] dt_1$$

$$+ \frac{1}{2!} \left[\int_0^1 dt_2 \int_0^1 f_{20}(t_1, t_2) \{ y(t_1) y(t_2) - 2 y(t_1) y'(t_2) + y'(t_1) y'(t_2) \} \right] dt$$

is invariant under (20).

THE RICE INSTITUTE

REDUCTIONS OF ENUMERATIONS IN HOMOGENEOUS FORMS*

BY E. T. BELL

1. *Introduction.* By carrying out the work in detail for the form $ax^2+by^2+cz^2$ we shall derive a useful set of reduction formulas, and illustrate a general process which can easily be applied to the reduction of the number $N(n=f)$ of representations of the integer n in any homogeneous form f of any degree in any number of variables. This set contains implicitly the complete set of corresponding reduction formulas for $Ax^2+By^2+Cz^2+\cdots+Et^2$, in any number of indeterminates $x, y, z, \ldots, t$. The formulas in no case yield by themselves a complete evaluation of $N(n=f)$ for any type of n, but in many instances they materially simplify the problem, either by making the evaluation for f depend upon that for a simpler form, or by reducing the n to be represented to a more tractable type. By means of the process developed here, combined with elliptic function expansions, I have recently obtained several new complete enumerations for special ternary and quinary quadratic forms; the results will be published in other papers.

Before proceeding to the main discussion it will be instructive to glance at what is known concerning $N(n=f)$ in the simplest case (other than f linear), viz., $f\equiv ax^2+by^2+\cdots$; when the degree of f exceeds 2 even partial evaluations of $N(n=f)$ are at present unknown. It seems fair to say that the simplest case of all, $N(n=x^2+by^2)$, $b>0$, is still far from complete; Dirichlet's well known general theorem† for the number of representations by the totality of a system of representative forms of determinant—b

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* Presented to the Society, San Francisco Section, April 5, 1924.

† Cf. Dickson's *History*, vol. 3, p. 19. References to the other citations of this introduction can be found by consulting the index to vol. 3, and running down the references to vol. 2.

does not of itself give a complete solution when the principal genus contains more than one class. For $f \equiv ax^2 + by^2 + cz^2$ there is the complete evaluation of $N(n = f)$ in the case $(a, b, c) = (1, 1, 1)$ by Gauss, an unproved statement of Liouville for $(a, b, c) = (1, 2, 3)$, partial results by Torelli for $(a, b, c) = (1, 1, 2)$, a special case of $(a, b, c) = (1, 2, 2)$ by Stieltjes, and beyond these apparently nothing detailed and specific for this $N(n = f)$. When $f \equiv ax^2 + by^2 + cz^2 + dt^2$, there is Jacobi's $N(n = f)$ for $(a, b, c, d) = (1, 1, 1, 1)$, several theorems of Liouville for $a = 1$ and each of b, c, $d = p^\alpha \cdot (\alpha = 0, 1; p = 3, 5)$ times a low power of 2, some similar results by Humbert when p = 11, or when $p = 3$, $\alpha = 2$, and Chapelon's evaluations when $p = 5$, $\alpha = 1, 2$. These appear to mark the limit of definite progress in this direction. Complete evaluations of $N(n = ax^2 + by^2 + \cdots)$ for more than 4 indeterminates $x, y, \ldots$ exist only for 5 and 7 squares. These remarks will indicate how far from satisfactory solutions even the simplest problems in the enumerative arithmetic of homogeneous forms still are.

The final formulas of this paper in § 5 have been checked. The nature of the work is such that this verifies all preceding formulas.

2. *Notation.* In all that follows p is prime, the integers n, a, b, c are prime to p, and a, b, c are coprime; k, M, A, B, C are arbitrary integers; α, β, γ, δ are integers $\geqq 0$. To simplify the printing we shall write

$$N(p^\alpha n = ap^\delta x^2 + bp^\beta y^2 + cp^\gamma z^2) \equiv (\alpha; \delta, \beta, \gamma),$$

in which δ, β, γ (also a, b, c, n) are regarded as given constants. Note that $p^\alpha n$ is any integer.

3. *Lemma.* Although it may be obvious that

$$(1)\quad N(kM = kAx^2 + kBy^2 + kCz^2) = N(M = Ax^2 + By^2 + Cz^2),$$

we shall prove it, as upon this depends all that follows. The $\sum$ on the left extending to all integers $x, y, z \gtreqless 0$, that on the right to all integers M,

$$\sum q^{kAx^2 + kBy^2 + kCz^2} \equiv \sum q^{kM} N(kM = kAx^2 + kBy^2 + kCz^2).$$

In this replace q by $\sqrt[k]{q}$:

$$\sum q^{Ax^2+By^2+Cz^2} \equiv \sum q^M N(kM = kAx^2+kBy^2+kCz^2).$$

In the original identity take $k=1$:

$$\sum q^{Ax^2+By^2+Cz^2} \equiv \sum q^M N(M = Ax^2+By^2+Cz^2).$$

By comparing the second identity with the third we get (1).

By the notation explained in § 2, the evaluation of $N(M = Ax^2+By^2+Cz^2)$ is equivalent to that of $(\alpha;\ \delta,\ \beta,\ \gamma)$. By the lemma, if $\alpha < \delta, \beta, \gamma, (\alpha;\ \delta, \beta, \gamma) = 0$; if $\alpha \geqq \delta, (\alpha;\ \delta, \delta, \delta) = (\alpha - \delta;\ 1, 1, 1)$; while if $\delta,\ \beta,\ \gamma$ are unequal, one of them, say δ, is not greater than either of the others, and if $\alpha \geqq \delta$, $(\alpha;\ \delta,\ \beta+\delta,\ \gamma+\delta) = (\alpha-\delta;\ 0,\ \beta,\ \gamma)$.

Hence the evaluation of $N(M = Ax^2+By^2+Cz^2)$ is reduced to that of $(\alpha;\ 0,\ \beta,\ \gamma)$, in which, without loss of generality, we may assume $\gamma \geqq \beta$. Evidently the inequality $\gamma \geqq \beta$ (by the definitions of $\beta,\ \gamma$ in § 2) can be eliminated by replacing γ by $\gamma+\beta$ wherever γ occurs. Eliminations of this kind simplify the final formulas. The further evaluation of $N(M = Ax^2+By^2+Cz^2)$ is now reduced to that of $(\alpha;\ 0,\ \beta,\ \beta+\gamma)$.

4. *Preliminary Reductions.* Let $s \geqq 0$ be an integer such that $\alpha - 2s,\ \beta - 2s \geqq 0$, and therefore also $\beta + \gamma - 2s \geqq 0$. Suppose for a moment that for some $s > 0$ we have $\alpha - 2s$, $\beta - 2s > 0$. If $(\alpha;\ 0,\ \beta,\ \beta+\gamma) > 0$, then must $x \equiv 0 \bmod p$, and therefore by s applications of the Lemma (§ 3),

$$(2) \qquad (\alpha;\ 0,\ \beta,\ \beta+\gamma) = (\alpha - 2s;\ 0,\ \beta - 2s,\ \beta+\gamma-2s),$$

which obviously remains true when $s = 0$ and when $(\alpha;\ 0,\ \beta,\ \beta+\gamma) = 0$. Choose for s the lesser of $[\alpha/2]$, $[\beta/2]$, where $[t]$ is the greatest integer $\leqq t$; when $\alpha = \beta$, take $s = [\beta/2]$. Clearly the reductions (2) can be performed precisely s times, s being as just chosen. Separating out the cases of (2) for even and odd values of β we get

(1. 1) $\alpha \leqq \beta$, $(2\alpha; 0, 2\beta, 2\beta+\gamma) = (0;\ 0,\ 2\beta - 2\alpha,\ 2\beta+\gamma-2\alpha)$;

(1. 2) $\alpha \geqq \beta$, $(2\alpha;\ 0,\ 2\beta,\ 2\beta+\gamma) = (2\alpha - 2\beta;\ 0,\ 0,\ \gamma)$;

(1. 3) $\alpha \leqq \beta$, $(2\alpha+1;\ 0,\ 2\beta,\ 2\beta+\gamma)$
$$=(1;\ 0,\ 2\beta-2\alpha,\ 2\beta+\gamma-2\alpha);$$
(1. 4) $\alpha \geqq \beta$, $(2\alpha+1;\ 0, 2\beta, 2\beta+\gamma)=(2\alpha+1-2\beta;\ 0, 0, \gamma)$;

and the complementary set,

(2. 1) $\alpha \leqq \beta$, $(2\alpha;\ 0,\ 2\beta+1,\ 2\beta+\gamma+1)$
$$=(0;\ 0,\ 2\beta+1-2\alpha,\ 2\beta+\gamma+1-2\alpha);$$
(2. 2) $\alpha \geqq \beta$, $(2\alpha;\ 0,\ 2\beta+1, 2\beta+\gamma+1)$
$$=(2\alpha-2\beta;\ 0,\ 1,\ \gamma+1);$$
(2. 3) $\alpha \leqq \beta$, $(2\alpha+1;\ 0,\ 2\beta+1,\ 2\beta+\gamma+1)$
$$=(1;\ 0,\ 2\beta+1-2\alpha,\ 2\beta+\gamma+1-2\alpha);$$
(2. 4) $\alpha \geqq \beta$, $(2\alpha+1;\ 0,\ 2\beta+1,\ 2\beta+\gamma+1)$
$$=(2\alpha+1-2\beta;\ 0,\ 1,\ \gamma+1).$$

Only those on the right having a pair of zeros in the symbol are irreducible. The further reduction of the rest is effected in a similar way, first powers of the prime p, instead of second, being now successively eliminated. The process is seen by examining the right of (1. 3), (2. 3). When $\alpha<\beta$ we have $2\beta-2\alpha \geqq 2$, $2\beta+\gamma-2\alpha \geqq 2$, and since pn is the number represented in the right of (1. 3), it follows that $x \equiv 0 \bmod p$. Applying the lemma, we get

$$(1;\ 0,\ 2\beta-2\alpha,\ 2\beta+\gamma-2\alpha)$$
$$=(0;\ 1,\ 2\beta-2\alpha-1,\ 2\beta+\gamma-2\alpha-1),$$

and this evidently vanishes (when $\alpha<\beta$). Similarly for (2. 3), and we have

(1. 31) $\alpha<\beta$, $(2\alpha+1;\ 0,\ 2\beta,\ 2\beta+\gamma)=0$;
(2. 31) $\alpha<\beta$, $(2\alpha+1;\ 0,\ 2\beta+1,\ 2\beta+\gamma+1)=0$,

which may replace (1. 3), (2. 3), since the cases $\alpha=\beta$ are included in (1. 4), (2. 4).

Similarly, provided that $\alpha-2s$, $\gamma-2s+1 \geqq 0$, we get

$$(\alpha;\ 0,\ 1,\ \gamma+1)=(\alpha-2s;\ 0,\ 1,\ \gamma+1-2s),$$

and, provided that $\alpha-1-2s$, $\gamma-2s \geqq 0$,

$$(\alpha;\ 0,\ 1,\ \gamma+1)=(\alpha-1-2s;\ 1,\ 0,\ \gamma-2s).$$

Upon separation of cases according to even, odd γ, these yield the formulas which enable us to complete the reduction of (1. 1)—(2. 4). It is unnecessary to preserve the very simple calculations. We find

(3. 1) $\alpha \leqq \gamma+1, \quad (2\alpha;\ 0, 1, 2\gamma+2) = (0;\ 0, 1, 2\gamma+2-2\alpha);$
(3. 2) $\alpha \geqq \gamma+1, \quad (2\alpha;\ 0, 1, 2\gamma+2) = (2\alpha-2\gamma-2;\ 0, 1, 0);$
(3. 3) $\alpha \leqq \gamma, \quad (2\alpha+1;\ 0, 1, 2\gamma+2) = (0;\ 1, 0, 2\gamma+1-2\alpha);$
(3. 4) $\alpha \geqq \gamma+1, \quad (2\alpha+1;\ 0, 1, 2\gamma+2) = (2\alpha-2\gamma-1;\ 0, 1, 0),$

and the complementary set

(4. 1) $\alpha \leqq \gamma, \quad (2\alpha;\ 0, 1, 2\gamma+1) = (0;\ 0, 1, 2\gamma+1-2\alpha);$
(4. 2) $\alpha \geqq \gamma+1, \quad (2\alpha;\ 0, 1, 2\gamma+1) = (2\alpha-2\gamma-1;\ 1, 0, 0);$
(4. 3) $\alpha \leqq \gamma, \quad (2\alpha+1;\ 0, 1, 2\gamma+1) = (0;\ 1, 0, 2\gamma-2\alpha);$
(4. 4) $\alpha \geqq \gamma, \quad (2\alpha+1;\ 0, 1, 2\gamma+1) = (2\alpha-2\gamma;\ 1, 0, 0),$

all of which are further irreducible. Note that since γ may take the value zero, $(\alpha;\ 0, 1, 2\gamma)$ is not necessarily reducible, while the type considered, $(\alpha;\ 0, 1, 2\gamma+2)$, is.

Apply (3. 1)—(4. 4) to (2. 2), (2. 4) after having first eliminated the condition $\alpha \geqq \beta$ by replacing α wherever it occurs by $\beta+\alpha$. The results are:

(5. 1) $\alpha \leqq \gamma, \quad (2\beta+2\alpha;\ 0, 2\beta+1, 2\beta+2\gamma+1)$
$= (0;\ 0, 1, 2\gamma+1-2\alpha);$
(5. 2) $\alpha \geqq \gamma+1, \quad (2\beta+2\alpha;\ 0, 2\beta+1, 2\beta+2\gamma+1)$
$= (2\alpha-2\gamma-1;\ 1, 0, 0);$
(5. 3) $\alpha \leqq \gamma, \quad (2\beta+2\alpha+1;\ 0, 2\beta+1, 2\beta+2\gamma+1)$
$= (0;\ 1, 0, 2\gamma-2\alpha);$
(5. 4) $\alpha \geqq \gamma, \quad (2\beta+2\alpha+1;\ 0, 2\beta+1, 2\beta+2\gamma+1)$
$= (2\alpha-2\gamma;\ 1, 0, 0);$

and the complementary set,

(6. 1) $\alpha \leqq \gamma+1, \quad (2\beta+2\alpha;\ 0, 2\beta+1, 2\beta+2\gamma+2)$
$= (0;\ 0, 1, 2\gamma+2-2\alpha);$
(6. 2) $\alpha \geqq \gamma+1, \quad (2\beta+2\alpha;\ 0, 2\beta+1, 2\beta+2\gamma+2)$
$= (2\alpha-2\gamma-2;\ 0, 1, 0);$
(6. 3) $\alpha \leqq \gamma, \quad (2\beta+2\alpha+1;\ 0, 2\beta+1, 2\beta+2\gamma+2)$
$= (0;\ 1, 0, 2\gamma+1-2\alpha);$
(6. 4) $\alpha \geqq \gamma+1, \quad (2\beta+2\alpha+1;\ 0, 2\beta+1, 2\beta+2\gamma+2)$
$= (2\alpha-2\gamma-1;\ 0, 1, 0).$

Examining (1. 1)—(2. 4) and (5. 1)—(6. 4) we see it is necessary to consider only

$$N(p^{\alpha}n = ax^2 + bp^{2\beta}y^2 + cp^{2\beta+\gamma}z^2) \equiv N_1(p^{\alpha}n), \tag{7}$$

$$N(p^{\alpha}n = ax^2 + bp^{2\beta+1}y^2 + cp^{2\beta+2\gamma+1}z^2) \equiv N_2(p^{\alpha}n), \tag{8}$$

$$N(p^{\alpha}n = ax^2 + bp^{2\beta+1}y^2 + cp^{2\beta+2\gamma+2}z^2) \equiv N_3(p^{\alpha}n), \tag{9}$$

in order, by the reductions in § 3, to obtain a complete set of reduction formulas for

$$N(M = Ax^2 + By^2 + Cz^2). \tag{10}$$

That the three sets in § 5 are exhaustive is evident by inspection on referring to the notation in § 2.

5. *Final Formulas.* For the N_i $(i = 1, 2, 3)$ see (7), (8), (9). From (1. 1)—(1. 4) and (1. 31), by eliminating the condition $\alpha \geqq \beta$, we find

(I) *Form* $ax^2 + bp^{2\beta}y^2 + cp^{2\beta+\gamma}z^2$:

$$\begin{aligned} \alpha \leqq \beta,\ & N_1(p^{2\alpha}n) = (0;\ 0,\ 2\beta - 2\alpha,\ 2\beta + \gamma - 2\alpha); \\ \alpha < \beta,\ & N_1(p^{2\alpha+1}n) = 0; \\ & N_1(p^{2\beta+\alpha}n) = (\alpha;\ 0,\ 0,\ \gamma). \end{aligned}$$

From (2. 1), (2. 31) with γ replaced by 2γ, and from (5. 1)—(5. 4) we find upon eliminating $\alpha \geqq \gamma + 1$, $\alpha \geqq \gamma$.

(II) *Form* $ax^2 + bp^{2\beta+1}y^2 + cp^{2\beta+2\gamma+1}z^2$:

$$\begin{aligned} \alpha \leqq \beta,\ & N_2(p^{2\alpha}n) = (0;\ 0,\ 2\beta + 1 - 2\alpha,\ 2\beta + 2\gamma = 1 - 2\alpha); \\ \alpha < \beta,\ & N_2(p^{2\alpha+1}n) = 0; \\ \alpha \leqq \gamma,\ & N_3(p^{2\beta+2\alpha}n) = (0;\ 0,\ 1,\ 2\gamma + 1 - 2\alpha); \\ & N_2(p^{2\beta+2\gamma+2\alpha+2}n) = (2\alpha + 1;\ 1,\ 0,\ 0); \\ \alpha \leqq \gamma,\ & N_2(p^{2\beta+2\alpha+1}n) = (0;\ 1,\ 0,\ 2\gamma - 2\alpha); \\ & N_2(p^{2\beta+2\gamma+2\alpha+1}n) = (2\alpha;\ 1,\ 0,\ 0). \end{aligned}$$

From (2. 1), (2. 31) with γ replaced by $2\gamma + 1$, and from (6. 1)—(6. 4) upon elimination of $\alpha \geqq \gamma + 1$ we find

(III) *Form* $ax^2 + bp^{2\beta+1}y^2 + cp^{2\beta+2\gamma+2}z^2$:

$$\begin{aligned} \alpha \leqq \beta,\ & N_3(p^{2\alpha}n) = (0;\ 0,\ 2\beta + 1 - 2\alpha,\ 2\beta + 2\gamma + 2 - 2\alpha); \\ \alpha < \beta,\ & N_3(p^{2\alpha+1}n) = 0; \end{aligned}$$

$$\alpha \leqq \gamma + 1,\ N_3(p^{2\beta+2\alpha}n) = (0;\, 0,\, 1,\, 2\gamma + 2 - 2\alpha);$$
$$N_3(p^{2\beta+2\gamma+2\alpha+2}n) = (2\alpha;\, 0,\, 1,\, 0);$$
$$\alpha \leqq \gamma,\ N_3(p^{2\beta+2\alpha+1}n) = (0;\, 1,\, 0,\, 2\gamma + 1 - 2\alpha);$$
$$N_3(p^{2\beta+2\gamma+2\alpha+3}n) = 2\alpha + 1;\, 0,\, 1,\, 0).$$

In all of the above no further reduction is possible.

6. *Successive Reductions.* Let D be the greatest common divisor of B, C in (10), and assume without loss of generality (§ 3) that M, A, B, C are relatively prime in their totality. Let $M = M'p^\alpha$, where p is any prime divisor of D, and M' is prime to p. Apply (I)—(III) of § 5. Repeat the process on the results for each remaining prime divisor of D, obtaining finally a system of formulas analogous to (I)—(III) in which (10), for its several possible cases according to the prime factors of D, is replaced by a corresponding $N(M'p^{\alpha'} = A'x^2 + B'y^2 + C'z^2)$ in which no further reduction with respect to B', C' is possible. This system of formulas may conveniently be written as a set of equalities between r-rowed matrices, where r is the number of distinct prime factors of D. To each pair of A, B, C in (10) will correspond such a system of equalities, and all three together give the complete reduction of (10). It would be of interest to discuss this set.

7. (IV) *Form* $Ax^2 + By^2 + Cz^2 + \cdots + Et^2$. As in § 3 the reduction for this form is referred to that of

$$N(p^\alpha n = x^2 + p^\beta y^2 + p^{\beta+\gamma} z^2 + \cdots + p^{\beta+\gamma+\cdots+\varepsilon}\, t^2),$$

where $\beta, \gamma, \ldots, \varepsilon$ are integers $\geqq 0$, and a precisely similar argument shows immediately that this N is reduced when $N(p^\alpha n = x^2 + p^\beta y^2 + p^{\beta+\gamma} z^2)$ is reduced. The complete set of reduction formulas can be written down from § 5.

8. *General Form.* When the degree of f is $3 + \alpha$, the process of reducing $N(n = f)$ is evident from the foregoing; the discussion now depends upon $[k/(3+\alpha)]$ instead of $[k/2]$.

THE UNIVERSITY OF WASHINGTON

THE SCIENTIFIC WORK OF JOSEPH LIPKA

BY W. C. GRAUSTEIN

Joseph Lipka, Associate Professor of Mathematics at the Massachusetts Institute of Technology, died January 15, 1924, in his forty-first year and the sixteenth year of his service at the Institute. He was a mathematician of proven worth, intensely interested in his science, and unremitting in his efforts to forward its progress. The last few years of his life were especially remarkable for their productivity. During this time he contributed what would normally be considered, both in volume and value, as the creditable work of a decade, and gave abundant promise of continued fruitfulness. That he has been taken in his prime is indeed a source of great regret and a distinct loss to mathematics.

During Lipka's student days at Columbia, Professor Kasner was busily applying differential geometry to dynamics and developing, in euclidean spaces of two and three dimensions, the geometric properties of dynamical trajectories and related systems of curves. It was this field of investigation in which Lipka was later initiated and did his work for the doctorate. His thesis consisted in a generalization, to a curved space of n dimensions, of results obtained by Kasner concerning a natural family of curves, that is, a family of trajectories in a conservative field of force corresponding to a given constant of energy, or more generally, any family of extremals resulting from a problem in the calculus of variations of the form,

$$\int e^{\varphi} ds = \text{minimum}, \tag{1}$$

where ds is the linear element of the space in question, and φ is a function of the coordinates of the space.

Lipka's later work, though largely confined to the one field, falls naturally into two periods. The earlier of these sees the generalization of many of Kasner's theories, first to ordinary surfaces and then to curved spaces of n dimensions. Noteworthy is the exhaustive study of dynamical trajectories on a surface for any positional field of force, and also the investigation of the geometrical properties of trajectory and related systems in Riemannian n-space. But perhaps the most important achievement of this period was the paper which established the validity, in all cases, of the Thomson-Tait criterion for a natural family. According to the theorem of Thomson and Tait, the ∞^{n-1} curves of a natural family meeting an arbitrary hypersurface orthogonally form a normal hypercongruence, i. e., admit of ∞^1 normal hypersurfaces. That this property is characteristic of natural familes (for $n > 2$) had been proved by Kasner for a euclidean space of three dimensions. In the paper now under discussion, this was proved for a euclidean space, first of four and then of n dimensions, and finally for a general curved space.

During his career Lipka made two trips abroad, and both of them he turned to good account. The first, during the summer of 1913, brought him into contact with Professor Whittaker and the mathematical laboratory at Edinburgh and thus furnished the inspiration which led him to found at the Institute a laboratory course and later to write the text on *Graphical and Mechanical Computation.*

The summer of 1921 and the following academic year Lipka spent on leave in Europe, principally in Rome. This sojourn abroad marked the beginning of a new period in his scientific work. The stimulus and inspiration which he derived from his new associations, especially from those with Professor Levi-Civita, are clearly evident in his writings. His work is now permeated with the methods of the absolute calculus and governed largely by new ideas. As a first illustration may be mentioned the use of Hamilton's canonical equations and infinitesimal contact transformations to prove anew, in a brief and elegant fashion, the Thomson-Tait characteristic property of natural families, and to extend this property so that it applies to an irreversible system; in the extension, which of course applies merely to a conservative field of force, it is no longer the curve which is orthogonal to the hypersurfaces in question, but the resultant of the velocity vector along the curve and the vector whose components are the coefficients of the (new) linear term in the Lagrangian function.

Another group of papers revolve about Levi-Civita's notion of parallelism, or parallel motion of a direction along a curve, in a Riemannian space. By use of this concept Lipka gives a simple intrinsic definition of geodesic curvature, which is the analogue of that of ordinary curvature in euclidean space. Thus, if P' is a point on a curve C neighboring to a given point P of C, and if $d\omega$ is the angle between the tangent at P' and the parallel at P' to the tangent at P, the limit of the ratio of $d\omega$ to the arc PP' is the geodesic curvature of C at P. In a later paper a similar process is employed to obtain the curvature of C at P relative to any curve C' tangent at P to C. This new type of relative curvature finds application in dynamics when the curve C' is taken as the trajectory of a natural family.

From equation (1) it is clear that natural families are a generalization of geodesics and that every natural family is the conformal representation of the geodesics in a second space. It was doubtless these facts which led Lipka to generalize Levi-Civita's concept of parallelism by replacing the geodesics at the basis of it by the trajectories of a natural family, and which suggested for the new type of parallelism the name "conformal parallelism". This idea, by its very nature, is of value in the study of dynamical problems. For example, a trajectory is now characterized by the fact that its direction at any point is conformally parallel to its initial direction. Again, the curves orthogonal to the ∞^1 trajectories on an ordinary surface which issue at right angles from a given curve are conformally parallel.

Perhaps the aspect of conformal parallelism which will make the widest appeal is its close connection with the second differential parameter. Levi-Civita has shown that the Gaussian curvature of a surface at a point P is the ratio of the angle between the initial and final positions of a direction which moves by parallelism completely around an infinitesimal cycle through P, to the area of the cycle. Lipka shows that, in the case of conformal parallelism, this ratio is equal to the difference between the curvature at P and the value at P of $\Delta_2\varphi$, where φ is the "characteristic" function given in (1). If, then, φ is any point function on the surface, the value at P of $\Delta_2\varphi$ can be interpreted as the ratio of the angle between the two final positions of a direction which moves first by parallelism and then by conformal parallelism (with the characteristic function φ) completely around the cycle, to the area of the cycle.

The substitution of trajectories for geodesics and trajectory surfaces for geodesic surfaces forms also the basis for the development, in another paper, of a generalization of the Riemannian theory. Thereby Riemannian curvature is replaced by trajectory curvature, that is, the Gaussian curvature of a trajectory surface at a point. Moreover, the theory developed by Ricci concerning principal directions and principal curvatures lends itself to complete generalization. The new theory contains the old as a special case, when φ is constant. The relationships between the fundamental quantities in the two theories are thus readily obtained and are quite remarkable in their simplicity and elegance.

Lipka's last published paper has to do with a system of invariants introduced by Ricci in his study of orthogonal congruences, and called by him coefficients of rotation, in light of the fact that he was able to show that they can be interpreted in terms of rotations in the euclidean space in which the given curved space is imbedded. Following a suggestion of Levi-Civita, Lipka proves that these invariants admit a simple intrinsic interpretation within the given space by means of the notion of parallelism. This interpretation enables him to give elegant characterizations of certain types of congruences, notably congruences of geodesics, in terms of the vanishing of certain of the invariants, and to obtain other results of equal importance. Finally, by introducing conformal parallelism instead of that of Levi-Civita, he develops a new set of invariants, which yield corresponding results bearing on trajectories.

It is a pleasure to read Lipka's writings. The habit of clear thinking and clear expression was ingrained in him. To this, as well as to his personal qualities, was due his success as a teacher.

Lipka was natural, unpretentious, always cheerful and sympathetic, ever willing to lend a hand, and enthusiastically devoted to his chosen work. And beneath these traits there was a definite, but more elusive, quality which vitalized them, a certain inner buoyancy, an ever forward pressing optimism, an intangible force which drew others to him.

LIST OF LIPKA'S WRITINGS

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1918

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1920

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On conformal parallelism. JOURNAL OF MATHEMATICS AND PHYSICS, Massachusetts Institute of Technology, vol. 2, No. 3, pp. 175–194.

Trajectory surfaces and a generalization of the principal directions in any space. PROCEEDINGS OF THE AMERICAN ACADEMY, vol. 59, No. 3, pp. 51–77.

On the relative curvature of two curves in V_n. This BULLETIN, vol. 29, No. 8, pp. 345–348.

1924

On Ricci's coefficients of rotation. JOURNAL OF MATHEMATICS AND PHYSICS, Massachusetts Institute of Technology, vol. 3, No. 1, pp. 7–23.

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VEBLEN ON ANALYSIS SITUS

The Cambridge Colloquium, 1916, Part II. *Analysis Situs*. By Oswald Veblen. New York, The American Mathematical Society, 1922. vii + 150 pp.

The present reviewer finds himself facing the infrequent experience of tackling the same work for the second time. That he derives no little satisfaction from it, is due to the fact that since the first review was written[*] he has made ample use of the book as an instrument of research, and hopes that the additional experience thus acquired will prove of value not only to himself but also to his readers.

Let us say at once that the *Lectures* have stood the test of usage very well indeed Hitherto a beginner attracted by the fascinating and difficult field of analysis situs, was obliged to wade through many widely scattered papers, beginning with Poincaré's classic of the 1895 in JOURNAL DE L'ÉCOLE POLYTECHNIQUE and its multiple *Compléments*. Difficult reasonings beset him at every step, an unfriendly notation did not help matters, to all of which must be added, most baffling of all, the breakdown of geometric intuition precisely when most needed. No royal road can be created through this dense forest, but a good and thoroughgoing treatment of fundamentals, notation, terminology, may smooth the path somewhat. And this and much more we find supplied by Veblen's *Lectures*. That few, if any, were better qualified than the author, by temperament and scientific past, to produce such a work, is well known among the devotees of analysis situs, whose number it will assuredly increase. As this field presents difficult problems in profusion, with but few general methods, such an increment is very much to be hoped for.

Two streams are found in analysis situs. the one related to point-set theory, the other primarily of a combinatorial nature, wherein are treated manifolds which for example in the case of two dimensions, are depictable upon polyhedra with a finite number of plane polygonal faces. To this latter branch these *Colloquium Lectures* are almost wholly devoted. Withal the author is perhaps at his mathematical best at the points of contact of the two streams, say with the type of question presented by the occurrence of Jordan curves on a representative polyhedron.

Veblen has made extensive and very systematic use of the Poincaré matrices of a manifold, whereby the incidence relations beween the k-cells and $(k-1)$-cells† are defined by the elements of a certain matrix. These elements are $0, \pm 1$, when the orientations of the cells are taken into consideration. In a notable paper (ANNALS OF MATHEMATICS, 1913) Veblen and Alexander showed that for many purposes the elements could be taken merely modulo 2. By comparing all these matrices and reducing them in the usual manner, one obtains the all important *topological invariants* and their relations. Our geometer friends will ruefully complain that the structure is much too analytical for their taste, but there is no help to it for the present, as nothing else comparably rigorous is now in print. To the above aspect of the theory are devoted four of the five Lectures. Of particular excellence are the elegant development of normal forms for two-dimensional manifolds, the treatment of questions of orientation, also of *covering complexes*,—one of the most useful features of the book. Everything pertaining to general manifolds immersed in a given one with assigned cellular structure is thereby thoroughly clarified. Of decided interest in this connection is the author's reduction of arbitrary k-circuits (k-cycles) on a manifold to those composed exclusively of k-cells of a particular cellular subdivision. Thus a Jordan curve on a polyhedron would be reduced to a polygon of edges. From this follows indeed that certain integers related to the representative matrices (Betti numbers, torsion-coefficients, etc.) are actually topological invariants.

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[*] See *Bulletin des Sciences Mathématiques* for 1922.

[†] A k-cell is any point-set continuously depictable upon the interior of a hypersphere in k-space.

The fifth and last Lecture differs in type from the others, and is more in the nature of what we have been accustomed to expect of such Lectures. It contains an excellent summary of several important questions: homotopy and isotopy, theory of the indicatrix, a fairly ample treatment of the group of a manifold, finally a bird's eye view of what is known and not known (mostly the latter) on three dimensional manifolds.

There is no fundamental criticism to level at the *Lectures*. An error in the argument pp. 142–143 has been pointed out by J. Nielsen of Copenhagen. And if one has some good important theorems, why refrain from displaying them appropriately by means of heavy type, titles, and the like? Of this sin the author is repeatedly guilty: it does not facilitate the beginner's task. We should also have been most thankful for a more complete bibliography.

These *Colloquium Lectures*, one of the best productions of a Series of which American mathematics may well be proud, deserve to be and will be well received by fellow workers in this field and by the general mathematical public.

S. LEFSCHETZ

GOURSAT ON PFAFF'S PROBLEM

Leçons sur le Problème de Pfaff. By Edouard Goursat, Paris, J. Hermann, 1922. 386 pp.

On the first page of Volume 3 of the *Institutiones Calculi Integralis,* Euler declares with reference to the equation $Pdx + Qdy + Rdz = 0$, that there must be some multiplier M by means of which the expression on its left-hand side will become an exact differential; "for unless such a multiplier existed, the proposed differential equation would become absurd and would mean nothing at all". If this opinion had remained unchallenged, the volume under review would not have had much reason for existence. For the problem of Pfaff with which it is occupied relates to differential expressions of which the linear form $\sum_{i=1}^{n} X_i\,(x_1, \cdots, x_n)\,dx_i$ is the simplest type, independently of whether or not there exists a multiplier which renders them exact.

In a memoir of 1814, J. F. Pfaff showed the essential equivalence of the partial differential equation of the first order

$$\frac{dz}{dx_1} = \psi\left((z, x_1, \cdots, x_n, \frac{dz}{dx_2}, \cdots, \frac{dz}{dx_n}\right)$$

and of the total differential equation

$$dz - \psi(z, x_1, \cdots, x_n, p_2, \cdots, p_n)dx_1 - p_2dx_2 - \cdots - p_ndx_n = 0.$$

A solution of the latter equation consists of a set of n equations in $z, x_1, \cdots, x_n, p_2, \cdots, p_n,$ which are solvable for $z, p_2, \cdots, p_n$ in terms of $x_1, \cdots, x_n$ and thus yield functions which reduce this equation to an identity in $x_1, \cdots, x_n$ and their differentials. It is a special case of the general equation $X_1dx_1 + \cdots + X_ndx_n = 0$, in which X_i are functions of $x_1, \cdots, x_n$. The reduction of this equation and of the differential form on its left-hand side has been the object of many writers, among whom are Gauss, Jacobi, Grassmann, Clebsch, Darboux, Frobenius. This question occupies nearly all of Part I of Forsyth's *Theory of Differential Equations,* to which M. Goursat refers for a historical account of the development of the subject. It also furnishes the material for E. von Weber's *Vorlesungen über das Pfaffsche Problem,* to which only very casual reference is made in M. Goursat's volume. Indeed the latter is very different in character, for it is not primarily concerned with an account of the methods for the solution of the problem developed by various writers, but rather with its extensions and generalizations. This indeed is its real reason for existence as a separate volume.

The first two chapters form a natural sequel to the author's *Leçons* *sur l'Intégration des Équations aux Dérivées Partielles du Premier Ordre* to which they are similar in style. In the first, we find a thorough treatment of the fundamental theorem of Pfaff on the reduction of the linear differential form $\sum^n X_i \, dx_i$ to one or the other of the canonical forms $z_1dy_1 + \cdots + z_pdy_p + dy_{p+1}$ or $z_1dy_1 + \cdots + z_pdy_p$ according as the class of the form is $2p + 1$ or $2p$, where $z_1, \cdots, z_p, y_1, \cdots, y_{p+1}$ are linearly independent functions of $x_1, \cdots, x_n$. The concept "class of a form" introduced by Frobenius and the reductions to the canonical forms are developed here by the very elegant methods of Frobenius and Darboux, which are based on a consideration of a bilinear covariant of ω, viz.

$$\omega' = \sum a_{ij} dx_i \delta x_j,$$

where

$$a_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i},$$

and of four associated covariant linear differential systems. The matrix of the a_{ij} being skew symmetric, it becomes apparent that the even- or oddness of the class c will be of importance. This distinction does indeed run throughout the whole theory of the forms of Pfaff. A good knowledge of completely integrable linear differential systems is necessary to understand this part of the book.

Chapter 2 is devoted to the problem from which the book derives its title, viz., that of solving the equation $\omega = 0$. A solution consists of p equations $f_i(x_1, \cdots, x_n) = 0$, $i = 1, \cdots, p$, $p < n$, such that from them together with those obtained from them by differentiation, the given equation follows. It is shown that the maximum number of dimensions of a manifold which in this sense satisfies the equation is $n - (\gamma + 1)/2$, where γ is the class of the equation, a number which is equal to c or to $c - 1$, according as c is odd or even; every integral manifold of lower dimensionality lies in this maximum-dimensional manifold; moreover, the maximum number of dimensions is never less than the integral part of $n/2$. The families of characteristic manifolds are determined by the γ solutions of one of the associated complete linear differential systems referred to above, and a general theorem is obtained establishing their relation to the integral manifolds.

The second division of the book consisting of Chapters 3, 4, and 5 deals with the generalization of the problem of Pfaff to differential forms of higher degree

$$\Omega \equiv \sum A_{\alpha_1 \ldots \alpha_p} \, dx_{\alpha_1} \ldots dx_{\alpha_p}$$

in which $dx_{\alpha_1} \ldots dx_{\alpha_p}$ is a symbolic product whose characteristic properties are to be determined. To justify this problem as an extension of the problem of Pfaff, the latter is restated in the following form: To determine in n-space all the r-manifolds ($1 \leqq r < n$) such that the line integral $\int \omega$ shall vanish if taken along any curve on such a manifold. The generalized problem is then to determine in n-space all r-manifolds $p \leqq r < n$ such that the $\int \Omega$ vanishes if taken over any p-manifold situated on them. The theory of the symbolic product $dx_{a_1} \ldots dx_{a_p}$, and of the symbolic forms Ω, as developed and applied in Chapters 3 and 4 is largely the work of MM. Cartan and Goursat. As remarked in the preface, the study of the symbolic products could readily be connected with the tensor calculus. It seems to the reviewer indeed a wise decision to have given the development independently of the general theory.

The bilinear covarient ω' introduced for the simple case by Darboux and Frobenius finds an extension in the derivative Ω' of the form Ω. As in the linear problem the vanishing of ω' was the necessary and sufficient condition for exactness, so $\Omega' = 0$ is the necessary and sufficient condition for the exactness of Ω. The concept "class of a form" is extended as are also the associated differential systems.

Of especial interest in connection with the results of Chapters 1 and 2 are the applications of the symbolic forms to the equation of Pfaff. The p^{th} derivative of a linear form ω being a symbolic form of degree $p+1$, defined by the recursion formulas

$$\omega^{(2m-1)} = \frac{1}{m!}(\omega')^m,$$

$$\omega^{(2m)} = \omega(\omega')^m,$$

it is found that if the c^{th} derivative is the first among the successive derivatives of a linear form ω which vanishes, then the class of ω is equal to c. The derivative forms furnish moreover in a very direct way a method for the reduction of a form of Pfaff to the canonical form discussed in Chapters 1 and 2. Chapter 4 closes with the application of these results to a system of partial differential equations and to the problem of contact transformations.

The fundamental properties of the integral invariants of a system of differential equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} = dt$$

are shown in Chapter 5 to follow quite naturally from the properties of the symbolic forms considered in the preceding chapter. The treatment followed complements in an interesting manner that of Cartan in his recent book on Integral Invariants (see this BULLETIN, vol. 29, p. 140). There are a variety of beautiful results: the relation between the invariant character of $\int \omega$ and of $\int \omega'$, ω being a form of any degree; the theorem that every relative integral invariant is the sum of an absolute integral invariant and the integral of a symbolic total differential, etc.

The last three chapters are devoted to recent results in the study of systems of Pfaff, a great part of which is due to Cartan. The first question considered in Chapter 6 is as to the maximum number of dimensions ρ of an integral manifold of a system of r forms of Pfaff.

There is a rich amount of material, too varied to lend itself to brief discussion, on systems of linear differential expressions, contact transformations, derived systems, the problem of Monge, second order partial differential equations, etc. The final chapter is devoted to the classification of the integral elements of a system of Pfaff and to the existence theorem.

The book affords an excellent introduction to the study of a problem which occupies a very central position in the theory of differential equations. It gives a detailed survey of the classical theory, of its connections with other domains, of its most modern delelopments, and of the directions in which further advances may be made.

A. Dresden

WATSON ON BESSEL FUNCTIONS

A Treatise on the Theory of Bessel Functions. By G. N. Watson. Cambridge, University Press, 1922. viii + 804 pages.

The purpose of this book is twofold: to develop certain applications of the fundamental processes of the theory of functions of complex variables for which Bessel functions are admirably adapted; and secondly, to compile a collection of results which shall be of value to the increasing number of mathematicians and physicists who encounter Bessel functions in the course of their researches. The author believes that the existence of such a collection is demanded by the greater abstruseness of properties of Bessel functions (especially of functions of large order) which have been required in recent years in various problems of mathematical physics.

In his exposition the author has endeavoured to accomplish two specific results: to give an account of the theory of Bessel functions which a pure mathematician would regard as fairly complete; and to include all formulas, whether general or special, which, although without theoretical interest, are likely to be required in practical applications. An attempt is made to give the latter results, as far as possible, in a form appropriate for use in the applications. These exalted aims the author seems to have achieved with a remarkable success. The great amount of material thus to be included and the necessity for keeping the size of the book within bounds have made necessary the employment of a concise and compact exposition: but this has been attained without undue sacrifice of clarity. Nearly all parts of the book are as easily read as one has a right to expect in the case of material of this sort. The general theory on which the special results in this volume are based is to be found in the *Course of Modern Analysis* by Whittaker and Watson. It is greatly to the reader's convenience to have a single volume for reference for the basic theory on which the exposition is founded.

Concerning the choice of canonical functions the author speaks as follows in his preface: "It is desirable to draw attention here to the function which I have regarded as the canonical function of the second kind, namely the function which was defined by Weber and used subsequently by Schläfli, by Graf and Gubler, and by Nielsen. For historical and sentimental reasons it would have been pleasing to have felt justified in using Hankel's function of the second kind; but three considerations prevented this. The first is the necessity for standardizing the function of the second kind; and, in my opinion, the authority of the group of mathematicians who use Weber's function has greater weight than the authority of the mathematicians who use any other one function of the second kind. The second is the parallelism which the use of Weber's function exhibits between the two kinds of Bessel functions and the two kinds (cosine and sine) of trigonometric functions. The third is the existence of the device by which interpolation is made possible in Tables I and III at the end of Chapter XX, which seems to me to make the use of Weber's function inevitable in numerical work."

In connection with each section references are given to the memoirs in which the results were first announced but the methods of proof employed are often different from those of the original investigators. A very complete bibliography of thirty-six pages is given at the end of the book.

Space does not allow us to give an analysis of the contents of so large a volume. But we may give a bare outline of the contents by means of chapter headings. These are as follows: Bessel functions before 1826 (pages 1–13), the Bessel coefficients (14–37), Bessel functions (38–84), differential equations (85–131), miscellaneous properties of Bessel functions (132–159), integral representations of Bessel functions (160–193), asymptotic expansions of Bessel functions (194–224), Bessel functions of large order (225–270), polynomials associated with Bessel functions (271–307), functions associated with Bessel functions (308–357), addition theorems (358–372), definite integrals (373–382), infinite integrals (383–449), multiple integrals (450–476), the zeros of

Bessel functions (477–521), Neumann series and Lommel's functions of two variables (522–550), Kapteyn series (551–575), series of Fourier-Bessel and Dini (576–617), Schlomilch series (618–653), the tabulation of Bessel functions (654–664), tables of Bessel functions (665–752), bibliography (753–788), index of symbols (789–790), list of authors quoted (791–795), general index (796–804).

The author's work is well done throughout. The reviewer did not detect any omissions or serious blemishes of any kind. In some respects the exposition is probably more disjointed than is necessary. Some parts would profit by being brought into closer connection with related general theories relative to differential equations; but the author is right in avoiding the use of any large part of the general theory of differential equations. It appears to the reviewer that the exposition of the theory of the zeros of Bessel functions would profit by a closer connection with the general oscillation theory of the solutions of linear homogeneous differential equations of the second order, especially since that theory has taken such simple form in the hands of Bôcher. In a similar manner the general notions involved in the Birkhoff theory of expansions in terms of orthogonal and biorthogonal functions satisfying linear differential equations would serve to give greater unity to the four long chapters on expansions in terms of Bessel functions. But, even so, there is something to be said in favor of the direct treatment employed by the author.

Many of the results recorded appear to be of the nature of special instances of general propositions yet to be discovered. In the case of these a disjointed exposition is inevitable until the general theory has been brought into existence. Perhaps the book will serve a third important purpose in addition to the two which the author had in mind in preparing it. It is suited to suggest further researches in two ways: it will raise interesting questions concerning Bessel functions and the differential equation which they satisfy; it will also suggest general theories concerning linear differential equations, theories special instances of which are afforded by the Bessel equation. In this respect the expansion theory seems to hold out the greatest promise. We may well close the review by emphasizing the importance in this respect of the four excellent chapters devoted to expansions of functions in series of Bessel functions.

R. D. CARMICHAEL

BIRKHOFF ON RELATIVITY

Relativity and Modern Physics. By G. D. Birkhoff. Cambridge, Harvard University Press, 1923. xi + 283 pp.

Birkhoff's general plan for his book on relativity is excellent. He begins with a rapid survey of those branches of classical physics which are to be affected most seriously by the new theory. This survey shows that the old theories have been fundamentally limited by the underlying concept of a rigid body in empty space, and leads to the inquiry as to what sort of a theory can be built up by using for our physical model "a number of very small material particles in motion at comparatively great distances from each other in otherwise empty space".

For simplicity he limits attention at first to a one-dimensional case. The position of a particle B with respect to a particle A is determined by the time t_1 of emission and the time t_2 of return of a flash of light from A which is reflected by B.* Thus the events of a one-dimensional universe are seen to constitute a two-dimensional manifold. It is shown how the coordinates t_1 and t_2 may be replaced by the more conventional x, t, but the early part of the discussion is carried out in terms of the physically primordial t_1 and t_2.

By introducing certain assumptions (isometric, etc.) as to the character of space-time, attention is limited to two special cases which Birkhoff calls "aeolotropic" and "isotropic" respectively. The first leads to the classical and the second to the special relativity metric. The consequences of the two assumptions are developed far enough by the end of chapter four to give a clear general idea of the differences between the two cases. From this point on attention is restricted to the isotropic case.

Three chapters are now devoted to the development of the special relativity in a two-dimensional space-time, the main topics being the dynamics of a particle and of a system of particles and one-dimensional hydrodynamics. From this discussion there emerges that complex of ideas which Einstein has bound together in the energy-momentum tensor. Then comes a chapter on tensor analysis and Riemann geometry, followed by a chapter on gravitational theory in two dimensions.

On page 150 (there are 270 pages of text) we turn for the first time to physical problems of more than one space dimension. In two chapters the previous discussion of special relativity is generalized to four dimensions, so that on arriving at page 186 we have the kinematics, dynamics, and hydrodynamics of the special theory behind us. Reversing the historical order, the author now closes his discussion of the special relativity with an account of the electromagnetic equations.

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* This method of arriving physically at a coordinate system had previously been suggested by J. L. Synge, NATURE, vol. 108, p. 275 (October 27, 1921) and worked out to a certain extent for the three-dimensional case.

The remaining sixty odd pages of the book are devoted to the general relativity. They contain first the determination of the possible types of linear homogeneous tensor equations of the second order. Then follows an account of Einstein's partial differential equations for a gravitational field with or without matter. Then the Schwarzschild form for the solar field is deduced and followed by an account of the three famous astronomical applications.

As intimated above, the arrangement of the book seems to the reviewer to be a very happy one. It responds very well to the fact that although the relativity theory arose from a study of the electromagnetic equations it was obtained by revising our views of kinematics and matter. Also, the prolonged attention to the two-dimensional case accustoms the reader to the machinery of tensor analysis and to a long series of abstract conceptions while he is dealing with figures that are easily visualized. The reader is thus prepared to meet the difficulties due to the complicated set of relations in the four-dimensional case, with the most important general concepts already in mind. It is true, on the other hand, as pointed out by Eisenhart in a careful review of this book in *Science* for December 28, 1923, that the methods which suggest themselves first in the two-dimensional case cannot always be used in the four-dimensional one. But such cases are exceptional in this book.

The book is intended as an introduction to the subject which may be used as a text in an undergraduate course. From this point of view, the first four chapters seem very difficult reading. There are other books, as, for instance, that of Kopff, with which it would be easier to start. It seems likely that the book will find its place not as a textbook in the ordinary sense but as one of two or three texts to which the student may refer for different points of view.

The reviewer has heard the opinions of various people who have read Birkhoff's book, or parts of it, without being previously well acquainted with the Einstein theory. In general, the accounts were more favorable from those who were not professional mathematicians than from those who were. The distinction seems to be due to the fact that readers of the latter class were disturbed by what they regarded as lapses from mathematical elegance or clarity of statement. In many cases they were handicapped by their own ignorance of physics. No one in either class regarded the book as easy reading, but all were impressed with the author's originality in point of view and with his grasp of the subject.

The reviewer's own opinion is that the real merit of the book is that it will convey new ideas and stimulating points of view to a reader who has already devoted a good deal of thought to the subject. The derivation of the Schwarzschild form without the hypothesis that the field is static and irreversible is obviously an important contribution. So is the chapter on linear tensor equations. But there are many less prominent matters which show great originality of method or content. The reviewer has dipped into the book at several places and always found something worth thinking about. See, for example, the treatment of statistical mass on page 72, or the form for the equations for gravitation on page 229, or the account of the tensor Φ^{i}_{jkl} (which is in fact the second extension of g_{v}) on page 123.

The postulational treatment to which the author calls special attention in the preface would better, in the opinion of the reviewer, be characterized as prolegomena to a set of postulates. The term "postulational treatment" has come in recent years to have a very precise significance. It implies an explicit list of undefined terms and a sharp statement of the logical form of the unproved propositions. It usually implies also an investigation of independence proofs. Such a study would doubtless have been impracticable in this book, but it would be interesting to see it carried out on some other occasion along the lines indicated by Birkhoff.

In various places Birkhoff brings out either explicitly or implicitly the great role played by considerations of symmetry and simplicity in the derivation of the equations of physics. One is tempted to feel that many of the other reasons advanced for adopting the expressions actually used in physical theories merely cover up the fact that these expressions are the simplest and most symmetrical ones which do not flagrantly contradict known facts. This reflection is also induced frequently on reading Einstein's book, *The Meaning of Relativity*, and it is doubtless capable and worthy of much more explicit development.

OSWALD VEBLEN

BOREL'S THEORY OF FUNCTIONS.

Méthodes et Problèmes de la Théorie des Fonctions. By Émile Borel. (Collection de monographies sur la théorie des fonctions.) Paris, Gautier-Villars, 1922. ix + 148 pp.

The volume under review, which Borel states in the introduction is to be the last of this justly celebrated collection of monographs to appear under his own name, is designed to supplement the earlier volumes of the series by bringing together in their original form a number of his notes and memoirs on the theory of functions. The twenty-seven articles included in this volume appeared in a number of journals during the years 1895—1912. They cover a wide range of subject matter and indicate clearly the breadth, originality, and versatility which have been the source of the very great influence which Borel has exerted on the progress of analysis in the last thirty years.

The memoirs presented are grouped in four chapters: *Les domaines et la théorie des ensembles; Les opérations et les développements en séries; La théorie de la croissance et le rôle des constants arbitraires; Les fonctions de variable complexe, en général, et les fonctions particulières.*

The introduction is devoted to an interesting analysis of the many striking analogies between biology and the theory of functions, both in content and in historical order of development. As man learned to use animals and plants before he had acquired a thoroughgoing knowledge of anatomy or physiology so mathematicians discovered and utilized the elementary functions, algebraic, circular, exponential. In biology the study of the structure of organisms led to the analysis of cellular life and organic chemistry; in the theory of functions the analysis of the properties of functions brought into existence the modern theories of aggregates, numbers, sets of points.

In the first chapter Borel remarks that the domains and ensembles are to functions as the tissues are to living organisms and calls attention to the importance which this part of the theory of functions has come to assume. Of the seven papers included in this chapter three are brief notes, one on the representation of discontinuous functions as limits of continuous functions and two on the theory of measure. The last of these outlines a procedure which may be substituted for the method of Lebesgue in the theory of integration. Three articles are devoted to the analysis and classification of sets of measure null. It is significant that the discoverer of the role in analysis of the sets of measure null should be the pioneer in the study of their structure and the closely related theory of monogenic non-analytic functions.

This chapter also contains a memoir entitled, *"Sur les définitions analytiques et sur l'illusion du transfini"*, which presents ably and forcibly the author's views regarding the limitations which should be placed upon arguments involving the transfinite. As is well known, no finite collection of words and symbols is adequate to the representation of all the transfinite ordinals of the second class. Borel calls attention to the relation between these ordinals α and the corresponding types of increasing function $\varphi_\alpha(n)$. In terms of these functions $\varphi_\alpha(n)$ it is possible to define a function $f(x)$ which is not representable analytically, but the complete definition of the function $f(x)$ rests upon the assumed definition of the $\varphi_\alpha(n)$ and the latter definition is impossible in a finite number of words.

The second chapter, on operations and developments in series, contains a short memoir on the conditions governing a change in the order of the terms of a conditionally convergent series, a memoir on the representation of functions of two real variables, and one on the integration of unbounded functions and constructive definitions. In his discussion of constructive definitions Borel expounds in detail his view that the only entities which may properly enter a mathematical discussion are those which are "well-defined". The continuum of Cantor and Dedekind is not "well-defined" in the sense of Borel since the numbers which form the continuum are not all definitely given. Hence, to be entirely consistent, it would be necessary for Borel either to abandon the use of the concept of the continuum and the related theory of functions or else so to transform these theories that they rest upon a "well-defined" basis. In spite of the admonishments of Borel and other adherents of his views, the number of those who call themselves mathematicians and who devote themselves to the study of concepts which are not "well-defined", such as non-measurable functions, transfinite cardinals and ordinals, measurable sets which are not Borel sets, curves in non-metrical analysis situs, is large and is increasing. It is not possible for any one group to guide or stem the tide of mathematical investigation or to say that here is mathematics; thus far and no farther may it go.

In the preliminary remarks to the third chapter Borel points out that in biology the greater interest and value lies in the study of the existent and normal species rather than in the study of hypothetical non-existent or abnormal species. By analogy, in mathematics the subject of functions which increase exponentially (to which that chapter is devoted) is relatively of great importance. The chapter contains ten notes, of which three are on the theory of the increase of integral functions and three are on various modes of approximation. One of the remaining four notes is concerned with an analogy between the number e and the algebraic numbers, one with an analytic partial differential equation

$$\frac{\partial^2\varphi}{\partial x^2} - a^4 \frac{\partial^2\varphi}{\partial y^2} = \psi(x, y)$$

which has for certain values of a a solution $\varphi(x, y)$ which is nowhere analytic in x or y; another to a formula for any analytic integral of a linear partial differential equation with analytic coefficients, and the last to the periods of abelian integrals and a related generalization.

The fourth chapter contains seven notes on the theory of functions of a complex variable which appeared in the COMPTES RENDUS from 1897—1912. If $\varphi(z)$ is a function with zeros at $a_1, a_2, \ldots$, the formula

$$f(z) = \sum \frac{c_n \varphi(z)}{(z-a_n)\varphi'(a_n)}$$

determines, in the case of convergence, a function $f(z)$ which assumes prescribed values c_n for $z = a_n$. In the first note Borel considers conditions that φ and so f be unique and brings out the intimate relationship between the theory of the zeros of integral functions and the problem of determining a function $f(x)$ uniquely by an enumerable infinity of conditions. The next three notes consider the relations between the coefficients of a power series and the singularities of the corresponding function. The fifth note, entitled *Sur l'étude asymptotique des fonctions méromorphes* (1902), is of historical interest. In this note Borel applied the method of exclusion of singularities to the case of functions whose singularities are everywhere dense in a region and was thereby led by a natural sequence of ideas to the discovery of the importance of the sets of measure null. The sixth and seventh notes are concerned respectively with a remarkable and unexplained relation between the coefficients of certain differential equations and the invariants of binary forms and with the variation of an analytic function in the neighborhood of an essential singularity.

In a brief conclusion Borel recommends that mathematicians seek to classify systematically those mathematical entities which are well-defined. He shows clearly that this work will rest upon a profound study and classification of those incommensurable numbers which are truly defineable, and remarks that although the subject is very difficult it is one in which the least conquest is infinitely precious because of its reactions; one attains here the living substance itself of which all mathematical entities are made; nothing is more important than the properties of number.

E. W. CHITTENDEN

SHORTER NOTICES

Vorlesungen über die Theorie der Algebraischen Zahlen. By E. Hecke. Leipzig, Akademische Verlagsgesellschaft, 1923. 264 pp.

This book had its origin in lectures delivered by the author in Basel, Gottingen, and Hamburg. As stated in the preface the aim of the book is to bring the reader to a comprehension of the questions which at present form the summit of the theory of algebraic number fields, without presupposing any knowledge of the theory of numbers. In reading the book the reviewer was particularly impressed by its richness in content. By the skillful coordination of important notions, the author gives, in the 264 pages, an elegant and comprehensive account of the modern theory of algebraic numbers.

The first chapter contains an introductory account of the theory of rational numbers. This is followed by a chapter of 28 pages on the theory of groups with special consideration of abelian groups, both finite and infinite. The purpose of this chapter is to familiarize the reader with those properties of abelian groups which may be advantageously used in the further development of the theory of rational numbers as well as the theory of algebraic numbers.

In the third chapter the group properties developed in the second chapter are used in the further study of the rational numbers. The chapter contains the theory of solutions of congruences; the theory of power residues; and in particular that of quadratic residues leading up to a statement of the law of quadratic reciprocity. The proof of this law is, however, deferred to Chapter eight, which contains the author's proof of the general law of quadratic reciprocity in a general algebraic number field.

Chapter four contains the algebraic development of the theory of algebraic numbers, and the algebraic properties of fields defined by the roots of algebraic equations.

Chapter five is the longest chapter of the book and contains a very comprehensive account of the arithmetic of an algebraic number field. The general theory of ideals is developed and theorems regarding the factorization of rational primes are given for certain special types of fields. This is followed by Minkowski's theorem on linear forms, with its application to the study of the units of a field.

A brief discussion of rings is given, and by the use of fractional ideals the author develops that part of the theory which deals with differents and discriminants of fields, and conductors of rings.

The sixth chapter is an introduction to the use of transcendental methods in the theory of algebraic number fields. It contains the development of the expression for the density of the prime ideals of a class, and its application to the problem of the determination of the number of classes. This is followed by a consideration of the zeta function and some of its applications.

Chapter seven is devoted to a more detailed consideration of quadratic fields. Norm residues are considered and the properties of Hilbert's norm residue symbol are developed. The separation of the ideals into genera is studied and the number of genera determined.

The further topics considered in this chapter are the zeta function in a quadratic field; the expression for the class number in a quadratic field (this without the use of the zeta function); the Gaussian sum; and the relations existing between ideals of a quadratic number field and binary quadratic forms.

Chapter eight contains a new proof of the most general law of quadratic reciprocity in an arbitrary algebraic number field. In this development the author makes use of his generalization of the Gaussian sum. The chapter ends with a proof of the existence of class fields of relative degree two. This is a consequence of the general law of reciprocity.

G. E. WAHLIN

Die Philosophischen Grundlagen der Wahrscheinlichkeitsrechnung. By E. Czuber. No. XXIV of Wissenschaft und Hypothese. Leipzig and Berlin, B. G. Teubner, 1923. 8 + 343 pp.

Czuber's book on the philosophy of probability is in essence a clarification of the fundamental concepts and theorems of the theory of probability by means of analysis and criticism of attacks directed against the theory in certain recent writings. The author's patient analysis and kindly criticism center chiefly on Meinong's rather labored *Über Möglichkeit und Wahrscheinlichkeit* and Marbe's naive ideas on probability expressed in his *Die Gleichförmigkeit in der Welt.* The concept and definition of probability, the addition and multiplication theorems, the theorems of Bernoulli, of Poisson, and of Bayes, the relation of probability to our knowledge of nature, all come in for detailed discussion.

Czuber's book, in the opinion of the reviewer, is intended primarily for non-mathematicians. Students of mathematical probability will find in the author's admirable *Wahrscheinlichkeitsrechnung* (3d edition, 1914) the substance of nearly all his views on the fundamentals of probability found in the *Grundlagen.* One instance of departure from his earlier work may be worth noting here: the obtaining (in Chap. VI) of the theorem of Bayes as a simple corollary of the multiplication theorem for dependent events (compare Boole's *Laws of Thought*, p. 254, III).

B. A. BERNSTEIN

Die Arithmetik der Quadratischen Formen, zweite Abteilung. By P. Bachmann. Leipzig, B. G. Teubner, 1923. xxii + 537 pp.

Bachmann's series of books, which together aim to cover the entire field of the theory of numbers, are recognized as the standard from the standpoints both of scholarship and clearness. Hence the publication of a new book in this series is an important event to the many devotees of this field. Although the new volume forms the second part of the work bearing the same title, it is really independent of the first part (which appeared in 1898), as well as of the other volumes of the series. This self-contained book employs largely the methods of the geometry of numbers initiated by Minkowski, and treats mainly the problem of the reduction of quadratic forms.

There are sixteen chapters, the number of pages in each of which varies only slightly from the average 33. The topics of the chapters are as follows: Binary quadratic forms; Lattices and continued fractions; Reduction of indefinite binary quadratic forms; Their minima; Geometrical interpretation of binary quadratic forms by means of lattices in a plane; Lattices in space and positive ternary quadratic forms; Geometry of numbers in n-dimensional space; Positive quadratic forms in n unknowns; Their reduction; Complete forms of Voronoi, limit (or extreme) forms; Equivalence and classes of positive quadratic forms; Decomposable forms; Quadratic and cubic irrationalities; Generalized algorithm of continued fractions; Minkowski's criterion for algebraic numbers; Indefinite quadratic forms in more than two unknowns.

The book was edited by Haussner, but follows essentially the manuscript left by Bachmann at his death in 1920.

L. E. Dickson

Analytische Geometrie des Raumes. By Salmon-Fiedler, revised, by Brill-Kommerell. Leipzig, Teubner, 1922. 590 pp.

Those familiar with the original Salmon-Fiedler will at once recognize that about ninety per cent of the revised edition is identical with the original. The remaining ten per cent of the book has been rewritten in such a manner as to make the revision a real contribution as a text book on the subject. The changes have been made in the form of more detailed explanations of certain rather obscure articles, the addition of many cuts and the addition of new subject matter. A great deal of the projective treatment has been omitted and analytic proofs substituted. The subject of quadratic forms has been entirely rewritten and precedes the transformations of surfaces. By changing the order the latter subject is introduced by means of the former. Many new examples have been added and several of the original exercises have been omitted. As a text I consider it superior to the revised English edition.

F. M. Morgan

On a Special Polyadic of Order n—p which can be Derived from any p Independent Vectors in an n-Dimensional Space and which can be Regarded as a Generalization of the Vector Product. By Almar Naess. (Videnskapsselskapets Skrifter, I. Mat.-Naturw. Klasse, 1922, No. 13.) Kristiania, Jacob Dybwad, 1923. 53 pp.

This beautifully printed pamphlet of 53 pages is an important contribution to generalized vector algebra. It deals with the "space-complement" of a set of p vectors, proved invariant under orthogonal transformation, and distributive, i. e. the space-complement of a sum of polyadics is the sum of the space-complements of the terms. The *space-complement of the space-complement* of p vectors is their alternant product, a fact whose importance in the logic of the subject can hardly be overestimated, — e. g. in 3-space the space-complement of two vectors is their vector product, while their alternant product is a skew-symmetric tensor often confounded with the vector product. Many other illustrations are given of which the generalized divergence and curl are especially noteworthy.

F. L. Hitchcock

The Atom and the Bohr Theory of its Structure. By H. A. Kramers and Helge Holst. New York, Knopf, 1923. xiii + 210 pages.

This volume contains an excellent English translation by R. B. Lindsay and Rachel T. Lindsay of the Danish original by the outhors named. There is an introduction by Sir Ernest Rutherford. The latter recommends the book as a clearly written and accurate account of the development of our ideas on atomic structure. My purpose in writing a note about the book in this place is to call the attention of mathematicians to the fact that we have here a self-contained introduction to the important ideas of Bohr and his followers concerning the atomic structure of matter. This book itself is free of mathematical detail. But the theories to which it gives an introduction involve problems of great difficulty and importance requiring the combined skill of the mathematician and the physicist for their resolution. After a brief account of the earlier development of the atomic theory the authors set forth the ideas of Bohr in some detail for the case of the hydrogen atom. The nature of the problems which arise become manifest here in their simplest form. Then several applications are made to various problems of atomic structure. The book will serve to orient the reader in the whole field. Details are not given of the mathematical problems; but one acquainted with as much of the theory as this book contains will find it comfortable to proceed to the memoirs. The Bohr theory opens up a great opportunity for the cooperation of mathematics and physics in solving one of the most fundamental problems of all natural science.

R. D. Carmichael

Vector Analysis. By C. Runge. Vol. I. Translated by H. Levy. New York, Dutton and Co., 1923. vi + 226 pages.

This translation contains the vectors of three-dimensional space. The original was reviewed in this BULLETIN, vol. 27, ser. 2, p. 464. There is little to add with regard to the translation. Heavy black letters have been substituted for German letters, which is more in conformity with prevailing customs, and tends to legibility, and the pages have been printed in a more open manner. The book is printed on thick paper which is a disadvantage as it makes it somewhat clumsy to use, and is not likely to be durable. The reviewer wonders why the translator saw fit (p. 211) to use cogradience and contragradience instead of the common terms cogrediency and contragrediency. The book will be of little use to students of physics particularly.

J. B. SHAW

Champ de Gravitation d'une Sphère Matérielle et Signification Physique de la Formule de Schwarzschild. By Jean Becquerel. Paris, J. Hermann, 1923. 32 pp.

La Théorie Einsteinienne de la Gravitation. By Gustave Mie (translated by J. Rossignol). Paris, J. Hermann, 1922. xi + 118pp.

The first of these publications contains an interesting interpretation of Schwarzschild's equation by means of an orthogonal projection of the non-euclidean solar field upon an asymptotic Minkowskian plane four-space tangent to the solar field at infinity.

This problem is also treated in an appendix which lends distinction to an otherwise undistinguished pamphlet by Mie.

C. N. REYNOLDS, JR.

Annuaire pour l'An 1923. Publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1923. 654 pp. + Appendices.

Only minor changes are noticeable in the Annuaire for 1923, although these are all of a nature to keep the publication up to date. Since the volume no longer appears well in advance of the year of date, astronomical information for the succeeding year is given in a supplement.

Besides two "Notices", one by P. Appell on Gabriel Lippmann and the other by A. Jobin on Jules Carpentier, a long one entitled *Le climat de France: l'eau atmosphérique* is contributed by G. Bigourdan. This gives in readable form the main features of what we call weather and climate with the physical principles which are to be used in the discussion of them. Particular attention is paid to rainfall, its distribution in time and space, with tables for various departments of France.

E. W. BROWN

NOTES

The concluding number of volume 24 of the TRANSACTIONS OF THIS SOCIETY (December, 1922) contains the following papers: *Representations of a complex point by pairs of ordered real points,* by W. C. Graustein; *Non-loxodromic substitutions and groups in n dimensions,* by E. B. Van Vleck; *Birational transformations simplifying singularities of algebraic curves,* by G. A. Bliss; *A symbolic theory of formal modular covariants,* by Olive C. Hazlett; *On the mean-value theorem corresponding to a given linear homogeneous differential equation,* by Georg Pólya. The third number of volume 25 of the TRANSACTIONS (July, 1923) contains: *Discontinuous boundary conditions and the Dirichlet problem,* by Norbert Wiener; *A type of differential system containing a parameter,* by F. H. Murray; *On a remarkable class of entire functions,* by J. I. Hutchinson; *Note on an ambiguous case of approximation,* by Dunham Jackson; *Expansions in terms of solutions of partial differential equations. Second paper: Multiple Birkhoff series,* by C. C. Camp; *Circular plates of constant or variable thickness,* by C. A. Garabedian; *Permutable rational functions,* by J. F. Ritt; *On approximation by functions of given continuity,* by Dunham Jackson.

The concluding number of volume 45 of the AMERICAN JOURNAL OF MATHEMATICS (October, 1923) contains: *Sylow subgroups in the group of isomorphisms of prime power abelian groups,* by H. A. Bender; *The isodyadic quintic,* by J. C. Glashan; *On class number relations for bilinear forms in four variables,* by E. T. Bell; *Relative inclusiveness of certain definitions of summability,* by D. S. Morse; *Modular invariants of a binary group with composite modulus,* by Constance R. Ballantine; *Determination of a surface by its curvatures and spherical representation,* by W. C. Graustein. The first number of volume 46 (January, 1924) contains: *On the theory of numbers and generalized quaternions,* by L. E. Dickson; *A study of the rational involutorial transformations in space which leave a web of sextic surfaces invariant,* by J. O. Osborn; *On the reduction of differential parameters in terms of finite sets, with remarks concerning differential invariants of analytic transformations,* by O. E. Glenn; *On the isodyadic septimic equations,* by J. C. Glashan.

The first International Congress for Mechanics was held at Delft, Holland, April 22–25, 1924. The congress met in three sections, for the presentation of papers on rational mechanics, theory of elasticity and rigidity, and hydro- and aerodynamics, including aviation. General sessions were also held.

In connection with the International Mathematical Congress to be held in Toronto on August 11-16, the American Section of the International Mathematical Union has extended its membership as provided under the general regulations governing that body. The previous members were as follows:

The President of this Society, ex officio: O. Veblen.

The Secretary of this Society, ex officio: R. G. D. Richardson.

Other representatives of this Society: L. P. Eisenhart, E. V. Huntington, H. F. Blichfeldt.

Representative of the Mathematical Association of America: H. L. Rietz.

The additional members of the Section recently appointed to serve from March to November, 1924, are as follows:

By this Society: A. B. Coble, Arnold Dresden, L. E. Dickson, Virgil Snyder.

By the Mathematical Association of America: R. C. Archibald, E. H. Moore, J. W. Young.

By the National Academy of Sciences: E. H. Moore.

By the American Astronomical Society: E. W. Brown.

By the American Physical Society: Leigh Page.

By the American Association for the Advancement of Science: G. A. Miller.

The Section thus enlarged has elected the following persons from among its members to act as delegates to represent the United States at the quadrennial meeting of the International Mathematical Union, to be held at Toronto during the Congress: Coble, Eisenhart, Huntington, Richardson, Rietz, Snyder. The present officers of the American Section are E. V. Huntington, Chairman; H. L. Rietz, Secretary.

The Royal Society of London will celebrate, on July 10 and 11, the centenary of Lord Kelvin's birth. Sir J. J. Thomson will deliver a memorial address.

Steps are being taken to found a History of Science Society, for encouragement of the study of this subject and for the support of the journal Isis, now edited by Dr. George Sarton. Correspondence with regard to this Society may, for the present, be addressed to Professor David Eugene Smith, 525 West 120th Street, New York City.

The gold medal of the Royal Astronomical Society has been awarded to Professor A. S. Eddington, for his work on star-streaming, on the internal constitution of a star, and on generalized relativity.

Dr. Elihu Thomson, of the General Electric Company, has received the Kelvin gold medal, which was founded in 1914 by British and American engineers and is awarded triennially by the presidents of representative British societies.

Professors Niels Bohr and Albert Einstein have been elected foreign members of the Göttingen Academy of Sciences.

Announcement has been made in this BULLETIN (p. 90, Jan.–Feb., 1924) that the fellowships administered by the National Research Council for physics and chemistry had been extended to include mathematics, and that the extension to mathematics had been placed in effect at once by a supplementary action. Under this action, the following persons have been appointed to National Research Fellowships in mathematics for the academic year 1924–1925 by the Research Fellowship Board: L. M. Graves (Ph. D., Chicago, 1924), Harry Levy, (Ph. D., Princeton, 1924), J. H. Taylor (Ph. D., Chicago, 1924), J. M. Thomas (Ph. D., Pennsylvania, 1923). The following persons have been reappointed as Fellows in mathematical physics: Herman Zanstra (Ph. D., Minnesota, 1923), T. Y. Thomas (Ph. D., Princeton, 1923).

At the request of the Commission on New Types of Examination of the College Entrance Examination Board, Professor L. P. Eisenhart of Princeton University has formed a committee to report on the validity of certain statistical methods used in the investigations of the Commission. The other members of the committee are Professors R. W. Burgess, W. L. Crum, E. V. Huntington, H. H. Mitchell, H. L. Rietz, and J. H. M. Wedderburn.

Mr. W. E. Byrne, of Rensselaer Polytechnic Institute, has been awarded an American Field Service Fellowship for 1924–1925, for the study of mathematics at the University of Paris. Applications for such fellowships for 1925–1926 should reach Dr. I. L. Kandel, 525 West 120th Street, New York City, not later than December 15, 1924.

Dr. Philip Franklin, now Benjamin Peirce teaching fellow at Harvard University, has been appointed instructor in mathematics at the Massachusetts Institute of Technology for the year 1924–1925.

At the University of Chicago, Associate Professor A. C. Lunn has been promoted to a full professorship of mathematics. Dr. Mayme I. Logsdon, instructor in mathematics, has recently been appointed to a deanship in the College of Science.

Dr. Pauline Sperry, instructor in mathematics at the University of California, has been promoted to an assistant professorship.

The death is announced, at the age of ninety-one years, of Gustave Eiffel, the French engineer, known for his work in aerodynamics.

Professor R. E. Wilson, of Northwestern University, died December 30, 1923, at the age of fifty-two. Professor Wilson had been a member of the American Mathematical Society since 1903.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

Appell (P.). Education et enseignement. Notices et discours. Paris, Alcan, 1922. 2 + 299 pp.

Atanassiévitch (X.). La doctrine métaphysique et géométrique de Bruno, exposé dans son ouvrage "De triplici minimo." Paris, Les Presses Universitaires de France, 1923. 156 pp.

Bohr (H.) og Mollerup (J.). Lærebog i matematisk Analyse. 4: Grundtrek af Læren om de kompleske Tal. København, Gjellerup, 1923. 130 pp.

Borel (E.). Eléments de la théorie des probabilités. 3e édition, revue et augmentée. Paris, Hermann, 1924. 6 + 246 pp.

Collingwood (E. F.). See Valiron (G.).

Cunningham (A. J. C.). Binomial factorizations giving extensive congruence-tables and factorization-tables. Volume 1 and volume 4 (supplement to volume 1). London, Francis Hodgson, 1923. 96 + 288 + 6 + 160 pp.

Drews (A.). Geschichte der Philosophie. Band 9: Die deutsche Philosophie der Gegenwart und die Philosophie des Auslandes. (Sammlung Göschen.) Berlin, de Gruyter, 1922. 148 pp.

Euler (L.). Opera omnia. Series I: Opera mathematica. VII: Commentationes algebraicæ ad theoriam combinationum et probabilitatum pertinentes. Edidit L. G. du Pasquier. Leipzig, Teubner, 1923. 58 + 580 pp.

Gordon (W.). See Russell (B.)

Goursat (E.). Cours d'analyse mathématique. 4e édition revue et augmentée. Tome I. Paris, Gauthier-Villars, 1923. 674 pp.

Gumbel (E. J.). See Russell (B.).

Hadamard (J.). Lectures on Cauchy's problem in linear partial differential equations. New Haven, Yale University Press, 1923. 10 + 316 pp.

Hart (I. B.). Makers of science: mathematics, physics, astronomy. London, Oxford University Press, 1923. 320 pp.

Hilbert (D.). See Russell (B.).

Hjelmslev (J.). Tre Foredrag over Geometriens Grundlag. København, Gjellerup, 1923. 68 pp.

Hoffmann (A.). René Descartes. 2te, verbesserte Auflage. Stuttgart, Fr. Frommans Verlag, 1923. 198 pp.

Junker (F.). Hohere Analysis. Band 2: Integralrechnung. 4te verbesserte Auflage. (Sammlung Göschen.) Berlin, de Gruyter, 1923.

von Kerékjártó (B.). Vorlesungen über Topologie. Berlin, Springer, 1923. 8 + 270 pp.

KNESER (A.). See SCHLOMILCH (O.).

KNOPP (K.). Aufgabensammlung zur Funktionentheorie. Teil 1. (Sammlung Göschen.) Berlin, de Gruyter, 1923. 136 pp.

KONORSKI (B. M.). Die Grundlagen der Nomographie. Berlin, Springer, 1923. 3 + 86 pp.

LEBESGUE (H.). Notice sur les travaux scientifiques de M. Henri Lebesgue. Toulouse, Privat, 1922. 4to. 92 pp.

MALMQUIST (A. J.). Föreläsningar i matematik. Stockholm, Teknologernas Handelsforenings Förlag, 1923. 16 + 704 pp.

MATHEMATICAL ASSOCIATION. The teaching of geometry in schools. A report prepared by the (British) Mathematical Association. London, Bell, 1923. 74 pp.

MOLLERUP (J.). See BOHR (H.).

MUICA (I.). Théorème de Fermat. 3e édition. Bucarest, Cartea Romaneasca, 1923. 8 pp.

NÖRLUND (N. E.). Videnskabelige Causerier. København, Gjellerup, 1923. 125 pp.

DU PASQUIER (L. G.). See EULER (L.).

PEET (T. E.). The Rhind mathematical papyrus: British museum 10057 and 10058. Introduction, transcription, translation and commentary. Liverpool, University Press, 1923. 4 + 136 pp. + 24 plates.

PICKEN (D. K.). The number system of arithmetic and algebra. Melbourne, University Press, 1923. 8 + 76 pp.

RUSSELL (B.). Einfuhrung in die mathematische Philosophie. Ins Deutsche ubertragen von E. J. Gumbel and W. Gordon. Mit einem Vorwort von David Hilbert. München, Drei Masken Verlag, 1923. 8 + 212 pp.

SCHALLER (J. G.). Beweis der Richtigkeit des "Grossen Fermatschen Satzes". Grabow, Selbstverlag, 1922. 23 + 14 + 15 pp.

SCHLOMILCH (O.). Kompendium der höheren Analysis. Band 1. In 6te Auflage bearbeitet von A. Kneser. Braunschweig, Vieweg, 1923. 10 + 619 pp.

SILBERSTEIN (L.). Synopsis of applicable mathematics. New York, Van Nostrand, 1923. 11 + 250 pp. $4.50

SOMMERVILLE (D. M. Y.). Analytical conics. London, Bell, 1924. 7 + 310 pp.

TANNERY (P.). Mémoires scientifiques. Tome V: Sciences exactes au moyen âge. Paris, Gauthier-Villars, 1922. 26 + 384 pp.

TONELLI (L.). Fondamenti di calcolo delle variazioni. Volume 2. Bologna, Zanichelli, 1923. 8 + 660 pp.

VALIRON (G.). Lectures on the general theory of integral functions. Lectures given at the University College of Wales (Aberystwyth) in 1922, translated by E. F. Collingwood. Preface by W. H. Young. Toulouse, Privat, and Cambridge, Deighton Bell, 1923. 11 + 208 pp.

YOUNG (W. H.). See VALIRON (G.).

PART II. APPLIED MATHEMATICS

Amoroso (L.). Lezioni di matematica finanziaria. Volume II. Napoli, Majo, 1923. 466 pp.

Andoyer. (H.) et Lambert (A.). Cours d'astronomie. 2e édition entièrement refondue. Partie 2: Astronomie pratique. Paris, Hermann, 1924. 316 pp.

Baudisch (H.). See Grossl (R.).

Bachmann (L.). Das Schachspiel und seine historische Entwicklung. Leipzig, Teubner, 1923.

Biles (J. H.). The design and construction of ships. Volume 1: Calculations and strength. 3d edition. Volume 2: Stability, resistance, propulsion, and oscillation of ships. 2d edition. London, C. Griffin, 1923. 448 + 428 pp.

Borel (E.). Traité du calcul des probabilités et des ses applications. Tome III, fascicule 1: Assurances sur la vie. Calcul des primes. Par H. Galbrun. Paris, Gauthier-Villars, 1924. 311 pp.

Born (M.). Atomtheorie des festen Zustandes. (Dynamik des Kristallgitter.) Leipzig und Berlin, Teubner, 1923.

Tychonis Brahe opera omnia. Edidit I. L. E. Dreyer. Tomus 10. Hauniæ, Gyldendal, 1923.

Callandreau (E.). See Creager (W. P.).

Castelnuovo (G.). Spazio a tempo, secondo le vedute di Einstein. Bologna, Zanichelli, 1923.

Chwolson (O. D.). Lehrbuch der Physik. Band 3, Abteilung 2: Die Lehre von der Warme. Herausgegeben von G. Schmidt. Braunschweig, Vieweg, 1923.

Coleman (G. S.). Hydraulics applied to sewer design. London, C. Lockwood, 1923. 8 + 150 pp.

Corvisy (A.). See Nernst (W.).

Creager (W. P.). La construction des grands barrages en Amérique. Traduit de l'anglais par E. Callandreau. Paris, Gauthier-Villars, 1923. 243 pp.

Davey, (N.). Studies in tidal power. London, Constable, 1923. 13 + 255 pp.

Dreyer (I. L. E.). See Tycho Brahe.

Duncan (J.) and Starling (S. G.). A text book of physics. Parts 2–3: Heat and light. London, Macmillan, 1923.

Eddington (A. S.). Raum, Zeit und Schwere; ein Umriß der allgemeinen Relativitätstheorie. Ins Deutsch übertragen von W. Gordon. Braunschweig, Vieweg, 1923. 8 + 204 pp.

Fleming (J. A.). Electrons, electric waves and wireless telegraphy. London and New York, Wireless Press, 1923. 8 + 326 pp.

FORESTIER (A.). L'énergie rayonnante. Paris, A. Blanchard, 1923. 64 pp.

FRANKLIN (W. S.) and MACNUTT (B.). Heat. A text book for colleges and technical schools. Lancaster, Pa., 1923. 8 + 187 pp.

GALBRUN (H.). See BOREL (E.).

GANDILLOT (M.). L'illusion d'Einstein. Paris, Vuibert, 1923. 18 pp.

GIBBS (W. E.). Clouds and smokes. The properties of disperse systems in gases and their practical applications, London, Churchill, 1923. 13 + 240 pp.

GORDON (W.). See EDDINGTON (A. S.).

GRAETZ (L.). Die Atomtheorie in ihrer neuesten Entwicklung. Sechs Vorträge. 4te, vermehrte Auflage. Stuttgart, Engelhorn, 1922. 8 + 100 pp.

GROSSL (R.) und BAUDISCH (H.). Mechanik der flüssigen Korper. 2te Auflage. Leipzig und Wien, Deuticke, 1922. 9 + 280 pp.

HOLST (H.). See KRAMERS (H. A.).

HUDDY (E. H.). Mechanical engineering formulae. London, Spon, 1923. 167 pp.

JANET (P.). Leçons d'électrotechnique générale. Tome 3. 5e édition. Paris, Gauthier-Villars, 1923.

JARRE (V.). Dualité de la matière. Essai sur le mécanisme du renouvellement des mondes. Paris, Alcan, 1923. 304 pp.

KRAMERS (H. A.) and HOLST (H.). The atom and the Bohr theory of its structure; an elementary presentation. Translated from the Danish by R. B. Lindsay and R. T. Lindsay. London, Copenhagen, and Christiania, Gyldendal, and New York, Knopf 1923. 13 + 210 pp.

LACOUR (A.). Die Gleichstrommaschine. Band 1: Theorie und Untersuchung. Bearbeitet von J. L. LaCour. 3te vollstandig umgearbeitete Auflage. Berlin, Springer, 1923. 12 + 728 pp.

LACOUR (J. L.). See LACOUR (A.).

LADENBURG (R.). Die Grundlagen der Quantentheorie und ihre experimentelle Prüfung. Leipzig, Barth, 1923. 45 pp.

LAMBERT (A.). See ANDOYER (H.).

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LILIENTHAL (G.). Vom Gleitflug zum Segelflug. Flugstudien auf Grund zahlreicher Versuche und Messungen. Berlin-Charlottenburg, Volckmann, 1923. 159 pp.

LINDSAY (R. B.). See KRAMERS (H. A.).

LINDSAY (R. T.). See KRAMERS (H. A.).

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MACNUTT (B.). See FRANKLIN (W. S.).

MARCOLONGO (R.). Relatività. Messina, Principato, 1921. 7 + 192 pp.

NERNST (W.). Traité de chimie générale. Traduction A. Corvisy. 2e édition française complètement refondue d'après la 10e édition allemande. Tome 1: Les propriétés générales des corps. Atome et molécule. Tome 2: Transformations de la matière et de l'énergie. Paris, Hermann, 1922–1923. 8 + 620 + 500 pp.

NICHOLSON (J. W.). See SCHUSTER (A.).

PEARL (R.). Introduction to medical biometry and statistics. Philadelphia and London, W. N. Saunders, 1923. 379 pp.

PETRASCHEK (K. O.). Der Grundwiderspruch in der speziellen Relativitätstheorie und seine Folgen. Leipzig, Hillmann, 1922. 76 pp.

PLANCK (M.). Vorlesungen über die Theorie der Wärmestrahlung. 5te, abermals umgearbeitete Auflage. Leipzig, Barth, 1923. 10+221 pp.

PRINGSHEIM (P.). Fluorescenz und Phosphorescenz im Lichte der neueren Atomtheorie. 2te, verbesserte Auflage. Berlin, Springer, 1923. 8 + 228 pp.

QUINT (H.). Die Relativitatstheorie. Ein Blick in diė Welt Einsteins. Leipzig und Wien, Anzengruber Verlag, 1923(?). 96 pp.

ROBINSON (G.). See WHITTAKER (E. T.).

ROSE (W. N.). Line charts for engineers. London, Chapman and Hall, 1923. 12 + 95 pp.

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SCHERESCHEWSKY (P.) et WEHRLÉ (P.). Les systèmes nuageux. Paris, Chiron, 1923. 16 + 77 pp.

SCHOENFLIES (A.). Theorie der Kristallstruktur. Ein Lehrbuch. Berlin, Gebrüder Borntraeger, 1923. 12 + 555 pp.

SCHROETER (J.). Haandbog i kronologi. D. 1. Kristiania, Cammermyers Boghandel, 1923. 4 + 210 pp.

SCHUMANN (W. O.). Elektrische Durchbruchfeldstarke von Gasen. Theoretische Grundlagen und Anwendung. Berlin, Springer, 1923. 7 + 246 pp.

SCHUSTER (A.). An introduction to the theory of optics. 3d edition, revised and enlarged by the author and J. W. Nicholson. London, Arnold, 1924. 15 + 397 pp.

SEWARD (H. L.). See HEWES (L. I.).

SHAW (N.). Forecasting the weather. 2d edition, revised and enlarged. London, Constable, 1923. 43 + 584 pp.

SMITH (W. G.). Engineering kinematics. New York and London. McGraw-Hill, 1923. 10 + 282 pp.

SMITS (A.). La théorie de l'allotropie. Traduit par J. Gillis. Paris, Gauthier-Villars, 1923. 19 + 523 pp.

SOLOVINE (M.). See THIRRING (H.).

STANTON (T. E.). Friction. London, Longmans, 1923. 14 + 183 pp.

STARK (J.). Prinzipien der Atomdynamik. 2te, umgearbeitete Auflage. Teil 1: Die elektrischen Quanten. Leipzig, Hirzel, 1922. 10 + 95 pp.

STARLING (S. G.). See DUNCAN (J.).

STEFFENSEN (I. F.). Matematisk Iagttagelslaere. København, Gads Forlag, 1923. 268 pp.

SULLIVAN (J. W. N.). Atoms and electrons. London, Hodder and Stoughton, 1923. 188 pp.

TALLQVIST (H.). Nyare strålnings- och atomfysik. Stockholm, Albert Bonniers Forlag, 1923. 8 + 487 pp.

THIRRING (H.). L'idée de la théorie de la relativité. Traduit de l'allemande par Maurice Solovine. Paris, Gauthier-Villars, 1923. 188 pp.

THOMSON (J. G.). The cycloid in astronomy. Edinburgh, Allen, 1923. 14 pp.

THOMSON (J. J.). The electron in chemistry. Being five lectures delivered at the Franklin Institute, Philadelphia. Philadelphia, Franklin Institute, 1923. 5 + 144 pp. $1.75.

TILLIEUX (J.). Essai d'un traité élémentaire de physique selon les théories modernes. Paris et Liège, Béranger, 1921.

VON TYSZKA (C.). Statistik. Teil 1: Theorie, Methode und Geschichte der Statistik. Jena, Fischer, 1924.

VARIČAK (V.). Darstellung der Relativitätstheorie im dreidimensionalen lobatscheskijschen Raume. Zagreb, Zaklada Tiskare Narodnih Novina, 1924. 10 + 104 pp.

WEATHERBURN (C. E.). Advanced vector analysis with application to mathematical physics. London, Bell, 1924. 16 + 222 pp.

WEHRLÉ (P.). See SCHERESCHEWSKY (P.).

WERKMEISTER (P.). Das Entwerfen von graphischen Rechentafeln (Nomographie), Berlin, Springer, 1923. 8 + 194 pp.

WHETHAM (W. C. D.). The theory of experimental electricity. Cambridge, University Press, 1923. 11 + 349 pp.

WHITTAKER (E. T.) and ROBINSON (G.). The calculus of observations. A treatise on numerical mathematics. London, Blackie, 1924. 16 + 395 pp.

WINES (W. E.). Strength of materials. London, McGraw-Hill, 1923. 20 + 241 pp.

YERMOLOFF (M.). Y a-t-il continuité dans le monde physique? Paris, Doin, 1923. 48 pp.

ZEHNDER (L.). Der Aufbau der Atome aus Uratomen. Tübingen, Lauppsche Buchhandlung, 1922. 23 pp.

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THE FORTY-THIRD REGULAR MEETING OF THE SAN FRANCISCO SECTION

The forty-third regular meeting of the San Francisco Section of this Society was held in Room 72, Mathematics Building, Stanford University, on Saturday, April 5, 1924, with Professor A. F. Carpenter in the chair. The following twenty-six members of the Society were present.

Alderton, Allardice, Bell, Blichfeldt, Buck, Cajori, A. F. Carpenter, De Cou, Green, Griffin, Growe, Haskell, Hewes, Hoskins, Lehmer, S.H.Levy, McFarland, W. A. Manning, Moreno, F. R. Morris, Noble, T. M. Putnam, Pauline Sperry, Stromquist, A. R. Williams, Wong.

After a discussion of the question as to what form of organization would best serve the interests of mathematics on the Pacific Coast, it was voted that the San Francisco Section continue to have two regular meetings a year, as heretofore, one at Stanford University, and one at the University of California; that a third regular meeting be held annually early in August in the Pacific Northwest; that a fourth regular meeting should be held in Southern California when this should be desirable.

A motion that the Section should not participate in the 1924 meeting of the Pacific Division of the American Association for the Advancement of Science was carried.

Titles and abstracts of the papers read at this meeting follow. The papers by Professors Eells, James, and Milne were read by title.

1. Professor E. T. Bell: *Reductions of enumerations in homogeneous forms.*

This paper appeared in the July number of this Bulletin.

2. Professor E. T. Bell: *The class number relations implicit in the Disquisitiones Arithmeticae.*

This paper appeared in full in the May-June number of this Bulletin.

3. Professor E. T. Bell: *On the inversion of arithmetic products.*

The Dedekind inversion of a finite product which yields the equation (or congruence) of primitive roots is shown to be a very special case of a general inversion process stated and proved here.

4. Professor E. T. Bell: *The numbers of representations of integers in certain ternary quadratic forms.*

The number $N(n=ax^2+by^2+cz^2)$ of representations of the integer $n > 0$ in the indicated form is determined in terms of the binary quadratic class number in each of the nine cases $(a,b,c)=(1,1,2)$, $(1,2,2)$, $(1,2,4)$, $(1,1,4)$, $(1,4,4)$, $(1,2,8)$, $(1,1,8)$, $(1,4,8)$, $(1,8,8)$. Hitherto there have been given only the results for $(1,1,1)$ (Gauss), and incomplete statements for $(1,1,2)$ (Torelli), $(1,2,2)$ (Stieltjes).

5. Professor E. T. Bell: *The class number relations implicit in Jacobi's theta formula.*

The relations are of a wholly new type involving an arbitrary even function whose arguments are linear functions of the indeterminates representing a fixed integer, whose negative is the argument of the class number functions involved, in certain quaternary quadratic forms.

6. Professor E. T. Bell: *Certain products involving the divisors of numbers.*

Certain very general identities between infinite products of the type $\prod(1-ax^{bn})^{g(n)}$, in which the $\prod$ extends to $n=1,2,\ldots$, a, b are constants, and $g(n)$ is a single valued function of n, are obtained and expanded in series. The identities yield many new theorems in partitions, particular cases of which have been given by Jacobi and others.

7. Professor E. T. Bell: *Complete class number expansions for the elliptic theta constants of degree* 3.

The nine possible expansions of the type $\vartheta_\alpha^a\vartheta_\beta^b(a+b=3;$ $\alpha,\beta=0,1,2,3)$ are found by elementary processes. The coefficients are class number functions; the series for $\vartheta_0^2\vartheta_3$, $\vartheta_0\vartheta_3^2$ have not previously been given. The rest were obtained at greater length by Kronecker and Hermite.

8. Professor Florian Cajori: *Note on our sign of equality.*

Professor E. Bortolotti has sent the author a photograph of a page of an undated manuscript now kept in the library of the University of Bologna, written by Pompeo Bolognetti, who before 1568 (possibly before the appearance of Recorde's algebra) used the sign $=$ for equality in the solution of equations.

9. Professor Florian Cajori: *History of the use of pictographs and algebraic symbols in elementary geometry.*

An account is given of the 250-year struggle between writers friendly and those hostile to the use of symbols in geometry. There still persists a deplorable lack of uniformity in notation.

10. Professor Florian Cajori: *Notes on Luca Pacioli's Summa.*

In this paper the editions of 1494 and 1523 of the *Summa* are compared. It is found that Pacioli's use of ℞ as an algebraic symbol has been misinterpreted by certain European and American writers on the history of notations. Pacioli sometimes used a dash to denote equality.

11. Professor Florian Cajori: *Despiau's Select Amusements.*

Despiau's *Select Amusements in Philosophy and Mathematics,* brought out at London in 1800 in French and in 1801 in English translation, is a unique book which has escaped the net of nearly all large libraries and of bibliographies of mathematical recreations. It is described in this paper.

12. Professor A. F. Carpenter: *Cone-cubic configurations of a ruled surface.*

Associated with each line element g of the general ruled surface S there are two cone rays. The points in which these rays cut the flecnode tangents of S drawn at the flecnode points of g are centers of perspective for certain configurations of cone-cubic points and cone-cubic osculating planes. The dual property holds for the planes determined by the cone rays and the flecnode tangents. These perspectivities lead to many interesting theorems which are indicated in this paper.

13. Professor W. C. Eells: *On a standard definition of the standard deviation.*

In this paper the author examines critically numerous definitions of the *standard deviation* (σ). The paper will appear in the JOURNAL OF EDUCATIONAL RESEARCH.

14. Professor F. L. Griffin: *Note on the yield of serial bonds issued at split rates of interest.*

To determine the yield on bonds, purchased at any market price, maturing serially in unequal amounts, and carrying two or more different rates of interest, requires the solution of a complicated equation of high degree. The author points out a convenient adaptation of Newton's method in connection with the ordinary bond tables, and discusses some fallacious methods commonly used by investors.

15. Professor Glenn James: *On the solution of algebraic equations with rational coefficients.*

A generalization of the ordinary factor theorem is used to determine quadratic factors. This paper will appear in the AMERICAN MATHEMATICAL MONTHLY.

16. Professor W. E. Milne: *Note on the flexure of an I-beam.*

The problem of determining the stresses in a beam bent by a terminal transverse load can be solved in terms of a function $\varphi(x, y)$, harmonic at all points of the cross section and having assigned values for the normal derivative on the boundary. The problem is easily solved if the boundary of the section is a circle, an ellipse, or a rectangle. The important case of an I-beam may be solved by means of a certain hyperelliptic integral. In this note it is shown how this integral may be evaluated to any desired degree of approximation by means of theta functions. It is further shown that for beams ordinarily used in structural work the constants are very small, so that the series used converges rapidly. The first two terms of the series will ordinarily give results with an error not exceeding one-thousandth of one per cent.

C. A. NOBLE,
Acting Secretary.

THE APRIL MEETING IN CHICAGO

The twenty-first regular Western meeting of the Society was held at the University of Chicago on Friday and Saturday, April 18 and 19, 1924. Nearly one hundred persons attended this meeting; among them were the following seventy-six members of the Society:

R.W. Babcock, W. S. Beckwith, H. A. Bender, Bliss, Blumberg, Bradshaw, O. E. Brown, Bunyan, C. C. Camp, Carlson, Carmichael, Crathorne, Curtiss, Denton, Dickson, Dostal, Dowling, Dresden, Emch, Escott, W. B. Ford, Gibbens, Gossard, Gouwens, L. M. Graves, W. A. Hamilton, E. R. Hedrick, Hildebrandt, Ingraham, Dunham Jackson, Kennedy, Kinney, Lane, Lefschetz, Logsdon, Luby, Lunn, McGavock, J. V. McKelvey, W. D. MacMillan, T. E. Mason, B. I. Miller, Miser, E. H. Moore, E. J. Moulton, F. R. Moulton, Nowlan, C. I. Palmer, Pepper, Pettit, Plapp, R. G. D. Richardson, Rider, H. L. Rietz, Roever, Roth, Schottenfels, Sellew, Sharpe, Shohat, W. G. Simon, Skinner, Slaught, Snedecor, Virgil Snyder, Stouffer, J. H. Taylor, Teach, T. Y. Thomas, J. S. Turner, Wainwright, Wait, K. P. Williams, F. E. Wood, J. W. A. Young, Ziwet.

At the meeting of the Council, the following thirty-seven persons were elected to membership in the Society:

Professor Eli Allison, Brenau College;
Professor Normal Herbert Anning, University of Michigan;
Professor Ernest Aubrey Bailey, LaGrange College;
Mr. Ben Charles Bellamy, Civil Engineer, Laramie, Wyo.;
Mr. Jacob Alfred Benner, Lafayette College;
Mr. Samuel Fletcher Bibb, University of North Dakota;
Dr. Laura Brant, Instructor in Physics, Vassar College;
Mr. Richard Lucius Cary, Baltimore, Md.;
Mr. Luther Jonathan Deck, Muhlenberg College;
Professor Aloysius F. Frumveller, Marquette University;
Professor Arthur Eugene Gault, Bradley Polytechnic Institute, Peoria, Ill.;
Dr. William Albert Hamilton, University of Wisconsin;
Professor Louis Alan Hazeltine, Stevens Institute of Technology;
Professor Guy Hildebrand Hunt, Southern Branch, University of California;
Professor Byron Ingold, Culver-Stockton College;
Miss Rosa Lea Jackson, Instructor, Northwestern University;
Professor Martha Myrtle Knepper, State Teachers College, Cape Girardeau, Mo.;
Mr. Claiborne Green Latimer, Tulane University;
Professor Otto B. Loewen, Ottawa University, Ottawa, Kansas;
Mr. William Hayward McEwen, Assistant, University of Minnesota;
Mr. Charles Edward Manierre, New York City;
Mr. Gaylor Maish Merriman, University of Cincinnati,

Miss Inez Morris, Kansas State Teachers College, Emporia;
Professor Albert George Rau, Moravian College;
Mr. Pardon Sheldon Remington, The Principia, St. Louis;
Professor David Henry Richert, Bethel College;
Dr. Daniel Stucker Robbins, State College, New Mexico;
Mr. Edwin Lawrence Rose, California Institute of Technology,
Professor Bernice Sanders, Wilberforce University;
Professor Thomas Francis Smith, Little Rock College;
Professor Mary Goode Stallworth, Alabama College;
Mr. Clarence George Tobias Stipe, University of South Dakota;
Professor Mildred Browning Stone, State Normal School, Salem, Mass.,
Professor Ralph Sylvester Underwood, Alabama Polytechnic Institute, Auburn, Ala.;
Mr. William Edwin Van de Walle, Harvard University;
Miss Frances Christine Wenrich, Kansas City Junior College;
Mr. George Andrew Yanosik, New York University;
Mr. Louis Yurow, Washington, D.C.

Thirty-seven applications for membership were received.

The Council accepted the invitation of Cornell University to hold the summer meeting and colloquium in Ithaca in 1925 and expressed to the members of the Mathematical Department of Cornell University its appreciation of the invitation. The speakers and subjects for the colloquium will be Professor L. P. Eisenhart, *The differential geometry of general surfaces*, and Professor Dunham Jackson, *The theory of approximation.*

The Assistant Secretary reported the following appointments made by President Veblen since the last meeting of the Society: as delegate to the celebration of the fiftieth anniversary of the founding of the Société Mathématique de France at the Sorbonne on May 22–24, 1924, Professor E. B. Van Vleck; to represent the Society at the semicentennial of the founding of Purdue University on May 1–3, Professor William Marshall; Committee on Arrangements for the summer meeting and colloquium of 1925: Professors J. H. Tanner, W. B. Ford, W. A. Hurwitz, R. G. D. Richardson, and H. L. Rietz; Committee to nominate trustees, officers and members of the Council to be elected in December, 1924, Professors G. A. Bliss (Chairman), Arnold Dresden, and C. L. E. Moore.

The Council voted to recommend to the Society for adoption at its meeting on May 3 amendments to the By-Laws, to eliminate the ephemeral matter relating to the transfer of business from the unincorporated Society to the incorporated body; to make possible the transaction by mail of some of the Council's business; and to create a sustaining membership.

The Council adopted resolutions of thanks to Professor O. D. Kellogg for valuable assistance rendered the Endowment Committee, to the Carnegie Corporation and to the firm of Allyn and Bacon for their generous contributions to the endowment fund. The Committee on Endowment was authorized to enter into agreements with industrial firms for a service of information and to proceed with the collecting of data necessary to maintain such service.

Those who were present at the meeting on Friday afternoon united in sending a cablegram extending to Professor Felix Klein their greetings and congratulations on the occasion of his seventy-fifth birthday.

On Friday evening a dinner was held at the Quadrangle Club at which sixty-eight persons were present. The need for continued support of the work of the Endowment Committee was emphasized. It was reported that over $23,000 had been contributed by 362 members of the Society. While this total amount was considered very gratifying, it was thought highly desirable that a much larger percentage of the membership take part in the subscription.

The scientific sessions of Friday and Saturday mornings were presided over by Vice-President Hildebrandt. He was relieved by Professor D. R. Curtiss on Friday afternoon; at this session Professor H. L. Rietz presented the symposium lecture on *Certain topics in the mathematical theory of statistics*; this paper appears in full in the present number of this BULLETIN.

The papers read at this meeting are listed below. Miss Kearney, Mr. Jensen, and Mr. Mickelson were introduced to the Society by Professor Dunham Jackson, and Mr. Reinsch by Professor Carmichael. The papers of Baker, Davis, Gehman, Kearney, Mickelson, Steimley, and Williams (first paper) were read by title.

1. Professor E. P. Lane: *The canonical eight-point quadric of a space curve.*

An eight-point quadric of a space curve, at a point P of the curve, is the limiting position of a quadric surface through P and seven neighboring points of the curve as these points approach P along the curve. There is a single infinity of eight-point quadrics at each point P of the curve. Among these are two of special interest. One is the osculating, or nine-point, quadric. The other, which the author calls the *canonical* eight-point quadric, is characterized by being tangent at P to the osculating plane of the curve at P. This quadric has an equation of simple form and is intimately connected with the curve. It may be used in defining a local coordinate system, in some respects analogous to the moving trihedron employed in the metric theory of space curves.

2. Professor E. P. Lane: *A characterization of surfaces of translation.*

If there exists on a surface a conjugate net with indeterminate ray curves, then the ray points of every surface point, with respect to this conjugate net, lie in a fixed plane called the ray plane of the net. The author characterizes surfaces of translation as the only surfaces that have the property that on each of them there exists a conjugate net with indeterminate ray curves and with ray plane at infinity.

3. Professor Arnold Emch: *Some problems of closure connected with the Geiser transformation.*

As is well known, the Geiser transformation in a plane is an involutory Cremona transformation defined by a special net of cubics through 7 independent points. Two cubics of the net intersect in a pair of corresponding points (PP') of the transformation. The seven points A_i are the base points of the transformation. To each of these corresponds every point of the rational cubic with this base-point as a node and passing through the six remaining base-points. The jacobian of the net is the so-called Aronhold curve, a definite point-wise invariant sextic with nodes at the base-points. This paper considers the classes of invariant curves as loci of pairs of corresponding points, and certain class-curves from which they are generated. The mapping of the Geiser transformation upon a general cubic surface leads to interesting problems of closure of sets of certain invariant curves.

4. Professor F. R. Sharpe: *The Jonquière space transformation of Godeaux.*

In a recent memoir (Royal Belgian Academy 1921) Godeaux has considered a type of birational transformation in which to any line of a rational congruence and a plane of a pencil in one space corresponds a similar line and plane in a second space. The author shows that the type is included in a type given by Montesano thirty-three years earlier and that a slightly more general form exists. It is also shown that certain cases which Godeaux concludes to be impossible are possible without restriction.

5. Dr. Gladys Gibbens: *A study of the relations between the focal surfaces of two congruences obtained from certain functions of a complex variable.*

This paper develops two rectilinear congruences by means of the Riemann-spherical representation of Wilczynski. They are obtained from an arbitrary function of a complex variable and its osculating linear fractional function. The focal sheets of the latter congruence are always quadrics. The author finds that each of these two quadrics not only osculates the corresponding focal surface of the original congruence, but is the osculating quadric of this surface. The point of osculation on each surface is the point corresponding to the origin, where the original linear fractional function osculates the arbitrary complex function.

6. Professor Virgil Snyder: *Non-monoidal involutorial transformations which leave each monoid of a web invariant.*

The author constructs a web of monoids of order n having a curve and a number of simple basis points in common, and having the property that any three surfaces of the web intersect in two variable points. This pair of conjugate points defines an involution, the properties of which are determined. In addition to the monoidal involutions with 7 basis points, there are six non-monoidal types for every value of $n \geqq 3$. The entire system can be expressed completely in terms of a single set of formulas. A partial list of the possible types for $n=3$ was already known. The other results are new.

7. Mr. J. A. Nyswander: *A direct method of obtaining the solutions of systems of linear differential equations having constant coefficients.*

The author gives a direct and comparatively simple method for obtaining the solutions of any linear differential system of the form $x_i' = a_{i1}x_1 + \cdots + a_{in}x_n$, $(i=1, \ldots, n)$. When the characteristic equation $D(\lambda) = 0$ has distinct roots, there is no difficulty. When $\lambda = \lambda_k$ is a multiple root of $D(\lambda) = 0$, all solutions of the form

$$x_i = \left(A_i^{(1)} t^{\omega} + \cdots + A_i^{(\omega+1)}\right) e^{\lambda_k t}$$

are obtained directly from $\omega + 1$ sets of linear equations in the $A_i^{(j)}$ which are sufficient conditions for the existence of such a solution. The characteristic determinant $D(\lambda)$ affords a set of n identities in λ which, when differentiated with respect to λ, yield (for $\lambda = \lambda_k$) exactly $\omega + 1$ sets of relations involving minors of $D(\lambda_k)$ and their λ-derivatives, whose coefficients are identical with the coefficients of the $A_i^{(j)}$ in the foregoing sets of linear equations; thus the desired values of the $A_i^{(j)}$ are given directly by these sets of relations. All solutions not of the form $x_i = A_i e^{\lambda_j t}$ are of the above type. A general form for the solutions of this type is set up, from which the n solutions of the system can be written at once in terms of minors of the determinant $D(\lambda_k)$ and of their λ-derivatives.

8. Dr. H. A. Bender: *Prime power groups containing only one invariant subgroup of every index which exceeds this prime number.*

If a group G of order p^m, p being an odd prime, is to contain but one invariant subgroup of every index greater than p, then each of these invariant subgroups must be non-cyclic whenever G is non-cyclic. For $p = 2$ the subgroup of order p^{m-1} must be cyclic. In either case the invariant subgroup of index p^2 must be the commutator subgroup of G. Furthermore, G contains a subgroup of index p which includes all of its operators whose orders exceed p^2, and every operator not in this subgroup transforms each operator of this subgroup into itself multiplied by an operator in the preceding major co-set.

It is shown that a necessary and sufficient condition that this group of order p^m contains but one invariant subgroup of every index greater than p, is that its $(m-2)$th commutator subgroup is of order p. The index of the largest invariant abelian subgroup can not exceed $p^{(p+1)/2}$, and the pth power of every operator of G must be in the central of G_{m-1}.

An interesting system of such groups is the Sylow subgroup of order p^{p+1} contained in the symmetric group of degree p^2.

9. Mr. L. M. Graves: *The derivative as independent function in the calculus of variations.*

The integrals considered are of the form

$$I = \int_{x_1}^{x_2} f\,(x, y, + \int_{x_1}^{x} z\,dx, z)$$

where z is a function $z(x)$ which is bounded and integrable in the sense of Lebesgue on the interval $x_1 \leqq x \leqq x_2$ and satisfies the additional condition $\int_{x_1}^{x_2} z\,dx = y_2 - y_1$. Sets of necessary and of sufficient conditions are obtained for a minimum of the integral I relative to the class of all such functions.

10. Mr. J. Shohat (Jacques Chokhate): *On the development of a continuous function in a series according to Tchebycheff's polynomials.*

Let $f(x)$ be a continuous function on the *finite* interval (a, b), and $\varphi_n(x)$ a normal orthogonal system of Tchebycheff polynomials on (a, b) with the *characteristic function* $p(x) \geqq 0$ in (a, b). The author investigates the convergence of the development $\sum^{\infty} A_i \varphi_i(x)$, where $A_i = \int_b^a p(x) f(x) \varphi_i(x)\,dx$. Writing $f(x) = \sum^{n} A_i \varphi_i(x) + r_n(x)$, and using a method given by D. Jackson in the TRANSACTIONS for 1912, he is able to evaluate $r^n(x)$, if certain assumptions are made concerning the function $\varphi_i(x)$, as e.g. $|\varphi_n(x)| < \tau n^{\sigma} (n = 1, 2, \ldots)$. The method is illustrated by two examples, one of which generalizes results obtained in Jackson's paper.

11. Mr. J. Shohat: *Note on the asymptotic expression of certain definite integrals.*

In his second paper, the author considers the integrals $\int_0^1 f(x) x^n dx$ and $\int_0^1 f(x)[1-(x-\xi)^2]^n\,dx$ under various hypotheses concerning the character of $f(x)$. Formulas are developed which enable him to find asymptotic expressions for a large number of definite integrals, and also an important inequality for $\sum_{i=0}^{n}\varphi_i^2(\xi)$, where $\varphi_i(x)$ are a normal orthogonal system of Tchebycheff polynomials on (0, 1) with the characteristic function $f(x)$ vanishing for $x=\xi$. This note was suggested by an article of P. Funk, *Beiträge zur Theorie der Kugelfunktionen*, MATHEMATISCHE ANNALEN, vol. 77 (1916).

12. Miss Elizabeth Carlson: *On the convergence of certain methods of closest approximation.*

The first part of this paper contains a discussion of conditions on a function $f(x)$ which are sufficient to insure the uniform convergence of $S_n(x)$ to the value of $f(x)$, where $S_n(x)$ is the Sturm-Liouville sum of order n which gives to the integral $\int_0^\pi |f(x)-S_n(x)|^m dx (m\geqq 1)$ its minimum value. In the second part of the paper, the same problem is discussed with the Sturm-Liouville sum replaced by a linear combination of the first n characteristic solutions of a certain third order differential equation with a particular set of boundary conditions. In this case it is necessary first to derive theorems concerning the magnitude of $|S_n'(x)|$. Two theorems are proved: If M is the maximum of $|S_n(x)|$ in $0\leqq x\leqq\pi$, then throughout this interval, $|S_n'(x)|\leqq n^2pM$, where p is independent of n and the coefficients in $S_n(x)$. Throughout the interval $\varepsilon\leqq x\leqq\pi-\varepsilon$, $|S_n'(x)|\leqq nqM$, where q is independent of n and the coefficients in $S_n(x)$ but depends upon ε.

13. Mr. C. M. Jensen: *The approximate representation of a function by a Sturm-Liouville interpolating formula.*

The approximate representation of a function by means of a Sturm-Liouville interpolating formula gives rise to problems similar to those which have been solved in connection with the representation by the trigonometric interpolating formula. The situation is complicated by the fact that, for the Sturm-Liouville functions, we have only an *approximate* analogue of the trigonometric formulas of the form

$$\sum_{k=1}^{n}\cos\frac{2\pi ik}{n}\cos\frac{2\pi jk}{n}=0,\ (i\neq j).$$

The main theorem obtained is that the Sturm-Liouville interpolating formula, of order n, applied to a Sturm-Liouville sum of order n, makes an error of the order $(\log n)/n$, the coefficients being subject to certain conditions. If $f(x)$ satisfies a Lipschitz condition, it can be represented by the Sturm-Liouville interpolating formula of order n with an error of order $(\log n)/n$. If $f(x)$ is merely continuous, its approximate representability by the trigonometric interpolating formula necessitates its representability by the corresponding Sturm-Liouville formula, and, for any continuous function, there exists a Sturm-Liouville interpolating formula which will converge to the right value.

14. Miss Dora E. Kearney: *Some theorems on convex functions.*

A continuous function $f(x)$ is said to be convex in an interval (a,b) if it satisfies the relation

$$f[(1-c)x_1+cx_2] \leqq (1-c)f(x_1)+cf(x_2)$$

for every c in the interval $0 < c < 1$ and for every pair of values x_1, x_2 in (a,b). One usually assumes this relation for $c = 1/2$ as definition and proves it for an arbitrary c. This paper shows that if there is any c in the interval specified, for which the given relation holds, then it holds for every such c. A corresponding theorem is proved for what may be called properly convex functions, for which the equality sign is excluded. Other elementary theorems on convex functions are presented.

15. Mr. E. L. Mickelson: *Orthogonal polynomials for interpolation.*

This paper is concerned with the construction and tabulation of polynomials orthogonal over a finite set of points equally spaced in an interval. The tables will be directly serviceable in the fitting of parabolic curves by the method of least squares or the equivalent method of moments.

16. Professor P. R. Rider: *The correlation between two variates one of which is normally distributed.*

The paper determines the correlation between two variates x and y $(y = kx^n, k > 0)$, where x is distributed according to the so-called normal law of error. A problem of this type would arise if one wished to find the correlation between the diameters and the weights of a set of homogeneous spheres, either the diameters or the weights being normally distributed. A concrete example might be afforded by the apples on a tree. Rietz has given other practical illustrations in discussing the frequency distribution of the second variate (ANNALS OF MATHEMATICS, (2), vol. 23, pp. 292—300). In the present paper it is shown that the product-moment coefficient, r, for the variates x and y is zero for $n \leqq -1/2$ and for $n = 0$, is negative for $-1/2 < n < 0$ and positive for $n > 0$, and approaches the value zero as n approaches ∞; moreover r is equal to unity for $n = 1$ but is less than unity in absolute value for all other values of n.

17. Professor Dunham Jackson: *Elementary applications of the notion of angle in function space.* Preliminary communication.

The present paper, while making no claim to novelty of substance, calls attention to the fact that the essentials of the Pearson-Yule theory of correlation are identical, in the sense of general analysis, with the elementary geometry of function space, and suggests the importance of a coordinated study of two fields which have been developed to a great extent independently of each other.

18. Professor Dunham Jackson: *A symmetric coefficient of correlation for several variables.*

This paper shows how simple geometric considerations in space of n dimensions, or in function space, as the case may be, lead to the definition of a symmetric coefficient for measuring the degree of resemblance of three or more sets of data. As in other instances of correlation, the extreme values of the coefficient are $+1$ and -1. For the case of three variables, the formula is $(4/9)(r_{12} + r_{13} + r_{23}) - 1/3$, where r_{12}, r_{13}, r_{23} are the ordinary coefficients of correlation of the variables taken two at a time.

19. Professor K. P. Williams: *Notation in tensor analysis.*

In this note the author calls attention to a possible simplification of notation in tensor analysis.

20. Professor K. P. Williams: *Concerning a type of integral equation.*

Andreoli has studied to some extent the integral equation in which the upper limit is $g(x)$. In the present paper the author considers the existence of solutions of this equation and also the relation it bears to another type of functional equation.

21. Professor H. T. Davis: *Derivation of the Fredholm theory from a differential equation of infinite order.*

In this paper it is shown how the Fredholm solution of integral equations can be derived directly from a special differential equation of infinite order. The infinite differential equation corresponding to the Fredholm equation is obtained. By taking n successive derivatives of this equation and disregarding terms of order higher than n, a system of equations is obtained which can be solved by elementary means for the unknown function. In the limiting case the Fredholm solution is easily found.

22. Professor G. A. Bliss: *Algebraic functions and their divisors.*

A number of methods have been devised for the development of the theory of algebraic functions without the use of preliminary birational transformations simplifying the singularities of the function. Among the most interesting of these is the one presented by Hensel and Landsberg in their *Theorie der algebraischen Funktionen,* which is an amplification of methods suggested originally, and apparently independently, by Kroneker and by Dedekind and Weber. The present paper is an effort to make more accessible the solutions of two important problems in the Hensel and Landsberg theory: the construction of the three types of elementary abelian integrals and the proof of the Riemann-Roch theorem. It is believed that the proofs have been considerably simplified, and their various logical steps unraveled from a large amount of most interesting but to these problems relatively irrelevant material with which they are interlaced in the Hensel and Landsberg treatise. Incidentally the author of this paper wishes to call attention to a recent reference by him to the Riemann-Roch theorem (TRANSACTIONS OF THIS SOCIETY, vol. 24 (1922), footnote, p. 277), which is now justified by a much more concise presentation.

23. Mr. J. H. Taylor: *A generalization of Levi-Civita's parallelism and the Frenet formulas.*

In the n-dimensional space considered in this paper the arc length is defined by an integral whose integrand is a function $F(x_1, \ldots, x_n; x'_1, \ldots, x'_n)$ which satisfies the conditions of the so-called regular problem of the calculus of variations. In this space vectors can be normed and orthogonalized with respect to a quadratic manifold which osculates the *indicatrix* $F = 1$ for an arbitrary direction x'. Blaschke has associated with the well known covariant derivative of a Riemann space a differentiation process which he calls the θ-process. The author has generalized this process for the more general space considered by him. By means of this generalization he can associate with each point of a curve a system of n contravariant vectors which can also be replaced by an equivalent unitary orthogonal system of vectors. By means of this latter system the Frenet formulas are obtained and the expressions for the curvatures are deduced. If two movable vectors have their initial points on the same curve and satisfy the system of equations $\theta\,\xi = 0$ associated with the curve it is shown that the angle between them remains a constant. This is a generalization of a theorem by Levi-Civita.

24. Professor R. P. Baker: *The connection of sheets in mapping.*

This paper exhibits various types of connection of sheets of points in multiple mapping by algebraic equations not occurring in ordinary Riemann surfaces. If

$$\varphi\,(u, v : x, y) = 0, \quad \psi\,(u, v : x, y) = 0$$

are algebraic equations and the real points of the locus in $R_4(u, v, x, y)$ are all finite, the (generalized) Riemann surfaces over the (u, v) and (x, y) plane are two-sided and of the same connectivity. They may have removable singularities of various kinds as adhesions and nodal lines with no real neighborhood. If the locus is infinite and the closure of projective space be used, the surfaces may be one-sided. The connectivity is preserved.

The paper considers the theorem of Hadamard (JOURNAL DE MATHÉMATIQUES, 1898) concerning the connectedness of the spherical representation of a surface whose curvature is negative everywhere except at isolated points where it is zero.

In mapping a surface by parameters the connectivity may be changed. For instance, in the two-parameter representation of a general cubic, the mapping is on an anchor ring. Five lines being lost, the connectivity reduces from 8 to 3. A Cremona transformation further reduces it and maps the surface on a projective plane.

25. Mr. H. M. Gehman: *Necessary and sufficient conditions that every closed and connected subset of a continuous curve be a continuous curve.*

It is shown that a necessary condition that every closed and connected subset of a continuous curve M be a continuous curve, is that M contain no closed and connected subset N, containing a continuum of condensation W, such that $N-W$ is uniformly connected im kleinen. This condition is also sufficient, if every continu of condensation of M is a continuous curve, not of a certain type derived by considering the necessary and sufficient conditions that the removal of an arc from a continuous curve leave the remainder (on one side of the arc) not uniformly connected im kleinen. Examples are given showing the independence of each condition.

A study is also made of the effect of replacing "$N-W$ is uniformly connected im kleinen" by "$N-W$ can be expressed as the sum of a finite number of connected sets each of diameter less than a given ε" (the Sierpinski S-property). A method is obtained for constructing a continuous curve by adding to any closed and connected set a countable infinity of arcs. It is shown that this is impossible if the set of arcs be finite in number.

26. Professors E. R. Hedrick and Louis Ingold: *Conjugate functions in three dimensions.*

This paper is a continuation of previous papers dealing with three-dimensional analogs of facts connected with the ordinary theory of functions of a complex variable.

Conjugate functions in two dimensions are certain related pairs of solutions of Laplace's equation. In three dimensions it is found that the solutions of a certain pair of partial differential equations of the second order are related in sets of three, and these sets are the natural analogs in three dimensions of sets of conjugate functions in the plane. Certain theorems are also generalized.

27. Professor J. S. Turner: *On the representation of a positive integer by the form* ax^2-by^2.

This paper contains a simple extension of a theorem of Tchebychef (JOURNAL DE MATHÉMATIQUES, vol. 16 (1851), pp. 257–282). It is proved that if a positive integer m can be represented by the form ax^2-by^2, where a, b are positive integers, then there exists a representation with

$$0 \leqq x \leqq \sqrt{\frac{(T+1)m}{2a}}, \quad 0 \leqq y \leqq \sqrt{\frac{(T-1)m}{2b}},$$

where $x = T$, $y = U$, are the least positive integral solutions of

$$x^2-Dy^2=1, \ D=ab;$$

and that from two such representations factors of m can be found.

It follows that if D is a determinant for which each genus contains only one cycle of reduced forms, and if m is an odd number prime to D and having the quadratic character of ax^2-by^2, then m is prime if it has a single primitive representation by ax^2-by^2 within the limits specified above, and composite in all other cases.

28. Professor Oswald Veblen and Dr. T. Y. Thomas: *Extensions of relative tensors.*

This paper is an addendum to the authors' paper on the geometry of paths in the TRANSACTIONS OF THIS SOCIETY for October, 1923. It contains formulas for the rth extensions of relative tensors of arbitrary weight. These include the formulas for tensor densities.

29. Professor Dunham Jackson: *Note on arc and angle in function space.*

The interpretation of the coefficient of correlation of two functions as the cosine of an angle, discussed in a previous paper, is further illustrated in terms of a parametric formula for length of arc in function space. If $\varphi(x)$ and $\psi(x)$ are two normalized functions in an interval (a, b), represented in function space by two points P and Q at unit distance from the origin, the length of the circular arc from P to Q, with center at the origin, is found by direct calculation to be arc cos r, where

$$r = \int \varphi(x)\psi(x)\,dx.$$

30. Mr. B. P. Reinsch: *Expansion problems in connection with the hypergeometric differential equation.*

The object of this paper is the development of an expansion theory for hypergeometric functions analogous to the one worked out by Neumann and Gegenbauer for Bessel functions. Expansions in series of

$$x^{k+m}F(a+k+m, b, 1+k+m; x), \quad (m=0,1,2,\ldots),$$

solutions of the hypergeometric differential equation, are found for $x^{k+m}, x^k/t-x$; and (employing the last result) for an analytic function $f(x)$ by means of Cauchy's integral formula. The analogues of the Cauchy-Taylor and the Laurent expansions appear. The theory is extended to the generalized hypergeometric functions of Goursat and Pochhammer. The results are then extended to the expansion of analytic functions of several variables in multiple series of ordinary or generalized hypergeometric functions.

31. Dr. L. L. Steimley: *On the summability and regions of summability of a general class of series of the form* $c_0+\sum c_n g(nx)$.

The author considers the general class of series of the form

$$Y(x)=c_0+\sum_{n=1}^{\infty} c_n g(nx),$$

where $g(x)$ has the asymptotic character

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\cdots\right)$$

valid for x approaching infinity in some sector V bounded by two rays extending from zero to infinity. $P(x)$ and $Q(x)$ are polynomials in x, a_i are functions of the argument of x, varying in the sector V, c_i are independent of x.

This paper deals with the summability of the series $Y(x)$ by the method of Cesàro. The character of the regions of summability and of uniform summability are determined. The character as to summability of every point on the boundary of the region of summability is determined except for a set of isolated points which may occur under very restricted conditions. Summability numbers are determined. A necessary and sufficient condition for summability of $Y(x)$ is developed.

32. Dr. L. L. Steimley: *On a general class of integrals of the form* $\int_0^\infty \varphi(t)g(tx)\,dt$.

The author considers a general class of integrals of the form $K(x) = \int_0^\infty \varphi(t)g(tx)\,dt$, where $g(x)$ has the asymptotic character described in the previous paper. The function $\varphi(t)$ is finite and single-valued for finite $t \geqq 0$, $|\varphi(t)|$ has an upper bound in every finite interval (0τ) and $\varphi(t)$ is integrable in the sense of Riemann in every such interval. The character of the regions of convergence, of absolute convergence and of uniform convergence are determined. The properties as to analyticity of functions defined by integrals $K(x)$ are treated. The character of the points, as to convergence, on the boundary of the region of convergence is determined, except for an isolated set of points. The character of the isolated points is determined except in very special cases. Convergence numbers in terms of φ and g are determined. A necessary and sufficient condition for the convergence of $K(x)$ is established.

33. Professor Cornelius Gouwens: *Invariants of the linear group, modulo* $\pi = p_1^{\lambda_1} \cdots p_n^{\lambda_n}$.

The transformations congruent, modulo π to a given transformation T form a class $[T]_\pi$. These classes are the elements of a group Γ. This group is formed by composition of the subgroups G_i of classes $[T]_\pi$, $T \equiv 1$ $(\text{mod } m_i = \pi/p_i^{\lambda_i})$, $|T| \equiv 1$ $(\text{mod } p_i^{\lambda_i})$. A function invariant under Γ is invariant under every G_i. The subgroup G_i is simply isomorphic with the group H_i of classes $[S]_p$ of transformations S whose determinants are congruent to unity modulo $P = p_i^{\lambda_i}$. Every invariant of Γ is a sum of invariants each of which is expressible as a product of m_i by an invariant of the group H_i.

ARNOLD DRESDEN,
Assistant Secretary.

INVARIANCE OF THE POINCARÉ NUMBERS OF A DISCRETE GROUP

BY OSWALD VEBLEN

Some time ago Professor J. Nielsen of Copenhagen kindly pointed out to me that the last statement in § 30, page 142, of my *Colloquium Lectures on Analysis Situs* is incorrect; i. e., it is not true that "(12) is a necessary condition that (10) shall be a transformation to a new set of generators of G." For in solving equation (11) account must be taken of the generating relations.

This error invalidates the argument given in the next two sections for the invariance of the Poincaré numbers of a group G. It therefore seems worth while to point out the following rather obvious way of proving this invariance.

We are considering a group G determined by a finite number of generators

$$g_1, g_2, \cdots, g_n$$

and a finite number of generating relations

$$g_1^{a_{i1}} g_2^{a_{i2}} \cdots g_n^{a_{in}} g_1^{b_{i1}} g_2^{b_{i2}} \cdots g_n^{b_{in}} \cdots g_1^{j_{i1}} g_2^{j_{i2}} \cdots g_n^{j_{in}} = 1 \qquad (1)$$
$$(i = 1, 2, \ldots, k).$$

This group determines uniquely a commutative group $\tilde{G}$ defined by the condition that it has n generators subject to (1) and the condition that all operations of the group be commutative. In view of the commutativity, the generating relations of $\tilde{G}$ reduce to

$$g^{\gamma_{i1}} g^{\gamma_{i2}} \cdots g^{\gamma_{in}} = 1 \qquad (2)$$
$$(i = 1, 2, \ldots, k)$$

where

$$\gamma_{it} = a_{it} + b_{it} + \cdots + j_{it}.$$

For the group $\tilde{G}$, a new set of generators $h_1, h_2, \ldots, h_n$ can be found, as explained in § 31, for which the generating relations are

$$(3) \qquad h_1^{d_1} = 1, \quad h_2^{d_2} = 1, \quad \ldots, \quad h_r^{d_r} = 1$$

where those of the exponents $d_1, d_2, \ldots, d_r$ which are not 1 are the invariant factors of the matrix $|\gamma_{it}|$.

An operation of $\tilde{G}$ is of finite period if and only if it is in the group H generated by the operations $h_1, h_2, \ldots, h_r$ and the relations (3). Hence H contains all operations of $\tilde{G}$ of finite period. Moreover H is a finite group because it is commutative and generated by a finite number of operations each of finite period. The invariant factors of $|\gamma_{it}|$ are invariants of H (Kronecker, BERLINER MONATSBERICHTE, 1870, p. 885).

These invariant factors are what Tietze calls the Poincaré numbers of the group G. They are invariants of G because G determines the commutative group $\tilde{G}$ uniquely and $\tilde{G}$ determines the finite group H uniquely and H determines the invariant factors uniquely.

PRINCETON UNIVERSITY

ANALYTIC FUNCTIONS AND PERIODICITY*

BY J. F. RITT

This paper presents two theorems which show that the condition that a function be periodic can be analyzed, in different ways, into a set of requirements, from the satisfaction of only one of which, if the function is analytic, the periodicity of the function can be inferred. The theorems are

THEOREM A. *If $f(z)$ is a uniform analytic function, and if an $a > 0$ exists such that every z_1 at which $f(z)$ is analytic is the center of a circle of radius a on which a z_2 lies at which $f(z)$ is analytic and assumes the same value as at z_1, then $f(z)$ is periodic, and has a period of modulus a.*

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* Presented to the Society May 3, 1924.

THEOREM B. *If $f(z)$ is a uniform analytic function, and if every z_1 at which $f(z)$ is analytic belongs to a sequence of non-coincident points in arithmetic progression, at each of which $f(z)$ is analytic and assumes the same value as at z_1, then $f(z)$ is periodic.*

In A, the condition that $z_1 - z_2$ have a constant modulus can be replaced by the condition that it have a constant argument, or, for that matter, a constant real or pure imaginary part. Also, in both theorems, the condition that $f(z)$ be uniform is not essential, and is made only to keep cumbersome statements out of the proof.

Theorems similar to the above can be proved for other peculiarities than periodicity. Many types of functional equations can be studied from the point of view of this paper.

We shall use the following lemma.

LEMMA. *Given a non-constant uniform analytic function $f(z)$, and a transformation which carries every z_1 at which $f(z)$ is analytic into a z_2 at which $f(z)$ is analytic and assumes the same value as at z_1, there exists an analytic function $\varphi(z)$ such that $z_2 = \varphi(z_1)$ for a set of points z_1 dense in some area. For every z in this area, $f[\varphi(z)] = f(z)$.*

If $f(z)$ is analytic at z and if $f'(z) \neq 0$, there are circles with z as center inside of which $f(z)$ is analytic and assumes no value more than once. For any z, one of these circles —call it c—has a maximum (perhaps infinite) radius. Those maximum circles c for the center of which, $z = x + iy$, both x and y are rational, are countable. Let them be arranged in a sequence

$$c_1, c_2, \ldots, c_n, \ldots.$$

It is easy to see that every z for which $f(z)$ is analytic and $f'(z) \neq 0$ is interior to some c_n.

Take now any circle γ inside of which $f(z)$ is analytic and assumes no value more than once. For every z_1 in γ, the hypothesis of the lemma provides a z_2. Those points z_2 at which $f'(z_2) = 0$ are countable, for the zeros of an analytic function which is not identically zero are isolated. Every other z_2 lies within some c_n; associate with its z_1 the c_n of smallest n in which z_2 lies. In this way those points z_1 of γ for which $f'(z_2) \neq 0$ are separated into a countable number of sets. Not all of these sets can be nowhere dense, for their sum differs from all of γ by a countable set.

Hence there is a c_n and a set of points—call it M—dense in some area within γ, such that for any z_1 of M, z_2 lies within c_n.

Take, and hold fast momentarily, any z_1 of M which is the center of some circle—call it Γ—in which M is dense. As $f(z_1) = f(z_2)$, we can take Γ so small that $f(z)$ assumes no value in Γ which it does not assume within c_n. To associate with every point of Γ that point of c_n at which $f(z)$ assumes the same value is to define a $\varphi(z)$ which is analytic in Γ. Furthermore, for a set of points z_1 dense in Γ, $z_2 = \varphi(z_1)$. This proves the lemma.

It is now easy to prove Theorem A. Because $|z_2 - z_1|$ is constant, the modulus of $\varphi(z)$ is constant for a set of points dense in Γ. As the modulus of $\varphi(z)$ is continuous, it is constant throughout Γ. An analytic function with a constant modulus is a constant. Let h be the constant value of $\varphi(z)$; of course $h \neq 0$. Then $f(z+h) = f(z)$ for every z in Γ, and hence throughout the domain of existence of $f(z)$.

Considering Theorem B, suppose that, for every z_1 at which $f(z)$ is analytic, a number $h(z_1) \neq 0$ exists such that $f(z)$ is analytic, and equal to $f(z_1)$, at every $z_1 + mh(z_1)$ $(m = 1, 2, \ldots)$.

According to the lemma, there exists a circle Γ, a set M dense in Γ, and an analytic $\varphi(z)$, such that $\varphi(z_1) = z_1 + h(z_1)$ for every z_1 in M.

Suppose that $\varphi(z) - z$ is not constant. Barring out the trivial case in which $f(z)$ is constant, we may assume that neither $f(z)$ nor $\varphi(z) - z$ assumes any value more than once in Γ; this can always be brought about by replacing Γ by some circle within Γ. Let Γ' and Γ'' be two circles concentric with Γ, and of radii respectively one-half and one-quarter that of Γ.

Consider the function

$$\frac{z}{m}+[\varphi(z)-z],$$

where m is any positive integer. As m increases this function approaches $\varphi(z)-z$ uniformly in Γ, so that, for m large enough, it assumes no value more than once in Γ', but assumes in Γ' every value which $\varphi(z)-z$, assumes in Γ''. The values assumed in Γ'' certainly cover some circle Δ.

Consequently, for m large enough, the function

$$\psi_m(z)=z+m[\varphi(z)-z]$$

maps Γ' conformally, in a one-to-one manner, upon a region Σ_m which covers a circle Δ_m, of radius m times that of Δ.

Let z be in Γ'. Associate with $\psi_m(z)$ the number $f(z)$. This defines a $g_m(z)$, analytic in Σ_m. Now $\psi_m(z)$ maps those points of M which lie in Γ' into a set of points dense in Σ_m. Since $\psi_m(z_1)=z_1+mh(z_1)$ for z_1 in M, $f(z)$ and $g_m(z)$ are equal for a set of points dense in Σ_m, and therefore throughout Σ_m.

As $f(z)$ assumes no value more than once in Γ', $f(z)$ assumes no value more than once in Σ_m. Consider any z_1 of M such that $\psi_m(z_1)$ is very close to the center of Δ_m. Since $h(z)=\varphi(z)-z$ for z in M, $h(z)$ is bounded for z at once in M and in Γ'. Hence if m is large, so that Δ_m is large, $\psi_m(z_1)+h(z_1)$, at which $f(z)$ takes the same value as at $\psi_m(z_1)$. will lie in Δ_m, and so in Σ_m. This contradiction with the statement at the head of the paragraph proves that $\varphi(z)-z$ is constant; it is not zero, for no $h(z)$ is zero. As $f[\varphi(z)]=f(z)$ for z in Γ, $f(z)$ is periodic.

COLUMBIA UNIVERSITY

A CONVERGENCE PROOF FOR SIMPLE AND MULTIPLE FOURIER SERIES*

BY M G. CARMAN

The purpose of this paper is to establish the convergence of both the simple and the multiple Fourier series with a second order linear homogeneous differential equation as a starting point. The method of proof for the simple series is essentially that of Birkhoff† applied to a special case. In the extension of the theory to the multiple series, the argument is similar to that used by Camp‡ in connection with a first order equation. Apart from any relation with more general theory, the proofs given here are of interest because of the elegance with which they lead to important results.

THEOREM I. *Let $f(x)$ be made up of a finite number of pieces in the interval $-\pi \leqq x \leqq \pi$, each real, continuous, and with a continuous derivative. For $-\pi < x < \pi$ the Fourier series for $f(x)$ converges to $\frac{1}{2}[f(x-0)+f(x+0)]$. For $x = \pm\pi$ it converges to $\frac{1}{2}[f(-\pi+0)+f(\pi-0)]$.*

We start with the differential equation

$$y'' + \varrho^2 y = 0, \tag{1}$$

and the boundary conditions

$$y(-\pi) = y(\pi), \qquad y'(-\pi) = y'(\pi), \tag{2}$$

where x is the independent variable and ϱ^2 is a parameter.

It is easily seen that a necessary and sufficient condition that the system (1), (2) have a solution is that ϱ be a positive or negative integer, or zero. These integral values of ϱ

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* The means of approach to Fourier series which is employed in this paper was suggested to a class of graduate students by Professor R. D. Carmichael; the problem was solved completely or in part by several members of the group.

† TRANSACTIONS OF THIS SOCIETY, vol. 9 (1908), p. 390.

‡ TRANSACTIONS OF THIS SOCIETY, vol. 25 (1923), pp. 123–34.

are called the *characteristic values*. We now seek a function which, for all values of ϱ other than the characteristic values, satisfies conditions (2) and comes as near as possible to satisfying (1) throughout the interval $-\pi \leqq x \leqq \pi$. Under one interpretation of "as near as possible" this function, which we shall call $G(x, \xi; \varrho^2)$, is the function which satisfies the boundary conditions and also satisfies equation (1) at every point of the interval, $-\pi \leqq x \leqq \pi$, except at one point ξ, at which point $G(x, \xi; \varrho^2)$ itself is continuous but its derivative with respect to x has a jump of 1. We may compute $G(x, \xi; \varrho^2)$ as follows: let

$$G(x, \xi; \varrho^2) = \begin{cases} c_1 \cos \varrho x + c_2 \sin \varrho x, & \text{when } -\pi \leqq x < \xi, \\ c_3 \cos \varrho x + c_4 \sin \varrho x, & \text{when } \xi < x \leqq \pi, \end{cases}$$

where the c's are independent of x. There are four relations among the c's, which are sufficient to determine them uniquely; two imposed by the boundary conditions, one by the condition of continuity of $G(x, \xi; \varrho^2)$ at $x = \xi$, and the fourth by the condition of discontinuity of $G'_x(x, \xi; \varrho^2)$ at $x = \xi$. This last relation, written explicitly, is

$$G'_x(\xi + 0, \xi; \varrho^2) - G'_x(\xi - 0, \xi; \varrho^2) = 1,$$

or

$$-c_3 \varrho \sin \varrho\xi + c_4 \varrho \cos \varrho\xi - (-c_1 \varrho \sin \varrho\xi + c_2 \varrho \cos \varrho\xi) = 1.$$

In this way we find

$$(3) \quad G(x, \xi; \varrho^2) = \frac{1}{2\varrho}\left[\operatorname{sgn}(x - \xi) \sin(\varrho x - \varrho\xi) + \frac{\cos \varrho\pi \cos(\varrho x - \varrho\xi)}{\sin \varrho\pi}\right].$$

Considered as a function of ϱ, $G(x, \xi; \varrho^2)$ is analytic except at the characteristic values of ϱ. For ϱ a positive or negative integer, $G(x, \xi; \varrho^2)$ has a pole of the first order, while at $\varrho = 0$ it has a pole of the second order. In order to reduce this second order pole to one of the first order, multiply both members of (3) by ϱ and then consider the function $\varrho G(x, \xi; \varrho^2)$, which is analytic except for simple poles at the characteristic values of ϱ. Denoting by $R_n(x, \xi)$ the residue of $\varrho G(x, \xi; \varrho^2)$ at $\varrho = n$, one readily finds that

$$R_n(x,\xi)=R_{-n}(x,\xi)=\frac{\cos(nx-n\xi)}{2\pi},$$

whence

$$(4)\qquad \frac{1}{2\pi i}\int_{\Gamma_n}\varrho\, G(x,\xi;\varrho^2)\,d\varrho = \frac{1}{2\pi}(\cos nx\cos n\xi+\sin nx\sin n\xi),$$

where Γ_n is a circle in the ϱ-plane with center at $\varrho=n$ and radius $\frac{1}{2}$. The Fourier series for $f(x)$ is

$$(5)\qquad \tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

where

$$(6)\qquad \begin{aligned} a_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos n\xi f(\xi)\,d\xi,\\ b_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin n\xi f(\xi)\,d\xi, \qquad (n=0,1,2,\ldots). \end{aligned}$$

The general term of this series is

$$a_n\cos nx+b_n\sin nx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\cos nx\cos n\xi+\sin nx\sin n\xi)f(\xi)\,d\xi,$$

which, by virtue of (4), may be written in the form

$$a_n\cos nx+b_n\sin nx=\frac{1}{\pi i}\int_{\Gamma_n}\int_{-\pi}^{\pi}\varrho\, G(x,\xi;\varrho^2)\,f(\xi)\,d\xi\,d\varrho.$$

This same term will clearly be obtained if the contour integration is along Γ_{-n} instead of Γ_n. Hence, if C_k is a circle in the ϱ-plane with center at the origin and radius $k+\frac{1}{2}$, and if S_k denotes the sum of the first $k+1$ terms of (5), we have

$$S_k=\frac{1}{2\pi i}\int_{C_k}\int_{-\pi}^{\pi}\varrho\, G(x,\xi;\varrho^2)f(\xi)\,d\xi\,d\varrho.$$

The limit of this double integral is desired as k becomes infinite. For this purpose the integral is broken up into two parts, $I_k^{-\pi,x}$ and $I_k^{x,\pi}$, the limits of ξ being $-\pi$ and x in $I_k^{-\pi,x}$ and x and π in $I_k^{x,\pi}$, where $-\pi<x<\pi$.

Considering $I_k^{-\pi,x}$ and integrating by parts with respect to ξ, we have

$$I_k^{-\pi,x} = \frac{1}{4\pi i}\int_{C_k}\int_{-\pi}^{x}\frac{\cos\varrho(\pi - x + \xi)}{\sin\varrho\pi}f(\xi)\,d\xi\,d\varrho$$

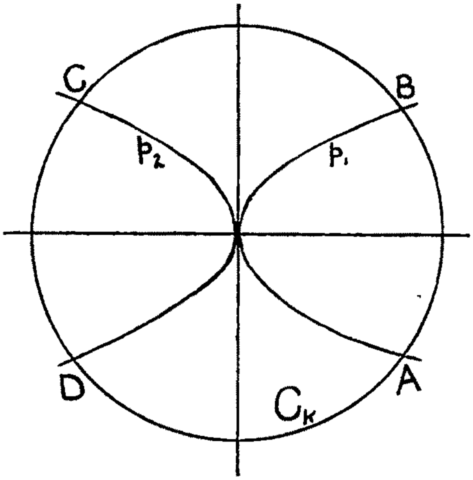
$$= \frac{1}{4\pi i}\int_{C_k}\left[\frac{1}{\varrho}f(x-0) + \frac{\sin\varrho x}{\varrho\sin\varrho\pi}f(-\pi+0)\right.$$

$$\left. - \frac{1}{\varrho}\int_{-\pi}^{x}\frac{\sin\varrho(\pi - x + \xi)}{\sin\varrho\pi}f'(\xi)\,d\xi\right]d\varrho.$$

As k becomes infinite each of the three parts of $I_k^{-\pi,x}$ contributes a certain amount to the sum of the series which we are seeking. The contribution of

$$\frac{1}{4\pi i}\int_{C_k}\frac{1}{\varrho}f(x-0)\,d\varrho$$

is easily seen to be $\frac{1}{2}f(x-0)$. We now show that the contribution of each of the other two parts is zero. In the ϱ-plane let p_1 and p_2 be two parabolas with vertices at the origin and axes coinciding with the axis of reals, p_1 lying in a right half-plane, meeting C_k in the points A and B, and p_2 lying in a left half-plane, meeting C_k in the points C and D. Then along the arcs AB and CD $|\sin\varrho x/\sin\varrho\pi|$ is bounded while the product of $|1/\varrho|$ and the length of the path of integration goes to zero as k becomes infinite. On the other hand, along



the arcs BC and DA $\lim_{k=\infty} | \sin \varrho x / \sin \varrho \pi | = 0$, since the imaginary part of ϱ becomes infinite with k, while the product of $| 1/\varrho |$ and the length of the path of integration is bounded. Therefore the contribution of the second part of $I_k^{-\pi,x}$ is zero. To see that the third part of $I_k^{-\pi,x}$ also contributes zero, take the absolute value of the integrand, replace $| f'(\xi) |$ by an upper bound M, integrate first with respect to ξ and then, breaking up C_k with the parabolas as before, integrate along C_k.

In precisely the same way one sees that as k becomes infinite the total contribution of $I_k^{x,\pi}$ is $\frac{1}{2} f(x+0)$. The theorem is now proved for interior points of the interval. It may be extended to include the end points $+\pi$ and $-\pi$ in the following way. Let $\Phi(x)$ be a function of period 2π, identical with $f(x)$ in the interval $-\pi \leqq x < \pi$. By the part of the theorem already proved $\Phi(x)$ is represented by its Fourier series in the interval $0 \leqq x \leqq 2\pi$ except possibly at $x = \pi$. Let $g(x) = \Phi(x+\pi)$. It may easily be shown that the Fourier constants of $g(x)$ as calculated directly from formulae (6) are identical with those obtained by replacing x by $x+\pi$ in the Fourier series for $\Phi(x)$. But since $g(x)$ satisfies all the conditions of the theorem, its Fourier series converges at $x = 0$ to $\frac{1}{2}[g(+0)+g(-0)]$. Hence at $x = \pi$, the Fourier series for $f(x)$ converges to

$$\begin{aligned} \tfrac{1}{2}[g(+0)+g(-0)] &= \tfrac{1}{2}[\Phi(\pi+0)+\Phi(\pi-0)] \\ &= \tfrac{1}{2}[f(-\pi+0)+f(\pi-0)]. \end{aligned}$$

Similarly at $x = -\pi$, the Fourier series for $f(x)$ converges to the same value.

THEOREM II. *Let* $f(x_1, x_2, \ldots, x_k)$ *be made up in the region* $-\pi \leqq x_j \leqq \pi$ $(j = 1, 2, \ldots, k)$ *of a finite number of pieces, each real, continuous, and with continuous partial derivatives with respect to* $x_1, x_2, \ldots, x_k$. *Then the Fourier series for* $f(x_1, x_2, \ldots, x_k)$ *converges to the value* $\frac{1}{2^k}\sum f(x_1 \pm 0, x_2 \pm 0, \ldots, x_k \pm 0)$, *where the summation sign means the sum of the* 2^k *terms obtained by taking all possible combinations of the positive and negative signs. If* $x_j = \pm\pi$, *instead of* x_j+0 *and* x_j-0 *read* $-\pi+0$ *and* $\pi-0$ *respectively.*

For the sake of simplicity the proof is given here for $k=2$. The method is clearly applicable to the general case.

We now use the system

$$\left.\begin{aligned}&y_j''+\varrho_j^2 y=0,\\ &y_j(-\pi)=y_j(\pi),\qquad y_j'(-\pi)=y_j'(\pi),\end{aligned}\right\}(j=1,2).$$

Calculating the residue of $\varrho_1 G(x_1,\xi_1;\varrho_1^2)$ at $\varrho_1=m$, and that of $\varrho_2 G(x_2,\xi_2;\varrho_2^2)$ at $\varrho_2=n$ and taking the product of these residues, one finds

$$\left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_n'}\int_{\Gamma_m}\varrho_1 G(x_1,\xi_1;\varrho_1^2)\varrho_2 G(x_2,\xi_2;\varrho_2^2)\,d\varrho_1 d\varrho_2$$

$$\begin{aligned}(7)\qquad =\frac{1}{4\pi^2}[&\cos mx_1\cos nx_2\cos m\xi_1\cos n\xi_2\\ &+\cos mx_1\sin nx_2\cos m\xi_1\sin n\xi_2+\sin mx_1\cos nx_2\sin m\xi_1\cos n\xi_2\\ &+\sin mx_1\sin nx_2\sin m\xi_1\sin n\xi_2],\end{aligned}$$

where Γ_m is a circle in the ϱ_1-plane about the pole $\varrho_1=m$ and Γ_n' is a circle in the ϱ_2-plane about the pole $\varrho_2=n$.

The Fourier series for $f(x_1,x_2)$ is

$$\begin{aligned}(8)\qquad \sum_{m,n=0}^{\infty}\varepsilon_{mn}[&A_{mn}\cos mx_1\cos nx_2+B_{mn}\cos mx_1\sin nx_2\\ &+C_{mn}\sin mx_1\cos nx_2+D_{mn}\sin mx_1\sin nx_2],\end{aligned}$$

where

$$\varepsilon_{mn}=\begin{cases}\frac{1}{4}, \text{ if } m=n=0,\\ \frac{1}{2}, \text{ if } m=0,\ n>0, \text{ or if } m>0,\ n=0,\\ 1, \text{ if } m>0,\ n>0,\end{cases}$$

and where

$$(9)\qquad\begin{aligned}A_{mn}&=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cos m\xi_1\cos n\xi_2 f(\xi_1,\xi_2)\,d\xi_1\,d\xi_2,\\ B_{mn}&=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cos m\xi_1\sin n\xi_2 f(\xi_1,\xi_2)\,d\xi_1\,d\xi_2,\\ C_{mn}&=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sin m\xi_1\cos n\xi_2 f(\xi_1,\xi_2)\,d\xi_1\,d\xi_2,\\ D_{mn}&=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sin m\xi_1\sin n\xi_2 f(\xi_1,\xi_2)\,d\xi_1\,d\xi_2.\end{aligned}$$

Combining formulas (7) and (9), we have

$$\begin{aligned}A_{mn}\cos mx_1\cos nx_2 + B_{mn}\cos mx_1\sin nx_2 \\ + C_{mn}\sin mx_1\cos nx_2 + D_{mn}\sin mx_1\sin nx_2\end{aligned}$$

$$(10)\qquad = \left(\frac{1}{\pi i}\right)^2\int_{\Gamma'_n}\int_{\Gamma_m}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\varrho_1 G(x_1,\xi_1;\varrho_1^2)\,\varrho_2 G(x_2,\xi_2;\varrho_2^2)\, f(\xi_1,\xi_2)\,d\xi_1 d\xi_2 d\varrho_1 d\varrho_2 .$$

The first member of this equation, which is the general term of (8), will also be obtained if Γ_m is replaced by Γ_{-m} or Γ'_n by Γ'_{-n}, or if both these changes are made simultaneously. Hence, if Γ_m and Γ'_n are replaced by C_μ and C'_ν respectively, where C_μ is a circle in the ϱ_1-plane with center at the origin and radius $\mu+\frac{1}{2}$, and C'_ν is a circle in the ϱ_2-plane with center at the origin and radius $\nu+\frac{1}{2}$, the second member of (10) gives four times the sum of the terms of (8) for which $m\leqq\mu$ and $n\leqq\nu$. Let $S_{\mu\nu}$ be this sum. Then we have

$$S_{\mu\nu} = \left(\frac{1}{2\pi i}\right)^2\int_{C'_\nu}\int_{C_\mu}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\varrho_1 G(x_1,\xi_1;\varrho_1^2)\,\varrho_2 G(x_2,\xi_2;\varrho_2^2)\, f(\xi_1,\xi_2)\,d\xi_1 d\xi_2 d\varrho_1 d\varrho_2 .$$

To find the limit of this four-fold integral as μ and ν become infinite, theorem I may be used, for by interchanging the order of the integrations with respect to ξ_2 and ϱ_1 the integral becomes simply a succession of two double integrals, each of the type evaluated in the proof of Theorem I. Integrating first with respect to ξ_1 and ϱ_1, and then with respect to ξ_2 and ϱ_2, we have

$$\begin{aligned}\lim_{\mu,\nu=\infty} S_{\mu\nu} = \tfrac{1}{4}[f(x_1-0,x_2-0)+f(x_1+0,x_2-0) \\ +f(x_1-0,x_2+0)+f(x_1+0,x_2+0)].\end{aligned}$$

THE UNIVERSITY OF ILLINOIS

ON CERTAIN TOPICS IN THE MATHEMATICAL THEORY OF STATISTICS*

BY H. L. RIETZ

1. *Introduction.* The mathematical theory of statistics dates back to the first publication† relating to Bernoulli's theorem in 1713. The line of development started by Bernoulli was carried forward by DeMoivre,‡ Stirling,§ Maclaurin,|| and Euler¶ culminating in the formulation of the Bernoulli theorem by Laplace** in substantially the form in which it still holds a fundamental place in mathematical statistics.

The *Théorie Analytique des Probabilités* of Laplace is undoubtedly the most significant publication at the basis of the development of mathematical statistics. Strangely enough, for a period of more than fifty years following the publication of the work of Laplace in 1812, little of importance was contributed to the subject. To be sure, the second law of error of Laplace was developed by Gauss and given its important place in the adjustment of observations, but there was on the whole relatively little progress. Perhaps a complex of causes was involved, but three fairly plausible reasons may be assigned for the lack of contributions to mathematical statistics at this period. First, Laplace left many of his results in the form of approximations

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* A Report presented by request of the Program Committee at the symposium held in Chicago, April 25, 1924.

† James Bernoulli, *Ars Conjectandi*, 1713, pp. 210-39 (published eight years after his death).

‡ A. DeMoivre, *Doctrine of Chances* (3rd ed. 1756) pp. 243-54. *Miscellanea Analytica*, 1730, pp. 191-97, Supplement.

§ J. Stirling, *Methodus Differentialis*, 1730, p. 135.

|| C. Maclaurin, *A Treatise on Fluxions*, 1742, p. 672.

¶ L. Euler, COMM. ACAD. PETROP. 6, 1732-33, ed. 1738, pp. 88-97.

** P. S. Laplace, *Théorie Analytique des Probabilités*, 3ième ed., 1820, vol. II, Chap III, pp. 280-85.

that would not form the basis for further development. Moreover, many of his theorems were not demonstrated with even a fair degree of rigor, and it required the work of Cauchy and others to supply proofs of the theorems. This was important work, but it did not, in general, lead to new results of importance in mathematical statistics. Second, the followers of Gauss promulgated the idea that the deviations from his law of error are due to lack of data, and this attitude was not conducive to the creation of generalized frequency functions. Third, Quetelet* was busy as a popularizer of mathematical statistics. His somewhat sensational language about the stability of social statistics, say of the number of suicides from year to year, caught the imagination; but unfortunately he often asserted the existence of stability on insufficient evidence. The activity of Quetelet cast upon statistics a suspicion of quackery, which still exists to some extent. Moreover, it will probably always be found that those statisticians who use mathematical formulas without guarding well their limitations are likely to have influence productive of an evil very similar to that produced by Quetelet.

An important step in advance was taken in 1877, only three years after the death of Quetelet, in the publication of the theory of Lexis† for the classification of statistical distributions with respect to normal, supernormal, and subnormal dispersion. This theory is based on urn schemata of different constitutions, and pays much attention to the degree of constancy of statistical ratios obtained from different parts of the field of observation. Charlier‡ states that this theory of Lexis is the first essential step forward in mathematical statistics since the days of Laplace. J. M.

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* A. Quetelet, *Lettres sur la Théorie des Probabilités Appliquée aux sciences, Morales et Politiques*, 1846.

† W. Lexis, *Zur Theorie der Massenerscheinungen in der menschlichen Gesellschaft*, 1877, p. 95.

‡ C. V. L. Charlier, *Vorlesungen über die Grundzüge der mathematischen Statistik*, 1920, p. 5.

Keynes[*] expresses a somewhat similar view. These may, however, be extreme views, when we take into account the fact that the inequality of Tchebychef was published earlier than the theory of Lexis.

The development of generalized frequency curves and a theory of correlation in the decade 1890 to 1900 started the period of activity in mathematical statistics in which we find ourselves at present.

After a first survey of various topics in the mathematical theory of statistics for consideration in this symposium, it seemed desirable to adopt some principle of selection or elimination of topics. But I have discovered no satisfactory principle of selection except a sort of principle of personal interest which leads me to restrict my remarks to certain points of interest under the following general headings:

I. *Generalized frequency curves.*
II. *Correlation.*
III. *Frequency surfaces.*
IV. *Theory of random sampling.*

No claim is made that the topics selected are more appropriate for consideration in this symposium than others which could easily be named, particularly if we should draw upon recent activities in the special fields of economic,[†] mortality,[‡] and stellar[§] statistics. My interest in the general theory of statistics would lead me to include the theory of Lexis with the recent generalization by J. L. Coolidge,[||] except for the fact that this theory has become readily accessible to members of the Society.

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[*] *A Treatise on Probability*, 1920, p. 393.

[†] Irving Fisher, *The Making of Index Number*, 1922.

[‡] James W. Glover, *U. S. Life Tables*, 1921.
Arne Fisher, *Frequency Curves*, 1922.

[§] K. G. Malmquist, *On some relations in stellar statistics*, ARCHIV FOR MATEMATIK, ASTRONOMI OCH FYSIK, vol. 16, No. 23, 1923.

[||] This BULLETIN, vol. 27 (1920–21), p. 439.

I. GENERALIZED FREQUENCY CURVES

2. *Introduction.* In the decade from 1890 to 1900, it became well established experimentally that the Gaussian probability function is inadequate to represent all frequency distributions which arise in biological data. When the problem of developing generalized frequency curves was finally attacked, the attack was made from several different directions. Thiele[*] and Charlier[†] in Scandinavian countries, Pearson[‡] and Edgeworth[§] in England, Fechner[||] and Bruns[¶] in Germany developed theories of generalized frequency curves from viewpoints which give very different degrees of prominence to the Gaussian probability curve in the development of a more general theory. Among other things, I hope to give a brief exposition of the most striking differences in these viewpoints while considering certain properties of the two systems of frequency curves to which I shall direct special attention—the Pearson system and the Charlier system.

3. *The Pearson System of Generalized Frequency Curves.* Pearson's first memoir[**] dealing with generalized frequency curves appeared in 1895. In this paper he gave four types of frequency curves in addition to the normal curve, with three sub-types under his Type I and two sub-types under his Type III. He published a supplementary memoir[††] in

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[*] T. H. Thiele, *Almindelig Iagttagelseslaere*, Copenhagen, 1889.

[†] C. V. L. Charlier, *Ueber das Fehlergesetz*, ARCHIV FÖR MATEMATIK, ASTRONOMI OCH FYSIK, vol. 2, No. 8, 1905, pp. 1-9.

——— *Die zweite Form des Fehlergesetzes*, ARCHIV, vol. 2, No. 15, 1905, pp. 1-8.

[‡] Karl Pearson, *Mathematical contributions to the theory of evolution*, PHILOSOPHICAL TRANSACTIONS, A, vol. 186 (1895), pp. 343-414.

[§] F. W. Edgeworth, *The asymmetrical probability-curve.* PHILOSOPHICAL MAGAZINE, vol. 41 (1896), pp. 90-99.

[||] G. T. Fechner, *Kollektivmasslehre* (edited by G. R. Lipps), 1897.

[¶] H. Bruns, *Ueber die Darstellung von Fehlergesetzen.* ASTRONOMISCHE NACHRICHTEN, vol. 143 (1897).

[**] Loc. cit., pp. 343-414.

[††] PHILOSOPHICAL TRANSACTIONS, A, vol. 197 (1901), pp. 443-56.

1901 which presented two further types. A second supplementary memoir* which was published in 1916 gave five additional types. Pearson's curves, which are widely different in general appearance, are so well known and so accessible that we shall take no time to comment on them as graduation curves for a great variety of frequency distributions, but we shall attempt to indicate the genesis of the curves with special reference to the methods by which they are grounded on or associated with underlying probabilities.

We shall consider a frequency function $y = F(x)$ of one variable, where $F(x)dx$ differs at most by an infinitesimal of higher order from the probability that x taken at random falls into the interval x to $x+dx$. Pearson's types of curves $y = F(x)$ are obtained by integration of the differential equation

$$(1) \qquad \frac{dy}{dx} = \frac{y(x+a)}{c_0 + c_1 x + c_2 x^2},$$

and by giving attention to the interval on x in which $y = F(x)$ is positive. Obviously, the Gaussian curve is given by the special case $c_1 = c_2 = 0$. We may easily obtain a clear view of the genesis of the system of Pearson curves in relation to laws of probability by following the early steps in the development of equation (1). The development is started by representing the probabilities of successes in n trials given by the terms of the symmetric point binomial $(1/2+1/2)^n$ as ordinates of a frequency polygon at intervals Δx. It is then proved that the slope $\Delta y/\Delta x$ of any side of this polygon is

$$\frac{\Delta y}{\Delta x} = -k^2 y(x+a),$$

where x and y, respectively, are the mean abscissa and the mean ordinate of the side of the polygon. By passing to the limiting situation, we may write

$$\frac{dy}{dx} = -k^2 y(x+a),$$

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* PHILOSOPHICAL TRANSACTIONS, A, vol. 216 (1916), pp. 431–57.

from which we obtain the Gaussian curve. The next step consists in dealing with the asymmetric point binomial $(p+q)^n$ in a manner analogous to that used in the case of the symmetric point binomial. This procedure gives the differential equation

$$\frac{dy}{dx}=\frac{y(x+a)}{c_0+c_1x},$$

from which we obtain the Pearson Type III curve

$$y=y_0\left(1+\frac{x}{a}\right)^{\gamma a}e^{-\gamma x}.$$

That is, with respect to the slope property, this curve stands in the same relation to the a priori most probable values given by the asymmetric binomial polygon as the normal curve does to a priori most probable values given by the symmetric binomial. Thus far the underlying probability of success has been assumed constant. The next step consists in taking up a probability problem in which the chance of success is not constant, but depends upon what has happened previously in a set of trials. Thus, the chances of getting $r, r-1, r-2, \ldots, 0$ black balls from a bag containing pn black and qn white balls in drawing r balls one at a time without replacements are given by the successive terms

$$\sum_{s=0}^{s=n}\frac{\binom{r}{s}(qn)_s\,(pn)_{r-s}}{(n)_r}$$

of a hypergeometric series. When the terms of this series are represented as ordinates of a frequency polygon, and the slope of a side is found in a manner analogous to that used in the case of the point binomial, we obtain the differential equation (1) from which the Pearson curves are obtained by integration.

The idea of obtaining a suitable basis for frequency curves in the probabilities given by terms of a hypergeometric series is the main principle which supports the Pearson curves as statistical probability or frequency curves rather than as mere graduation curves. That is to say, it is assumed in this system, that the distribution of statistical material may be likened to the law of probability represented by terms of a hypergeometric series. In examining the source of the Pearson curves, the fact should not be overlooked that the Gaussian probability curve can be derived from hypotheses containing much broader implications than are involved in a slope condition of the side of a symmetric binomial polygon. Can the generalized curves likewise be derived from hypotheses involving broader implications than are contained in the slope condition based on the probabilities given by a hypergeometric series? The answer seems to be unknown.

The method of moments plays an essential role in the Pearson system of frequency curves not only in the determination of the parameters but also in providing criteria for selecting the appropriate type of curve. Pearson has attempted to provide a set of curves such that some one of the set would agree with any observational or theoretical frequency curve of positive ordinates to the extent of having equal areas and equal first, second, third, and fourth moments of area about a centroidal axis.

Let μ_s be the sth moment coefficient about a centroid vertical taken as the y-axis. That is, let

$$\mu_s = \int_{-\infty}^{\infty} F(x) x^s dx.$$

Next, let $\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$. Then it is Pearson's thesis that the conditions $\mu_0 = 1$, $\mu_1 = 0$ together with the equality of the numbers μ_2, β_1, and β_2 for the observed and theoretical curves lead to equations whose solutions give such values to the parameters of the frequency function that we almost invariably obtain excellency of fit by using the appropriate one of the curves of his system to fit the data, and that badness of fit can be traced, in general, to heterogeneity of data, or to the difficulty in the determination of moments from the data as in the case of J and U shaped curves.

Let us next examine the nature of the criteria by which to pass judgment on the type of curve to use in any numerical case. Obviously, the form which the integral $y = F(x)$ obtained from (1) takes, depends on the nature of the zeros of the quadratic function in the denominator. An examination of the discriminant of this quadratic function leads to equalities and inequalities involving β_1 and β_2 which serve as criteria in the selection of the type of function to be used. A systematic procedure for applying criteria has been thoroughly developed.* The relations between β_1 and β_2 are represented by curves in the $\beta_1\beta_2$-plane. To illustrate, the normal curve corresponds to the point $\beta_1 = 0$, $\beta_2 = 3$ in this plane. Type III is to be chosen when the point (β_1, β_2) is on the line $2\beta_2 - 3\beta_1 - 6 = 0$; and Type V, when (β_1, β_2) is on the cubic

$$\beta_1(\beta_2 + 3)^2 = 4(4\beta_2 - 3\beta_1)(2\beta_2 - 3\beta_1 - 6).$$

In considering sub-types under Type I, the biquadratic

$$\beta_1(8\beta_2 - 9\beta_1 - 12)(\beta_2 + 3)^2 = (10\beta_2 - 12\beta_1 - 18)^2(4\beta_2 - 3\beta_1)$$

separates the area of J curves or modeless curves from the area of limited range modal curves and the area of U curves. Without going into further detail about criteria for the selection of the type of curve, we may summarize by saying that curves traced on the $\beta_1\beta_2$-plane provide the means of selecting the Pearson type of frequency curve appropriate to the given distribution in so far as the necessary conditions expressed by relations between β_1 and β_2 turn out to be sufficient to determine a suitable type of curve.

4. *Generalized Normal Curve.—Charlier System.* As indicated in § 3, the Pearson system of frequency curves assigns only a *point* in the $\beta_1\beta_2$-plane for the region of applications of the Gaussian law in fitting frequency distributions. It seems not unnatural, however, to have some

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* A. Rhind, BIOMETRIKA, vol. 7 (1909–10), pp. 127–35; cf. *Tables for Statisticians and Biometricians*, 1914, pp. ix–ixx, and pp. 66–67; see also, Karl Pearson, loc. cit., PHILOSOPHICAL TRANSACTIONS, A, vol. 216 (1916), pp. 429–57.

doubt about the Gaussian curve taking such a small place in the representation of frequency, and to turn with considerable interest to the Charlier system of representation of frequency which gives great prominence to the Gaussian probability function

$$y=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-b)^2}{2\sigma^2}}.$$

Among the early contributors to the theory of the generalized normal curve we find Gram,[*] Thiele,[†] Edgeworth,[‡] Bruns[§] and Charlier.[||] Of the various contributions to the subject, those of Charlier are particularly elegant and noteworthy. Charlier has shown by extensions of the Laplace theory based on the hypothesis of elementary errors that the law of error assumes one of the following two forms:

TYPE A.

$$F(x)=a_0\Phi(x)+a_3\Phi^{(3)}(x)+a_4\Phi^{(4)}(x)+\cdots+a_n\Phi^{(n)}(x)+\cdots,$$

where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},$$

and $\Phi^{(n)}(x)$ is the nth derivative of $\Phi(x)$ with respect to x.

TYPE B.

$$F(x)=c_0\Psi(x)+c_1\Delta\Psi(x)+c_2\Delta^2\Psi(x)+\cdots+c_n\Delta^n\Psi(x)+\cdots,$$

where

$$\Psi(x)=e^{-\lambda}\frac{\sin\pi x}{\pi}\left\{\frac{1}{x}-\frac{\lambda}{(x-1)1!}+\frac{\lambda^2}{(x-2)2!}-\frac{\lambda^3}{(x-3)3!}+\cdots\right\}$$
$$=\frac{e^{-\lambda}\lambda^x}{x!},$$

the Poisson exponential for non-negative integral values of x.

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[*] J. P. Gram, *On Raekkeudvidlinger, bestemte ved Hjaelp af de minste Kradvaters Methode* (Doctor's dissertation), Copenhagen, 1879.

[†] T. N. Thiele, *Almindlig Iagttagelseslaere*, 1889; cf. Thiele, *Theory of Observations*, 1903.

[‡] F. Y. Edgeworth, loc. cit, also *The law of error*, CAMBRIDGE PHILOSOPHICAL TRANSACTIONS, vol. 20 (1904), pp. 36–65, 113-41.

[§] H. Bruns, loc. cit, also *Wahrscheinlichkeitsrechnung und Kellektivmasslehre*, 1906.

[||] C. V. L. Charlier, loc. cit; also *Ueber Darstellung willkürlicher Functionen*, ARCHIV FOR MATEMATIK, ASTRONOMI OCH FYSIK, vol. 2, No. 20, 1905, pp. 1–35.

The Type A series represents an arbitrary function $F(x)$ subject to certain conditions of continuity[*] and vanishing at infinity. The coefficients a_n in Type A may be expressed in terms of moments of area under the given frequency curve because the functions $\Phi^{(n)}(x)$ and the Hermite polynomials $H_n(x)$ defined by the equation

$$\Phi^{(n)}(x) = (-1)^n H_n(x) \Phi(x)$$

form a biorthogonal system. Thus

$$a_n = \frac{(-1)^n \int_{-\infty}^{\infty} F(x) H_n(x)\, dx}{n!}.$$

Since $H_n(x)$ is a polynomial of degree n in x, the coefficients a_n are thus given in terms of moments of area under the frequency curve. Moreover, the value of any coefficient thus obtained is the same as that obtained by finding the best approximation to $F(x)$ in the sense of a certain least squares criterion by the first s terms of the series $(s > n)$. In the Type B series, the coefficients may be determined by the method of moments or by means of the semi-invariants[†] of Thiele.

To be of practical value in fitting given numerical distributions, it is essential that only a few terms of the series in Type A be required to fit the distribution. The closeness with which the first few terms will represent a given function $F(x)$ depends much on the extent to which the generating function $\Phi(x)$ is a fair approximation to the distribution. In case the generating function $\Phi(x)$ is not even a rough approximation to the distribution, it may be possible to introduce in place of $\Phi(x)$ a function of approximation $\Theta(x)$ as a generating function in the series. N. R. Jørgensen[‡] has used the function

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[*] Wera Myller-Lebedeff, MATHEMATISCHE ANNALEN, vol. 64 (1907), p 338; and H. Weyl, MATHEMATISCHE ANNALEN, vol. 66 (1908), p. 306.

[†] Arne Fisher, *The Mathematical Theory of Probabilities*, 1922. p. 271.

[‡] *Undersögelser over Frequensflader og Korrelation*, 1916, pp. 177–93.

$$\Theta(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\log x - m)^2}{2\sigma^2}}$$

as a generating function and has by this means succeeded in representing closely some remarkably skew distributions. H. C. Carver* and Emeterio Roa have done some work on this plan of representation by introducing various types of generating functions.

With respect to the selection of Type A or Type B of Charlier to represent given data, no criterion has been given which enables one to distinguish sharply between cases in which to apply one of these types in preference to the other, but Type B applies in general, to decidedly skew distributions, and particularly to those defined only at integral values of the variable when the Poisson exponential is the generating function. While the systematic procedure in fitting Charlier curves to data is thus not so well standardized as the methods used in fitting curves of the Pearson system to data, tables of $\Phi(t)$, where t is in units of standard deviation, of its integral $\int_0^t \Phi(t)dt$, and of its second to eighth derivatives are given to five decimal places for $t = 0$ to $t = 5$ at intervals of .01 by James W. Glover,† and tables of the function, its integral and first six derivatives are given by N. R. Jørgensen‡ to seven decimal places for $t = 0$ to $t = 4$.

The question naturally arises as to the arguments which support the Charlier system of representation in comparison with the Pearson system. The Charlier system is surely the better grounded in the theory of probability, and is adapted to the representation of an arbitrary function subject to reasonable conditions of continuity and vanishing at infinity. A disadvantage of the system is found in the fact that its application to numerical distributions is likely to be laborious, and the probable errors of the coefficients of the series are large if we find it necessary to use more than three or four significant terms. But even if the representation by the Charlier system proves to be so laborious that it is rarely used with the more common numerical distributions, the series is nevertheless of great value in the representation of laws of probability. To illustrate, take the problem of the distribution of the sum of n numbers selected at random from a uniform distribution. Laplace gave an approximate solution of the problem adapted to computation for the case when n is large, and applied the result of his theory to the question of the random distribution of the orbits of comets. The method of Laplace in obtaining the approximation is of doubtful validity. Cauchy* put Laplace's approximation on a much more rigorous basis in a memoir published in 1841. In a paper which the writer presented to the Society last April, it is shown that the representation is given by the Type A function of Charlier, and that each additional term improves the approximation in the sense of a certain least squares criterion. Again, B. H. Camp in a recent paper† on certain important problems in sampling has made much use of the Type A representation in his theory. The point in citing these illustrations is that the Charlier representation is likely to be found very useful in the general theory of probability, apart from the fitting of frequency curves to numerical data.

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* Cited in *Handbook of Mathematical Statistics,* by H. L. Rietz and others, 1924, p. 116.

† *Tables of Applied Mathematics,* 1923, pp. 392–411.

‡ Loc cit., p. 178–93

5. *Transformation of Frequency Functions.* Before leaving the subject of frequency functions of one variable, let us consider briefly the idea of regarding certain frequency functions as the result of transformation of the independent variable in simple frequency functions. F. Y. Edgeworth‡

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* A. L. Cauchy, JOURNAL DE L'ECOLE POLYTÉCHNIQUE, Cahier 28, vol. 21 (1841), pp. 147-248.

† JOURNAL OF THE AMERICAN STATISTICAL ASSOCIATION, vol. 18 (1923), pp. 964-77.

‡ JOURNAL OF THE ROYAL STATISTICAL SOCIETY, vol. 61 (1898), pp. 670-700.

and J. C. Kapteyn[*] proposed that skew frequency curves should be regarded as transformations of the Gaussian curve. It seems that this idea would accord with what happens in certain natural phenomena, although it is difficult to predict its generality. That is to say, if certain values are distributed normally, we inquire into the distribution of certain simple functions of these values. For example, the writer gave in a recent paper[†] certain properties of frequency curves obtained when the values of a normally distributed variable, x, are transformed by

$$x = kx'^n.$$

It is shown that the form and properties of the resulting distribution differ widely according as $n < 0$, $0 < n < 1$, and $n > 1$. As another example, N. R. Jørgensen has made $x = \log x'$ in the Gaussian frequency function thus obtaining

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\log x' - m)^2}{2\sigma^2}}.$$

He found the semi-invariants of Thiele for this function and showed that the function may replace the Gaussian function as a generating function in the Charlier series of Type A for the representation of certain skew distributions. Thus, it seems that functions arising from the transformation of variables may be found to be useful.

II. Correlation

6. *Simple Correlation.* It seems that Francis Galton[‡] was the first to deal with correlation among direct observations. If we should follow the historical development of correlation, we should simply give Galton's definition of correlation and his ideas of the regression of one variable on another before proceeding to correlation surfaces in three dimensions. For example, the mathematical solution of the special correlation problem proposed by Francis

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[*] *Skew Frequency Curves in Biology and Statistics,* Groningen, 1903.

[†] H. L. Rietz, Annals of Mathematics, vol. 23 (1922), pp. 292–300.

[‡] Proceedings of the Royal Society, vol. 40 (1886), p. 42.

Galton to J. D. Hamilton Dickson in 1886* consisted simply in giving the equation of a normal frequency surface to correspond to given standard deviations and regression lines. Furthermore, the early contributions of Karl Pearson to correlation theory involving the influence of selection were concerned with frequency surfaces.† But, beginning with a paper by G. Udny Yule in 1897, the theory of correlation has been developed without limitation to a particular type of frequency surface. It is of some interest that Yule returned very close to the primary ideas of Galton, by placing the emphasis on the lines of regression. This method of Yule may be appropriately called the regression method of approaching correlation in contrast to the frequency surface method. It is our purpose to present enough of the elements of the regression method to give a basis for a brief exposition of the nature of certain recent contributions to correlation from the regression standpoint. Special attention will be directed (1) to the general method of determining the successive terms of a non-linear regression equation, (2) to the connection of the correlation coefficient and regression curves with some simple problems of a priori probabilities, (3) to the development of a theory of correlation of n variables in the case of non-linear regression by means of multiple and partial correlation ratios.

To introduce a convenient notation let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_N, Y_N)$ be pairs of real numbers such that at least two of the X's are unequal and at least two of the given Y's are unequal. Let $\overline{X}, \overline{Y}$ be the arithmetic means of the given values of X's and Y's respectively.

Then let

$$x_i = \frac{X_i - \overline{X}}{\sigma_x}, \qquad y_i = \frac{Y_i - \overline{Y}}{\sigma_y},$$

where σ_x and σ_y are respectively standard deviations (root-mean-squares) of X's and Y's, so that each deviation is expressed by its ratio to the standard deviation of the system to which it belongs. Then the correlation coefficient r is defined by

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* PROCEEDINGS OF THE ROYAL SOCIETY, vol. 40 (1886), p. 63.

† PHILOSOPHICAL TRANSACTIONS, vol. 187 (1896), pp. 253–318.

$$r = \frac{1}{N}\sum_{1}^{N} x_i y_i. \tag{1}$$

By writing (1) in the form

$$r = 1 - \frac{1}{2N}\sum_{1}^{N}(x_i - y_i)^2, \tag{2}$$

and

$$r = -1 + \frac{1}{2N}\sum_{1}^{N}(x_i + y_i)^2, \tag{3}$$

it follows at once that $-1 \leqq r \leqq 1$, and that the X_i's and Y_i's are linearly related when $r = 1$ or -1. In papers by Huntington[*] and Jackson,[†] the significance of r is brought out in interesting ways by interpretations of $\sum_{1}^{N}(x_i - y_i)^2/(2N)$ in (2).

In the sense of a certain least squares criterion, we may obtain the values of y corresponding to assigned values of x more accurately from $y = rx$ than from any other linear equation. The line $y = rx$ is called the line of regression of y on x. The mean square of the errors involved in using values of $y = rx$ for the given y's is $s_y^2 = 1 - r^2$. Thus,

$$r^2 = 1 - s_y^2. \tag{4}$$

Let us now conceive of dividing the whole interval along the x-axis which includes our data into suitable equal class intervals Δx. Then the y's which correspond to the x's in the interval Δx are called an x-array of y's. When the means of the arrays of y's are on the line of regression $y = rx$, the regression of y on x is said to be linear. When the means of the arrays of y's are far from the line of regression, the value of the correlation coefficient is likely to be misleading. For example, we[*] may have $r = 0$ for variables x and y when y is a simple periodic function of x. To characterize correlation in such situations, Karl Pearson devised[†] a measure of correlation called the *correlation ratio*, which we shall now describe briefly. By analogy with (4), we define

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[*] E. V. Huntington, AMERICAN MATHEMATICAL MONTHLY, vol. 26 (1919), p. 425.

[†] Dunham Jackson, AMERICAN MATHEMATICAL MONTHLY, vol. 31 (1924), p. 117.

$$\eta_{yx}^2 = 1 - s_y'^2 \tag{5}$$

as the correlation ratio of y on x, where $s_y'^2$ is the mean square of deviations of y's from the *means* of arrays, and these means of arrays need not lie on or near a straight line. The s_y' in (5) agrees with the s_y in (4) only when the regression is linear and $\Delta x \to 0$. It is obvious that when s_y' is small in comparison to unity there is a tendency for the points of the scatter diagram to concentrate in a narrow band along the regression curve, and we have a high degree of correlation. It is an easy step from (5) to deduce

$$\eta_{yx} = \sigma_{\bar{y}_x}, \tag{6}$$

where $\sigma_{\bar{y}_x}$ is the standard deviation of means of x-arrays of Y's when the square of each deviation of a mean of an array is weighted with the number in the array. It is easily shown that

$$1 \geqq \eta_{yx}^2 \geqq r^2,$$

and that the equality $\eta_{yx}^2 = r^2$ holds only in the case of linear regression. It may be noted from (6) that the correlation ratio of y on x is the ratio $\sigma_{\bar{y}_x}/\sigma_y$, if the unit for measuring values of y is not σ_y.

As early as 1905, the parameters of the special regression curves given by polynomials $y = f(x)$ of the second and third degrees were determined in terms of power moments and product moments. In 1921, Karl Pearson* published a general method of determining successive terms of the regression curve of the form

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[*] See H. L. Rietz, QUARTERLY PUBLICATION, AMERICAN STATISTICAL ASSOCIATION, vol. 16 (1919), pp. 472–76.

[†] *On the general theory of skew correlation and non-linear regression*, DRAPERS' COMPANY RESEARCH MEMOIRS, BIOMETRIC SERIES II (1905), pp. 1–54.

$$y = f(x) = a_0\psi_0 + a_1\psi_1 + \cdots + a_n\psi_n, \tag{7}$$

where $a_0, a_1, \ldots, a_n$ are constants to be determined and ψ_s is an orthogonal function of x. That is,

$$\sum(N_x\psi_s\psi_{s'}) = 0, \qquad (s \neq s'),$$

if the summation $\sum$ be taken for all values of x corresponding to an arbitrary system of arrays with frequency in an x-array given by N_x.

7. *Simple Correlation and Probability.* Thus far in this lecture correlation has been discussed by means of averages, ratios of averages, and by the correspondence between an assigned value of one variable and an average value of another. Probability theory has not entered in explicit form. Before leaving simple correlation, I wish to say that it has seemed important to me to construct urn schemata which would give a meaning to the correlation coefficient in pure chance. In a paper† published in 1920, certain urn schemata were devised which give linear regression and very simple values for the correlation coefficient. Other schemata apparently equally simple give non-linear regression. The general plan of the schemata consisted in requiring certain elements to be common in successive random drawings. By means of partial correlation coefficients, J. R. Musselman‡ recently gave simple and interesting proofs of values of the correlation coefficients for those of my urn schemata in which the regression is linear. His method does not, however, replace my method because he assumes the existence of linear regression, which is proved in my

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* BIOMETRIKA, vol. 13 (1921), p. 296

† H. L. Rietz, ANNALS OF MATHEMATICS, vol. 21 (1920), pp. 307–22.

‡ JOURNAL OF THE AMERICAN STATISTICAL ASSOCIATION, vol. 18 (1923), pp. 908–11.

paper. It is hoped that further contributions to correlation involving urn schemata will give the theory of correlation a more prominent place in the pure theory of probability.

8. *Multiple Correlation.* Given N sets of values* of n real variables $x_1, x_2, \ldots, x_n$, where any variable, x_j, is referred to the arithmetic mean of its N given values as an origin, and is measured in units of the standard deviation, σ_j, of its N given values. Let r_{pq} be the correlation coefficient of the N given values of x_p and x_q. Then we seek to determine the parameters in the linear regression equation

$$x = b_{12}x_2 + b_{13}x_3 + \cdots + b_{1n}x_n + c \tag{8}$$

so that x computed from (8) will give the "best" estimates of the values of x_1 to correspond to assigned values of $x_2, x_3, \ldots, x_n$.

Adopting a least squares criterion,† we determine the coefficients in (8) so that

$$(x_1 - b_{12}x_2 - b_{13}x_3 - \cdots - b_{1n}x_n - c)^2$$

shall be a minimum. This gives for the regression equation of x_1 on $x_2, x_3, \ldots, x_n$

$$x = -\sum_{p=2}^{p=n} \frac{R_{p1}}{R_{11}} x_p, \ldots, \tag{9}$$

where R_{pq} is the cofactor of the pth row and qth column of the determinant

$$R = \begin{vmatrix} 1 & r_{12} & r_{13} & \cdots r_{1n} \\ r_{21} & 1 & r_{23} & \cdots r_{2n} \\ r_{31} & r_{32} & 1 & \cdots r_{3n} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots\cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots\cdot\cdot \\ r_{n1} & r_{n2} & \cdot\cdot & \cdots 1 \end{vmatrix}. \tag{10}$$

The correlation coefficient $r_{1\cdot 23\ldots n}$ between the observed values of x_1 and its corresponding estimated values x

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* We assume that at least two of the N given values of a variable x_i are unequal.

† G. Udny Yule, JOURNAL OF THE ROYAL STATISTICAL SOCIETY, vol. 60 (1897), p. 812.

calculated from the linear function (9) of $x_2, x_3, \ldots, x_n$ is called the multiple correlation coefficient of x_1 with the other $n-1$ variables. The multiple correlation coefficient $r_{1\cdot 23\ldots n}$ is expressible in terms of simple correlation coefficients by the formula

$$(11) \qquad r_{1\cdot 23\ldots n} = \sqrt{1-\frac{R}{R_{11}}}.$$

If the scatter $\sigma_{1\cdot 23\ldots n}$ of the observed values of x_1 from the regression hyperplane (9) is defined as the square root of mean square, that is,

$$\sigma^2_{1\cdot 23\ldots n} = \frac{\sum(x_1-x)^2}{N},$$

it can be proved that

$$(12) \qquad \sigma_{1\cdot 23\ldots n} = \sqrt{\frac{R}{R_{11}}},$$

and from (11) and (12) that

$$(13) \qquad \sigma^2_{1\cdot 23\ldots n} = 1 - r^2_{1\cdot 23\ldots n}.$$

9. *Partial Correlation.* It is often important to obtain the degree of correlation between two variables x_1 and x_2 when the other variables $x_3, x_4, \ldots, x_n$ have assigned values. To illustrate, we may have found a correlation between characteristics A and B. A plausible interpretation may be that the correlation thus found is due to the correlation of each of them with C. In this case we could remove the influence of C if we have an unlimited amount of data by restricting our data to a universe of A and B corresponding to an assigned C. But usually the data are not readily available for such a procedure. Instead of thus restricting data, we would often make use of a partial correlation coefficient. We define $r_{12\cdot 34\ldots n}$ as the partial correlation coefficient between x_1 and x_2 when we have eliminated the influences of the variables $x_3, x_4, \ldots, x_n$ in so far as they can be eliminated by means of a linear function of these variables. That is, the partial correlation $r_{12\cdot 34\ldots n}$ is the correlation between residuals

$$x_{1\cdot 34\cdots n}=x_1-b_{13}\cdot x_3-\cdots-b_{1n}\cdot x_n,$$

and

$$x_{2\cdot 34\cdots n}=x_2-b_{23}\cdot x_3-\cdots-b_{2n}\cdot x_n,$$

in estimating x_1 and x_2 by means of linear functions of $x_3, x_4, \ldots, x_n$ and this partial correlation is said to be of order $n-2$, the number of variables held constant. It would ordinarily involve a large amount of labor to apply this definition directly to data. Fortunately the partial correlation coefficient is expressible directly in terms of simple correlation coefficients by the formula

$$(14) \qquad r_{12\cdot 34\cdots n}=\frac{-R_{12}}{\sqrt{R_{11}R_{22}}}.$$

An important relation between partial and multiple correlation coefficients may now be derived. From (11) and (14),

$$1-r^2_{1\cdot 23\cdots n}=\frac{R}{R_{11}}, \qquad r_{12\cdot 34\cdots n}=\frac{R_{12}}{\sqrt{R_{11}R_{22}}}.$$

Hence we have

$$1-r^2_{12\cdot 34\cdots n}=\frac{R_{11}R_{22}-R^2_{12}}{R_{11}R_{22}}.$$

By a well known theorem of determinants,[*]

$$\begin{vmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{vmatrix}=R_{11}R_{22}-R^2_{12}=RR_{11\,22}.$$

Hence we have

$$(15) \quad 1-r^2_{12\cdot 34\cdots n}=\frac{R\cdot R_{11\,22}}{R_{11}R_{22}}=\frac{\dfrac{R}{R_{11}}}{\dfrac{R_{22}}{R_{11\,22}}}=\frac{1-r^2_{1\cdot 23\cdots n}}{1-r^2_{1\cdot 34\cdots n}}.$$

Thus we can express the partial correlation coefficient $r_{12\cdot 34\cdots n}$ of order $n-2$ in terms of the multiple correlation coefficient $r_{1\cdot 23\cdots n}$ of order $n-1$ and the multiple correlation coefficient $r_{1\cdot 34\cdots n}$ of order $n-2$.

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[*] Maxime Bôcher, *Introduction to Higher Algebra*, 1912, p. 33.

10. *Non-linear Regression in n Variables.—Multiple Correlation Ratio.* L. Isserlis* and Karl Pearson† have developed a theory of non-linear regression in the case of more than two variables. Consider the variables $x_1, x_2, x_3, \ldots, x_n$, and we have an array of observed values of x_1 whose mean value $\bar{x}_{1\cdot 23\cdots n}$ we may appropriately call the partial mean value of x's for constant $x_2, x_3, \ldots, x_n$. Then we may define the multiple correlation ratio $\eta_{1\ 23\cdot\ n}$ of x_1 on x_2, $x_3, \ldots, x_n$ by writing

$$(16)\qquad \eta^2_{1\cdot 23\cdots n} = \frac{\sum\limits_{x_2}\sum\limits_{x_3}\cdots\sum\limits_{x_n}\left(N_{23\cdots n}\bar{x}^2_{1\cdot 23\ \cdot n}\right)}{N}.$$

$$(17)\qquad = \sigma^2_{\bar{x}_{1\cdot 23\cdot\ n}}$$

where $\sigma_{\bar{x}_{1\ 23\cdot\ n}}$ is the standard deviation of the means of arrays computed by weighting the squares of the deviations of these means from the mean $\bar{x}_1 = 0$ of all x's with the number $N_{23\cdots n}$ in the array. It may be observed from (17) that $\eta_{1\ 23\cdots n}$ is the ratio $\sigma_{\bar{x}_{1\ 23\cdots n}}/\sigma_1$, if the unit for measuring values of x_1 is not σ_1. We now define $\sigma'_{1\cdot 23\cdot\ n}$ by the equation

$$(18)\qquad \sigma'^2_{1\cdot 23\ \cdot n} = \frac{\sum\limits_{x_2}\sum\limits_{x_3}\cdots\sum\limits_{x_n}\left\{N_{23\cdots n}(x_1 - \bar{x}_{1\ 23\ \ n})^2\right\}}{N},$$

the mean‡ square of standard deviations of arrays of x_1 for assigned values of other $n-1$ variables $x_2, x_3, \ldots, x_n$. From (16) and (18), we find

$$(19)\qquad \sigma'^2_{1\cdot 23\cdots n} = 1 - \eta^2_{1\cdot 23\ \cdot n}.$$

It may be recalled that our definition of η_{12} for two variables is such that $1-\eta^2_{12}$ is the mean square of standard deviations of the arrays of x_1 which correspond to assigned values of x_2. Next consider n variables, and let x_1 be limited to those $N_{34 \cdots n}$ values for which $x_3, x_4, \ldots, x_n$ are assigned. Then consider the expression

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* Biometrika, vol. 10 (1914-15), pp. 393-411.

† Proceedings of the Royal Society, A, vol. 91 (1915), pp. 492-98.

‡ The square of each standard deviation is weighted with the number in the array.

$$\sigma'^2_{1 \cdot 34 \cdots n}(1-\eta^2_{12 \cdot 34 \cdots n}), \tag{20}$$

where $\sigma'^2_{1\,34 \cdots n}$ is the mean square of standard deviations of arrays of x_1 for assigned values of $x_3, x_4, \ldots, x_n$. In this restricted universe, we deal with two variables, x_1 and x_2, and (20) is the mean square deviation for assigned values of $x_2, x_3, x_4, \ldots, x_n$, where $\eta_{12 \cdot 34 \cdots n}$ is the partial correlation ratio of x_1 on x_2 for constant $x_3, x_4, \ldots, x_n$. That is

$$\sigma'^2_{1 \cdot 34 \cdots n}(1-\eta^2_{12 \cdot 34 \cdots n}) = 1-\eta^2_{1 \cdot 23 \cdots n}. \tag{21}$$

Analogous to (19), we may write

$$\sigma'^2_{1\,34 \cdots n} = 1-\eta^2_{1\,34 \cdots n}. \tag{22}$$

Hence from (21) and (22), we have

$$1-\eta^2_{12 \cdot 34 \cdots n} = \frac{1-\eta^2_{1\,23 \cdots n}}{1-\eta^2_{1\,34 \cdots n}}. \tag{23}$$

From (23), we note that the partial correlation ratio of order $n-2$ can be expressed in terms of multiple correlations of order $n-1$ and $n-2$ in a form exactly analogous to that for expressing partial correlation coefficients in terms of multiple correlation coefficients. While the method of computing $\eta_{1 \cdot 23 \cdots n}$ is simple in principle, it is unfortunately laborious from the arithmetic standpoint. It is important as a next step in the investigation to discover a way of expressing multiple correlation ratios in terms of simple correlation ratios just as we know how to express multiple correlation coefficients in terms of simple correlation coefficients. L. Isserlis has taken a step in this direction by showing that, for a certain type of quadric regression surface, the direct calculation of the multiple correlation ratio may be replaced by the calculation of four simple correlation ratios. But the general problem of expressing the multiple correlation ratio in terms of simple correlation ratios is still unsolved.

III. FREQUENCY FUNCTIONS OF n VARIABLES—CORRELATION SURFACES

11. *Normal Correlation Surfaces.* The function

$$z = f(x_1, x_2, \ldots, x_n)$$

is called a frequency* function of the n variables $x_1, x_2, \ldots, x_n$ if

$$z\, dx_1\, dx_2 \cdots dx_n$$

gives to within infinitesimals of higher order the probability that a set values of $x_1, x_2, \ldots, x_n$ taken at random will fall respectively into the intervals x_1 to $x_1 + dx_1$, x_2 to $x_2 + dx_2, \ldots, x_n$ to $x_n + dx_n$. With the notation of § 7 on multiple correlation, the natural extension of the Gaussian frequency function of one variable to the case of n normally correlated variables $x_1, x_2, \ldots, x_n$ gives a frequency function of the exponential type

$$(1) \qquad z = z_0 e^{-\frac{1}{2}\Phi},$$

where Φ is a homogeneous quadratic function of the n variables and may be written in the form

$$(2) \qquad \Phi = \frac{1}{R}\left(R_{11}x_1^2 + R_{22}x_2^2 + \cdots + 2R_{12}x_1x_2 + \cdots\right),$$

the determinant R with correlation coefficients as elements and its cofactors R_{pp} and R_{pq} being defined in § 7.

Karl Pearson† published the general equation of this frequency surface in $n+1$ dimensions and dealt with certain of its important properties in 1896. F. Y. Edgeworth‡ had partially developed the theory of this surface as early as 1892. In three and four dimensions the form of the surface dates back to Bravais§ in 1846, but not as a surface of distribution of directly observed statistical measurements.

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* Charlier calls $f(x_1, x_2, \ldots, x_n)$ a correlation function. See ARCHIV FOR MATEMATIK, ASTRONOMI OCH FYSIK, vol. 8, No. 4 (1912).

† PHILOSOPHICAL TRANSACTIONS, A, vol. 187 (1896), pp. 253–318.

‡ PHILOSOPHICAL MAGAZINE, (5), vol. 34 (1892), pp. 190–204.

§ *Sur les probabilités des erreurs de situation d'un point*, MÉMOIRES PAR DIVERS SAVANTS, vol. 9 (1846), pp. 255–332.

Bravais considered the distribution of linear functions of independent errors in observed quantities rather than directly observed correlated variables. J. L. Coolidge* recently derived the Gaussian law of error for n variables by stating explicitly a set of underlying assumptions. He obtained a surface of the form (1), and determined the parameters by expressing moments of actual errors in terms of moments of residuals.

In our notation for simple correlation, § 6, the surface (1) in three dimensions takes the well known form

$$z = \frac{1}{2\pi\sqrt{1-r^2}}\, e^{-\frac{1}{2(1-r^2)}(x^2+y^2-2rxy)}. \tag{3}$$

The equal-frequency curves obtained by making z take constant values are an infinite system of homothetic ellipses, any one of which has an equation of the form

$$x^2+y^2-2rxy = \lambda^2. \tag{4}$$

When these ellipses are represented on the xy-plane (scatter-diagram plane), the probability that an (x, y) taken at random will lie within the ellipse (4) is given by

$$1-e^{-\frac{\lambda^2}{2(1-r^2)}}. \tag{5}$$

The particular ellipse of the system such that the probability that an (x, y) taken at random will fall within it is one-half, is called the *probable ellipse* and has been frequently discussed. In a paper published in 1912, the writer† defined the *ellipse of maximum probability* as that ellipse of the system along which, for a given small ring $d\lambda$, we expect a greater frequency than along any other ellipse of the system. This ellipse is given by making $\lambda^2 = 1-r^2$ in (4). It is a fact of some interest that this ellipse is the locus of parabolic points of the correlation surface.

One of the most interesting problems I have studied in connection with this surface relates to the determination of the locus along which the frequency or density of points on the plane of distribution (scatter diagram) bears a simple relation to the corresponding density under independence. Thus, we seek the curve along which dots of the scatter diagram are k times as frequent as they would be under independence where k is a constant. Equating z in (3) to k times the corresponding value of z when $r = 0$ in (3), we obtain the hyperbola

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* TRANSACTIONS OF THIS SOCIETY, vol. 24 (1922), pp. 135–43.

† H. L. Rietz, ANNALS OF MATHEMATICS, vol. 13 (1912), pp. 187–99.

$$(6) \qquad x^2 + y^2 - \frac{2}{r}xy = \frac{(1-r^2)}{-r^2}\log(k^2 - k^2r^2):$$

Karl Pearson had dealt* with the particular case of this curve for $k = 1$. I was impressed by the fact that the density of distribution at the centroid in (3) is $1/\sqrt{1-r^2}$ times as much as it would be under independence and was naturally inclined to inquire about the locus of all points for which $k = 1/\sqrt{1-r^2}$ in (6). It turns out that in this case the hyperbola degenerates into straight lines

$$(7) \qquad y = \frac{1}{r}x(1 \pm \sqrt{1-r^2}).$$

These lines separate the plane of distribution into four compartments such that 1/4 is the probability that a pair of values (x, y) taken at random will give a point falling into any prescribed one of these compartments. Although no further discussion of the properties of normal correlation surfaces will be attempted in this paper, certain properties analogous to those mentioned for the surface in three dimensions would probably follow rather readily in the case of the surfaces in higher dimensions. The system of ellipsoids of equal frequencies has been studied to some extent.† In a recent paper by James McMahon,‡ the connection between the geometry of the hypersphere and the theory of normal frequency functions of n variables is established by linearly transforming the hyperellipsoids of equal frequency into a family of hyperspherical surfaces, and by applying the formulas of hyperspherical goniometry to obtain theorems in multiple and partial correlations.

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* DRAPERS' COMPANY RESEARCH MEMOIRS, BIOMETRIC SERIES I, vol. 13, p. 10.

† See E. Czuber, *Theorie der Beobachtungsfehler*, 1891, pp. 355–82.

‡ BIOMETRIKA, vol. 15 (1923), pp. 192–208, paper edited by F. W. Owens after the death of Professor McMahon.

12. *Generalized Frequency Surfaces.* The history of the difficulties which have been encountered in efforts to reach general skew frequency surfaces has been given recently by Karl Pearson.* He tells us that in 1895 after the publication of his memoir on skew frequency curves, he proceeded to the problem of skew frequency surfaces, and gave much time to attempting its solution. He became convinced on experimental grounds that a generalized frequency surface could not be obtained by taking the product of two of his skew frequency functions and transforming coordinates. Pearson next approached the problem by endeavouring to determine surfaces which should grow out of the double hypergeometric series in a way analogous to that by which his skew frequency curves arise from the single hypergeometric series. He obtained the differential equations from the series, but he tells us he has failed to integrate them although he has returned to them again and again for nearly thirty years. In 1901, Pearson put the problem before L. N. G. Filon, who made some progress by obtaining certain special surfaces. Pearson had also obtained a special surface for linear regression and for what he calls parabolic variance. In 1914, Pearson put his differential equations before L. Isserlis. In a paper on the application of *Solid hypergeometric series to frequency distributions of space*†, Isserlis solved the problem of fitting a double hypergeometric series to certain distributions of two variables.

In 1923, Seimatsu Narumi—a Japanese mathematician—published‡ an important contribution to the solution of the problem. It appears that Pearson suggested* to Narumi the problem of working from functional equations, with assumed forms of regression and scedastic functions, back to the frequency surface. All I shall attempt here is to give a general idea of this method of approach, and to state a few of the most interesting results. Let the regression curve of y on x be $y=f_2(x)$, and that of x on y be $x=f_1(y)$. Narumi gives

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* BIOMETRIKA, vol. 15 (1923), p. 222.

† PHILOSOPHICAL MAGAZINE, (6), vol. 28 (1914), p. 279.

‡ BIOMETRIKA, vol. 15 (1923), pp. 77–88, pp. 208–221.

$$(8) \qquad \begin{aligned} z &= \Phi_1(y)\,\Psi_1[\{x-f_1(y)\}F_1(y)] \\ &= \Phi_2(x)\,\Psi_2[\{y-f_2(x)\}F_2(x)], \end{aligned}$$

as the general functional equation of the frequency surfaces. One way of regarding this equation is to consider $\Phi_1(y)$ as giving the relative frequency of values existing for a total array curve corresponding to an assigned y. With the array curve assigned, $\Psi_1[\{x-f_1(y)\}F_1(y)]$ would give relative frequencies at any point along the array curve in units on such a scale from array to array as to produce homoscedasticity because of the character of $F_1(y)$. That is, $F_1(y)$ and $F_2(x)$ are variable scales of measurement which when used to multiply standard deviations produce homoscedasticity. The method may be made clearer by dealing with some special cases.

(a) Given linear regression and constant homoscedasticity, we have $f_1(y)=m_1y+c_1$, $f_2(x)=m_2x+c_2$, where $F_1(y)$ and $F_2(x)$ are constants. The solution of the special functional equation leads to the normal correlation surface.

(b) Given linear regression and linear heteroscedasticity, we have $f_1(y)=m_1y+c_1$, $f_2(x)=m_2x+c_2$, and

$$\frac{1}{F_1(y)}=\lambda_1(y+a_1), \qquad \frac{1}{F_2(x)}=\lambda_2(x+a_2).$$

Then (8) takes the form

$$z=\Phi_1(y)\,\Psi_1\left\{\frac{1}{\lambda_1}\,\frac{x+g_1}{y+a_1}-\frac{m_1}{\lambda_1}\right\}=\Phi_2(x)\,\Psi_2\left\{\frac{1}{\lambda_2}\,\frac{y+g_2}{x+a_2}-\frac{m_2}{\lambda_2}\right\},$$

where $g_1=m_1a_1-c_1$ and $g_2=m_2a_2-c_2$.

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* BIOMETRIKA, vol. 15 (1923), p. 224.

The solution of the special functional equation leads to

$$(9) \qquad z = z_0(x+g_1)^{p_1}(y+g_2)^{p_2} \cdot \{(g_1-a_2)(y+a_1)+(g_2-a_1)(x+a_2)\}^q,$$

where p_1, p_2 and q are arbitrary constants. This is the general frequency surface with linear regression both ways and linear heteroscedasticity. The array curves both ways are Pearson curves of Type I. If $g_1-a_2 \rightarrow 0$ in (9), by simple transformations, the equation may be put into the form

$$(10) \qquad z = z_0(x+g_1)^{p_1'}(y+g_2)^{p_2} e^{q'} \frac{y+a_1}{x+g_1},$$

where $p_1' = p_1+q$, $q' = (g_1-a_2)q/(g_2-a_1)$. This special case has for arrays one way Pearson curves of Type III and the other way Pearson curves of Type VI.

(c) Given that the regression curves are certain equilateral hyperbolas both ways and that the standard deviations of arrays are of the form $1/(y+g_1)$ and $1/(x+f_2)$, respectively, it follows that the functional equation (8) takes the form

$$\begin{aligned} z &= \Phi_1(y)\Psi_1\{(y+g_1)x+f_1y+c_1\} \\ &= \Phi_2(x)\Psi_2\{(x+f_2)y+g_2x+c_2\}, \end{aligned}$$

and its solution is

$$(11) \qquad z = z_0(x+f_1)^{\gamma_1}(y+g_2)^{\gamma_2} e^{\frac{\gamma(x+f_2)}{y+g_1}},$$

which gives Pearson's Type III curves as array curves both ways.

The special cases we have given are to be regarded merely as illustrative. Narumi derives a considerable number of other surfaces. The publications of Narumi on this subject are to consist of Part I, II, and III. Part III has not thus far reached me.

13. *Extension of the Charlier System of Representation to Functions of Two Variables.* While great difficulties have been encountered in attempts to pass naturally from the Pearson system of generalized frequency curves to analogous surfaces for the characterization of frequency with respect to two variables, it appears that the way is

theoretically fairly clear for the extension of the Charlier system of representation to frequency functions of two or more variables. N. R. Jørgensen* has contributed to the solution of this problem. For simplicity, let us consider a continuous frequency function $F(x, y)$ of only two variables x and y measured from their respective mean values and in standard deviations as units as in § 6. Then the normal correlation function

$$\frac{1}{2\pi\sqrt{1-r^2}}\,e^{-\frac{1}{2}\Phi(x,y)},$$

where $\Phi(x, y) = (x^2+y^2-2rxy)/(1-r^2)$ plays a role in the representation of $F(x, y)$ analogous to that taken by the simple Gaussian function in the Charlier theory. The series to be considered is of the form

$$F(x, y) = \sum A_{m,n}e^{-\frac{1}{2}\Phi(x,y)}U_{m,n}, \tag{12}$$

where $U_{m,n}$ is a Hermite† polynomial defined by the equation

$$e^{-\frac{1}{2}\Phi(x-h,\,y-k)} = e^{-\frac{1}{2}\Phi(x,y)}\sum\frac{h^m k^n}{m!\,n!}U_{m,n}.$$

Let $\Psi(x, y) = (x^2+y^2+2rxy)/(1-r^2)$, and define $V_{m,n}$ by the equation

$$e^{hx+ky} = e^{\frac{\Psi(h,k)}{2(1-r^2)}}\sum\frac{h^m k^n}{m!\,n!}V_{m,n}.$$

In 1920, A. Guldberg‡ directed attention to the fact that the coefficient $A_{m,n}$ is readily expressible in moments of the given frequency function $F(x,y)$ by use of the well known fact that the functions $e^{-\frac{1}{2}\Phi(x,y)}\,U_{m,n}$ and $V_{m,n}$ form a biorthogonal system. That is, if we may assume the series (12) uniformly convergent, we find the coefficient

$$A_{m,n} = \frac{(1-r^2)^{m+n-\frac{1}{2}}}{4^{m+n}\pi m!\,n!}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F(x,y)\;V_{m,n}\,dx\,dy. \tag{13}$$

Since $V_{m,n}$ is a polynomial in x and y, the coefficients

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* Loc. cit., p. 86.

† Comptes Rendus, vol. 58 (Jan., 1864), cf. *Oeuvres de Charles Hermite,* vol. 2, 1908, pp. 301–5.

‡ Journal of the Royal Statistical Society, vol. 83, p. 127.

$A_{m,n}$ are expressible as moments of $F(x,y)$. The series represents the function subject to conditions of continuity and vanishing at infinity given by Wera Myller Lebedeff.* The question of representing data by the first few terms of the series has not been answered and can probably be answered only by experiments with a wide range of distributions.

IV. THE THEORY OF RANDOM SAMPLING

14. *Introduction.* The theory of random sampling deals with problems of drawing inferences concerning the constitution of a statistical aggregate or class of things from the nature of a representative part taken as a random sample. The drawing of such statistical inferences about a class of individuals from the analysis of a sample is fundamental in the applications of mathematical statistics in insurance, in biology, and in other branches of science. That is to say, past experience with limited samples has been applied very widely to predict future events among the class from which the sample was drawn. But when we undertake the exact formulation of the theory which supports the applications even in the case of the simplest statistical ratios, we may find ourselves involved in the disputes about the validity of the theory of inverse probabilities. It is not my purpose here to enter upon a discussion of this well known controversy. I mention it simply because it seems desirable to direct attention to two recent interesting papers on the subject by E. T. Whittaker† and Karl Pearson‡ both published in 1920 as well as to the renewed attack on the theory by J. M. Keynes in his book published in 1921.

The Pearson school of statisticians has accepted the theory of inductive probability, and the statisticians of this school have been very active in dealing with the problems of sampling errors in various kinds of averages, statistical coefficients and parameters of frequency curves. Among the various sampling problems dealt with recently, the most interesting to me are the problems of the distribution of certain averages and coefficients obtained from small samples, and the extensions of the theorem of Tchebychef. I shall give the remaining time to these two topics.

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* Loc. cit., p. 415.

† TRANSACTIONS OF THE FACULTY OF ACTUARIES, vol. 8 (1920), p. 163.

‡ BIOMETRIKA, vol. 13 (1920-21), p. 1.

15. *Fluctuations of Certain Averages obtained from Small Samples.* In the development of the theory of sampling, the assumption has usually been made that the sample contains a large number of individuals. But the lower bound of large numbers has remained poorly defined in this connection. For example, the usual probable error formulas have been applied to as few as ten observations. If there is a misapplication of formulas due to smallness of the sample, the source of the error would probably lie in the fact that the statistical constants from small samples are not distributed even roughly in accord with a Gaussian probability curve.

Beginning with a paper by Student[*] in 1908 there have been important experimental and theoretical results obtained on the distribution of arithmetic means, standard deviations, and correlation coefficients obtained from small samples. The material used in the experiments to which I refer consisted of 3000 pairs of measurements. The measurements were written on 3000 cardboards which were shuffled and from which 750 sets of 4 were taken. These provided two sets of 750 standard deviations each calculated from only four values, and 750 correlation coefficients each calculated from only four pairs of values. The distribution of the given 3000 values was roughly Gaussian in character. The simple inspection of the frequency distribution made it fairly obvious that the standard deviations experimentally obtained from sets of four were not distributed in accord with the Gaussian curve. Student found by empirical methods that the curve

$$(1) \qquad y = y_0 x^{n-2} e^{-\frac{nx^2}{2\sigma^2}}$$

seemed appropriate to give the distribution of standard deviations s obtained from samples of n, where σ is the standard deviation of the infinite population from which the samples are drawn.

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[*] BIOMETRIKA, vol. 5 (1908), p. 1.

In 1915, Karl Pearson* took an important step in advance by obtaining the distribution of the standard deviations of samples of n variates from an infinite population distributed in accord with the Gaussian curve. He obtained from theoretical considerations the distribution of s identical with that which Student found experimentally.

Moreover, by tabulating β_1, β_2 and the measure of skewness for integral values of n from 4 to 100, Pearson shows that (1) approaches the Gaussian curve as n increases provided we accept certain necessary conditions, that is $\beta_1 = 0$, $\beta_2 = 3$, and skewness equal to zero, as sufficient for practical approach to this curve.

From this table, Pearson concludes that for samples of 50 the usual theory of probable error of the standard deviation holds satisfactorily, and that to apply it to samples of 25 would not lead to any error of importance in the majority of statistical problems. On the other hand, if a small sample, $n < 20$ say, of a population be taken, the value of the standard deviation found from it will be usually less than the standard deviation of the population.

Turning next to the Student† experiment with the distribution of the 750 correlation coefficients each computed from 4 pairs of values mentioned above. The correlation coefficient of the whole 3000 pairs from which drawings were made was $r = .66$. He further selected samples of 8 and samples of 30 from the population of correlation $r = .66$. An examination of the experimental results makes it fairly obvious that the distributions for samples of 4 and 8 pairs are far from normal, and that the average value of r from these small samples is smaller than the $r = .66$ of the total population of 3000 from which the small samples are drawn. But with samples of 30, the correlation coefficient r approaches the correlation coefficient of the total population.

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* BIOMETRIKA, vol. 10 (1915), p. 522.

† BIOMETRIKA, vol. 6 (1908–9), p. 302.

In a paper published in 1913, H. E. Soper* obtained to a second approximation the mean and standard deviation of the distribution of the correlation coefficient r from samples of n from a population of correlation ϱ. He concludes that the mean value of the correlation coefficients obtained from small samples will be numerically less than the true correlation coefficient obtained from the aggregate and will be approximately represented by the formula

$$\varrho\left(1-\frac{1-\varrho^2}{2n}\right),$$

where n is the number in the sample.

In a paper published in 1915, R. A. Fisher† dealt with the frequency distribution of the correlation coefficient derived from samples of n pairs each taken at random from an infinite population distributed in accord with the normal correlation surface

$$z=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\varrho^2}}e^{-\frac{1}{2(1-\rho^2)}\left\{\frac{(x-m_1)^2}{\sigma_x^2}+\frac{(y-m_2)^2}{\sigma_y^2}-\frac{2\beta(x-m_1)(y-m_2)}{\sigma_x\sigma_y}\right\}},$$

where ϱ is the correlation coefficient. The frequency function obtained for the distribution of r is given by

$$(2)\quad y_n=f_n(r)=\frac{(1-\varrho^2)^{\frac{n-1}{2}}}{\pi(n-3)!}(1-r^2)^{\frac{n-4}{2}}\frac{d^{n-2}}{d(r\varrho)^{n-2}}\left[\frac{\operatorname{arc}\cos(\varrho r)}{\sqrt{1-\varrho^2r^2}}\right].$$

The derivation of this form of $f_n(r)$ would probably be found of special interest to those who are seeking applications of certain general conceptions derived from the geometry of n-dimensional space. The ordinates y_n cannot be readily calculated from (2) except for small values of n. Moreover (2) offers no rapid means of calculating the mean value r, the modal value of r, or the standard deviation σ_r to replace the approximations obtained by Soper. In order to investigate the approach of (2) to a normal curve as n increases, it seems necessary to provide methods for computing the ordinates y_n and moment coefficients for (2). This is accomplished in a joint memoir* by H. E. Soper, A. W. Young, B. M. Cave, A. Lee, and Karl Pearson. This memoir involves a tremendous amount of laborious numerical computation as well as the results of considerable theoretical work in adapting the function $y_n = f_n(r)$ and its moment coefficients to numerical calculation. The theoretical part consisted largely in obtaining series which converge with sufficient rapidity to be used in the numerical calculations. The tabulated values show the ordinates y_n at intervals of .05 for r from -1 to $+1$, at intervals of .1 for ϱ from 0 to .9, and for $n = 3, 4, 5, \ldots, 25, 50, 100$ and 400, making in all 260 frequency distributions. The values of β_1 and β_2 were computed for these distributions to study the approach to the normal curve.

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* Biometrika, vol. 9 (1913), p. 91.

† Biometrika, vol. 10 (1915), p. 507.

With respect to the approach of these distributions to the Gaussian form with increasing values of n, it is found that the necessary conditions $\beta_1 = 0$, $\beta_2 = 3$ for a Gaussian distribution are not well fulfilled for samples of 25 or even 50 whatever the value of ϱ. For samples of 100, the approach to the conditions $\beta_1 = 0$, $\beta_2 = 3$ is fair for low values of ϱ, but for large values of ϱ, say $\varrho > .5$, there is considerable deviation of β_1 from 0, and of β_2 from 3. For samples of 400, on the whole, the approach to the necessary conditions $\beta_1 = 0$, $\beta_2 = 3$ is close, but there is quite a sensible deviation from normality when $\varrho \geqq .8$. These results give us a striking warning of the dangers of applying the ordinary formula for the probable error of r when we have small samples.

In conclusion, it should not be forgotten that the assumption is made, in this theory of the distribution of r from small samples, that we have drawn samples from an infinite population well described by a normal correlation surface, so that the conclusions are not in the strictest sense applicable to distributions not normally distributed.

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* BIOMETRIKA, vol. 11 (1915-17), p. 328.

16. *The Tchebychef Theorem and its Recent Generalizations.* As the concluding topic of this paper, let us consider the recent extensions of the following remarkable theorem of Tchebychef which appeared* in the LIOUVILLE JOURNAL in 1867:

THEOREM. *If* $a, b, c, \cdots$ *represent the mathematical expectations of quantities* $x, y, z, \cdots$ *and* $a_1, b_1, c_1, \cdots$ *the mathematical expectations of their squares* $x^2, y^2, z^2, \cdots$ *the probability that the sum* $x+y+z+\cdots$ *is between*

$$a+b+c+\cdots+\lambda\sqrt{a_1^2+b_1^2+c_1^2+\cdots-a^2-b^2-c^2\cdots}$$

and

$$a+b+c+\cdots-\lambda\sqrt{a_1^2+b_1^2+c_1^2+\cdots-a^2-b^2-c^2\cdots}$$

will always be greater than $1-1/\lambda^2$, *whatever the value of* λ.

Tchebychef proved this theorem by simple algebraic methods. One great merit of the theorem lies in its freedom from restrictions with respect to the nature of the distribution of the variables.

To state the theorem in another form, let us assume the frequency distribution of an infinite population with standard deviation σ. If $P(\lambda\sigma)$ is the probability that a datum drawn at random from this distribution will differ in absolute value from the mean of the whole distribution by as much as $\lambda\sigma$, then.

$$(3) \qquad P(\lambda\sigma) \leqq \frac{1}{\lambda^2}, \qquad \text{or} \qquad 1-P(\lambda\sigma) \geqq 1-\frac{1}{\lambda^2}.$$

In 1919, Karl Pearson† published an important generalization of the theorem subject to the mathematical condition that the frequency function $F(x)$ is such that the integral

$$\int_a^b (x-\bar{x})^{2s} F(x)\,dx$$

exists, where a and b are the lower and upper bounds of the distribution. He found

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* *Des valeurs moyennes*, translated from the Russian by N. M. de Khanikof, JOURNAL DE MATHÉMATIQUES, (2), vol. 12, pp. 177–84.

† BIOMETRIKA, vol. 12 (1919), pp. 284–96.

$$1-P(\lambda\sigma) > 1-\frac{1}{\lambda^{2s}}\frac{\mu_{2s}}{\mu_2^s} = 1-\frac{\beta_{2s-2}}{\lambda^{2s}}$$

or

$$P(\lambda\sigma) \leqq \frac{\beta_{2s-2}}{\lambda^{2s}}, \tag{4}$$

where μ_{2s} is the $2s$th moment coefficient about the arithmetic mean of the area under $F(x)$. If we make $s=1$, we have the special case of the Tchebychef theorem. In his proof, Pearson deals with moments μ_n where n is even. Seimatsu Narumi* has pointed out that it is sufficient to assume n a positive constant provided the integral

$$\int_0^\infty x^n\{F(x)+F(-x)\}\,dx$$

exists.

It is Pearson's view that, although his inequality is in most cases a closer inequality than (3), it is usually not close enough to be of practical assistance in drawing conclusions. Hence, it becomes important to obtain closer inequalities by decreasing the right hand side of (4). This was accomplished in papers published almost simultaneously by Birger Meidel† and B. H. Camp‡ by placing certain restrictions on the nature of the distribution function $F(x)$. But the restrictions are so devised as to leave the distribution function sufficiently free to be useful in the actual problems of statistics. The main restriction placed on $F(x)$ by Camp is that it is to be a monotonic decreasing function of $|x|$ when $|x| \geqq c\sigma$, $c \geqq 0$. The severity of this restriction on $F(x)$ varies according to the value of c. Its general effect is to exclude distributions which are not represented by decreasing functions of $|x|$ at points a certain assigned distance from the origin.

With the origin so chosen that zero is at the mean, Camp reaches the generalized inequality

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* BIOMETRIKA, vol. 15 (1923), p. 246.

† COMPTES RENDUS, vol. 175 (1922), p. 806; cf. SKANDINAVISK ACTUARIETIDSKRIFT, 1922, p. 210–16.

‡ This BULLETIN, vol. 28 (1922), p. 427.

$$(5) \qquad P(\lambda\sigma) \leqq \frac{\beta_{2s-2}}{\lambda^{2s}} \frac{\left(\frac{2s}{2s+1}\right)^{2s}}{1+\varphi} + \frac{\varphi}{1+\varphi} P(c\sigma),$$

where

$$\varphi = \frac{\left(\frac{c}{\lambda}\frac{2s}{2s+1}\right)^{2s}}{(2r+1)\left(\frac{\lambda}{c}-1\right)}.$$

With $c=0$, the formula (5) is exactly Pearson's divided by $[1+1/(2s)]^{2s}$. With the origin chosen at the mode instead of the mean, and with moments defined with respect to the origin, Meidel* shows that

$$(6) \qquad P(\lambda\sigma) \leqq \frac{1}{\left(1+\frac{1}{n}\right)^{n}\lambda^{n}},$$

for any positive value of $n \geqq 1$. The general effect of the work of Camp and Meidel has been to decrease the larger number of the Tchebychef inequality by roughly fifty per cent. Even with the generalizations, the theorem of Tchebychef does not set such close limits on probabilities as the Gaussian law, but we should have regard for the fact that this theorem in its original form applies to any type of distribution, and in its generalized form to very general types of distribution.

In conclusion, it may be added that Markhoff† and Tschuprow‡ have in recent years given extensions of the work of Tchebychef on mathematical expectation and limiting values of probabilities along very different lines from those on which I have reported.

THE UNIVERSITY OF IOWA.

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* Loc. cit, p. 214.

† A. A. Markhoff, *Wahrscheinlichkeitsrechnung*, 1912, p. 67.

‡ A. A. Tschuprow, *Zur Theorie der Stabilität statistischer Reihen*, SKANDINAVISK AKTUARIETIDSKRIFT, 1919.

EISENHART'S TRANSFORMATIONS OF SURFACES

Transformations of Surfaces. By Luther Pfahler Eisenhart. Published with the cooperation of the National Research Council by The Princeton University Press, 1923. IX + 379 pp.

The results of a round five score of researches in three-dimensional differential geometry, generalized to a great extent to n-space and developed largely anew to form a unified theory governed by a central idea,—this is the offering before us. The researches, with few exceptions, are the product of investigations of the last quarter of a century carried on primarily by Bianchi, Darboux, Demoulin, Eisenhart, Guichard, Jonas, Koenigs, Ribaucour, and Tzitzeica. They deal, some directly and some rather indirectly, with transformations of surfaces of a given kind into surfaces of the same kind.

Of these transformations there are two general types, transformations F, the nature of which we shall describe later, and transformations in which the two surfaces are the focal surfaces of a W congruence. Both transformations appeared first in special forms, the latter in the particular case of pseudospherical surfaces developed by Bianchi (1879) and Backlund (1883), the transformations F in special cases discussed by Koenigs (1891), Darboux (1899). and subsequent writers. In fact, it was not until quite recently that the general transformation F was systematically studied, by Jonas, in 1915, and by Eisenhart, in 1917. It is this theory of F, or fundamental, transformations which forms the central and unifying theme of the book.

The scope of the book is broader than the title might first suggest. The material handled is not merely abundant—besides the text there are at the end of each of the ten chapters an average of twenty-five problems serving to a large extent to summarize the results of research,—but it is also highly diversified. More than a third of the book is devoted to congruences of spheres and of circles, rolling surfaces, and surfaces applicable to a quadric, subjects which one would not at first thought relate to the theory of transformations. To unify such a diversity of material in a natural and effective fashion is not simple, and the author is to be congratulated on the masterly way in which he has succeeded. Thereby he has not only given us in many cases new methods of arriving at known results, but has brought home to us in striking fashion the breadth and power of the central theory of F transformations.

The present book is a sequel to the author's *Differential Geometry* in the sense that it refers freely to the elementary treatise for facts and formulas. It is written, too, in much the same style. The development is concise and lucid, and the material in the separate chapters is well ordered. The references to the literature leave nothing to be desired, as soon as one comprehends that the many articles referred to by title only are to be attributed to the author himself.

J. L. Coolidge quotes Darboux as once having told him that in his opinion the essence of geometry consisted in finding in each individual problem the best method for its solution. He might well have added that this is a *sine qua non* in many a problem in differential geometry. For in this field, as much as, if not more than, in any other, a happy choice of method is essential. All the more then is Eisenhart, and those on whose researches he has drawn, to be complimented on the relative simplicity of the analytic formulation of problems. In proceeding now to more detailed discussions we hope to be able, in a few instances at least, to exhibit this analytic simplicity, as well as the elegance of the geometrical results.

When two surfaces in 3-space are in one-to-one point correspondence, there exists in general on each surface a (conjugate) net which corresponds to a (conjugate) net on the other surface. It is by means of these nets and the congruence of lines joining corresponding points on the two surfaces that the transformation is studied. A net is characterized by its point equation, written in the form

$$\frac{\partial^2\theta}{\partial u\partial v}=\frac{\partial\log a}{\partial v}\frac{\partial\theta}{\partial u}+\frac{\partial\log b}{\partial u}\frac{\partial\theta}{\partial v}. \tag{1}$$

Congruences of lines are restricted to those having two distinct families of developables. A congruence G and a net N are said to be conjugate if the curves of N lie on the developables of G, provided that N is not a focal net of G.

If N and N_1 are two nets in correspondence so that the lines joining corresponding points form a congruence G whose developables contain the nets, three possibilities arise. If N and N_1 are the focal nets of G, they are Laplace transforms of one another. If N is a focal net of G and N_1 is not, N_1 is called a Levy transform of N.

In the general case, when neither N nor N_1 is a focal net of G, G is conjugate to both N and N_1. This is the fundamental transformation, or transformation F.

To obtain a transformation F of a net N, we have but to choose first a congruence G conjugate to N and then a net N_1 conjugate to G; G is determined by a net N' parallel to N in that its direction parameters can be taken equal to the point coordinates of N'; then each net N_1 conjugate to G, and hence in relation F to N, is determined by a solution θ of the point equation (1) of N. The point coordinates of N_1 have the simple form $x_1=x-(\theta/\theta')x'$, where θ' is a solution, determined by θ, of the point equation of N'. Thus an F transform

N_1 of N involves two independent elements, a congruence G conjugate to N and a solution θ of (1).

Two nets N_1 and N_2, which are F transforms of the same net N by G_1, θ_1 and G_2, θ_2 respectively, are obviously F transforms of one another if G_1 and G_2 are the same congruence. If G_1 and G_2 are distinct, N_1 and N_2 will be in relation F if and only if $\theta_1 = \theta_2$, and then any two of the three nets, N, N_1, N_2, will be F transforms of the third by the same solution of the point equation of the third. The general case, in which G_1 and G_2 are distinct and $\theta_1 \neq \theta_2$, gives rise to the first and most important of the so-called theorems of permutability with which the book is studded. According to this theorem, if N_1 and N_2 are F transforms of N, as described, there exists a net N_{12} which is an F transform of both N_1 and N_2. Corresponding points of the four nets lie in a plane p which envelopes a net and corresponding tangent planes meet in a point P which generates a net.

One more important relationship is needed to complete the account of the general theory. A net N and a congruence G are said to be harmonic if the curves of N correspond to the developables of G and the focal points on a line of G lie on the tangents to the corresponding curves of N. This relationship is closely connected with both the transformations of Levy and the transformations F. In particular, it is found that, if two nets are in relation F, they are harmonic to the same congruence, i. e. their corresponding tangent planes meet in lines generating a congruence harmonic to both; this congruence is known as the harmonic congruence of the transformation F.

If the two non-parallel congruences G_1 and G_2 are harmonic to a net N, the locus of the point of intersection of corresponding lines of G_1 and G_2 describes a net, conjugate to G_1 and G_2 and known as a derived net of N. Conversely, if G_1 and G_2 are conjugate to a net N, the planes determined by corresponding lines of G_1 and G_2 envelope a net, harmonic to G_1 and G_2 and known as a derivant net of N. The net generated by the point P above mentioned is a derived net, and that enveloped by the plane p a derivant net, of each of the four nets entering into the theorem of permutability.

The general theory that we have outlined admits immediate extension to n-space, and it is in this form that it appears, in the first two chapters of the book. Chapter III deals with Laplace sequences in n-space; the analytic conditions that a sequence be periodic are established and hence it is shown that a periodic sequence of order p lies in a space of $p-1$ dimensions. In this and the two subsequent chapters homogeneous point and tangential coordinates are introduced and applied to suit the needs of the material. The developments of the latter chapters, IV and V, are for 3-space and have primarily to do with transformations of special type.

Of particular transformations F we mention two which are essentially duals of one another, namely transformations K and transformations Ω. A transformation K (Koenigs) is characterized by the property that a pair of corresponding points of the two nets separate harmonically the corresponding focal points of the conjugate congruence. In the case of a transformation Ω (Eisenhart), a pair of corresponding tangent planes to the two nets separate harmonically the corresponding focal planes of the harmonic congruence. In the first case the nets have equal point invariants and in the second, equal plane invariants.

An interesting correspondence is developed between transformations Ω of a net N into a net N_1 and transformations of Bianchi in which a surface Σ and its transform Σ_1 are the focal surfaces of a W congruence. The correspondence is characterized by the property that Σ and Σ_1 have the same spherical representations of their asymptotic lines as the curves of N and N_1 respectively. By means of it the theorem of permutability for transformations Ω is readily translated into a theorem of permutability for the transformations of Bianchi.

As further subjects treated in Chapters IV and V we should like to mention the transformations Ω of permanent nets (nets admitting an infinity of applicable nets), the theory of R nets (Demoulin and Tzitzeica), and that of ray congruences and ray curves (Wilczynski).

The interesting and important question of transformations of orthogonal nets (O nets) is considered exhaustively, first in n-space in Chapter VI and then in 3-space in Chapter VII. To discuss transformations F of O nets, it is necessary to know both the nature of the congruences conjugate to O nets and the nature of the nets conjugate to these congruences. It is shown (Guichard) that the latter nets are either O nets in the same space as the given one or projections in this space of O nets in higher spaces, and that the conjugate congruences are related to congruences of isotropic lines in a correspondingly simple manner. To Guichard is also due the introduction of the powerful analytical tool which renders this difficult subject tractable, namely, a suitably chosen moving polyhedral attached to the given O net.

Of the transformations F of O nets into O nets the most important are the transformations R, named for Ribaucour. These transformations have two fundamental aspects, which we shall describe for 3-space. In the first place, two O nets in relation R consist of the principal curves on the two sheets of the envelope of a congruence of spheres; the centers of the spheres form a net, called the central net, which is an F transform of each of the O nets and is itself the projection of an O net in 4-space. Again, the O nets of a one-parameter family consisting of a given O net, N, and its ∞^1 R transforms N_1 by the same θ possess a common harmonic congruence and admit an orthogonal congruence of circles (a cyclic system) whose axes are the lines of this congruence, a point of N and the corresponding points of the nets N_1 all lie on one of the circles.

There are two important subclasses of transformations R, the transformations D_m (Darboux and Bianchi) and the transformations E_m (Eisenhart and Bianchi). The first are the transformations R of isothermic O nets into isothermic O nets and are of type K; the second are the transformations R of O nets with isothermal spherical representation into O nets of the same kind and are of type Ω. In both cases the subscript m refers to a parameter in the transformation.

Chapter VIII deals with congruences of spheres and congruences of circles (in 3-space). A striking feature of this chapter is a correspondence established between congruences of spheres in 3-space and congruences of lines in 5-space, whereby the principal curves on the envelope of the spheres correspond to the developables of the congruence of lines and the pentaspherical coordinates of the spheres are the direction parameters of the lines. A congruence of circles of restricted type appears as consisting of the circles of intersection of the corresponding spheres of the two congruences which correspond to the congruences of tangent lines to a net in 5-space. In fact, this is the author's definition and from it he readily develops the characteristic property that a circle of the congruence is intersected by each of two infinitely near circles in two points. According to this definition, a congruence of circles and a net in 5-space correspond. Hence relationships between congruences of spheres and circles can be interpreted in 5-space as relationships between rectilinear congruences and nets, and vice versa. It was doubtless with this in mind that the author approached the relationships between congruences of spheres and circles previously developed by Guichard and justified the characterization of these as harmonic, conjugate, and orthogonal, on the basis that they correspond to like-named relationships in 5-space. Similarly, the transformations F of congruences of circles which are developed correspond to the transformations F of nets in 5-space. In particular, since to a cyclic system of circles corresponds an O net, the problem of F transformations of cyclic systems is equivalent to that of F transformations of O nets.

If S and $\bar{S}$ are two surfaces applicable to one another, and if S is held fast while $\bar{S}$ assumes all the (∞^2) positions in each of which a point of it is made to coincide with the corresponding point of S so that the corresponding directions at the two points also coincide, $\bar{S}$ is said to roll on S. In Chapter IX the author develops the theory of rolling surfaces, in the general case in which there are actually corresponding nets on S and $\bar{S}$, by applying the general theory of transformations F to nets admitting applicable nets. He thus obtains the theorems of Darboux and Guichard relative to elements invariably fixed to $\overline{S}$, as $\overline{S}$ rolls on S. As an example of the beautiful results deducible in this manner, we quote one theorem: When $\overline{S}$ rolls on S, the isotropic lines through a point invariably fixed to $\overline{S}$ meet the plane of contact of $\overline{S}$ with S in a circle which generates a cyclic system and whose points generate the O nets orthogonal to this system. The latter part of the chapter is devoted to the theory, due primarily to Bianchi, of transformations R deformable in the sense that their central nets admit applicable nets which can serve as central nets of new transformations R; in particular, deformable transformations D_m and E_m are discussed.

The problem of the deformation of a quadric, considered in Chapter X, is approached through the study of permanent nets on the quadric and the corresponding permanent nets on the deforms of the quadric. Two transformations of these nets are then investigated, a transformation F_k, of Guichard and Eisenhart, and a transformation B_k of Bianchi. The first of these is simply a transformation F of a permanent net on a quadric, Q, into a second permanent net on Q, or a resulting transformation F of a net applicable to Q into a similar net; for each value of the constant k there exist ∞^2 transformations F_k of a net $\overline{N}$ applicable to Q into ∞^2 nets $\overline{N}_1$ applicable to Q, and for each of certain special values of k the nets $\overline{N}_1$ can be arranged in one-parameter families possessing elegant properties.

Two nets in relation B_k are the focal nets of a W congruence; in other words, a transformation B_k is not of type F, but of the second type discussed at the beginning of the review. Moreover, as originally developed by Bianchi, the transformations B_k have apparently no connection with transformations F_k. Our author may, then, take justifiable pride in bringing the two into coordination. To this end, he introduces a new definition of a transformation B_k. Thereby, a B_k transform of a net $\overline{N}$ applicable to a net N on Q appears as the derived net, $\hat{N}$, of $\overline{N}$ by two suitably chosen solutions, θ_1 and θ_2, of the point equation of $\overline{N}$; $\hat{N}$ is, of course, applicable to Q, and the net on Q of which it is the deform is related in a simple manner to the derived net of N by θ_1 and θ_2, a net lying on a quadric confocal to Q. The connecting link between the transformations B_k, thus defined, and the transformations F_k consists in the fact that each of the functions, θ_1, θ_2 defines also an F_k transform of $\overline{N}$. This connection is not merely of interest in itself, but serves also to establish a theorem of permutability of transformations of the two types.

The major elements of the book are beyond criticism. The author has, however, a tendency on occasion to scorn minor conditions and subsidiary cases. For example, the definition of a net is so formulated that it includes systems of curves not generally regarded as nets, namely systems on a developable surface one of whose families consists of the asymptotic lines; the author doubtless intended to exclude systems of this type, but nowhere is there a statement to this effect.

In the definition of a net p,O (the projection in n-space of an O net in $(n+p-1)$-space), it appears essential that the $p-1$ complementary functions should be themselves linearly independent as well as linearly independent of the point coordinates of the net. Even then, there would exist nets, each of which belonged, under the definition, to every one of the categories p,O, $p \geqq 2$; the net of generators of a paraboloid, considered as a surface of translation, is of this type. It is of course possible to leave the definition so that a particular net admits various characterizations p, O, but this is not in keeping with the spirit of the subsequent investigations, in which always the minimum value of p is sought. Similar remarks hold for the congruences p, I.

On page 10 we find the theorem: If two non-planar nets correspond and the tangents to the parametric curves in one family are parallel, the nets are parallel. This theorem is true only in general and the exceptions to it are of some importance. For example, in the map of the two translation surfaces, $x = U(u) + V(v)$, $\bar{x} = U(u) + \bar{V}(v)$, the generators correspond and corresponding tangents to the generators $v = \text{const.}$ are parallel; the map, however, is not in general a parallel map. The slip in the proof comes from overlooking the fact that a certain equation may be illusory. Fortunately, the theorem does not figure in subsequent developments.

On page 24 it is stated that two solutions of the equation (1) are functionally dependent if and only if one is a linear function of the other with constant coefficients; this is obviously untrue in case either of the coefficients on the right hand side of (1) is zero. On page 198 the possibility of the vanishing of one or the other of two functions (q and r) is not considered, so that the results there obtained appear to be true only in general.

The Hamburg press, from which the work comes, has done a creditable piece of book manufacturing, except for the bright (now dirty) yellow of the binding. The proof reading of the text itself might have been done more thoroughly, but there are next to no errors in the formulas, so far as the reviewer has been able to ascertain by checking a reasonable selection from the thousand and more which the book contains.

As the first of what it is to be hoped will be many mathematical books published with the cooperation of the National Research Council, this work sets a worthy standard. The need of unified presentations of the results of research in special fields is becoming more and more pressing and one who has so successfully met this need in his own field as the present author may look back happily on his labors with the comforting realization that he has contributed greatly to his science.

W. C. GRAUSTEIN

THE STRASBOURG CONGRESS

Comptes Rendus du Congrès international des Mathématiciens (Strasbourg, 1920). Publiés par Henri Villat. Toulouse, Édouard Privat, 1921. xlvii + 670 pp.

The Strasbourg Congress is particularly interesting as the first congress after the War, and its proceedings indicate the results of researches made during the War as well as forecast those to be made later. If the reader will glance at the list of topics in the volume, he will notice that many of those topics have continued to develop since the time of the Congress, and that some of them (for instance that of general complex numbers) have since then reached a certain definitive stage.

The mathematical part of the volume before us consists of five general lectures, by Messrs. Larmor, Dickson, de la Vallée-Poussin, Volterra, and Norlund, and of 68 other papers and résumés of papers. Among the latter, 32 are in arithmetic, algebra and analysis, 10 are in mathematical physics and applied mathematics, and 9 are on questions philosophical, historical, and pedagogical. An adequate review of the volume would necessitate an analysis of each of these general lectures and papers; because of lack of space we give merely abstracts of the general lectures.

Larmor: *Questions in physical interdetermination.*

Sir Joseph Larmor's lecture is a masterly critical discussion of modern theories of spacetime, with particular reference to the two questions of physical interdetermination and of the postulation of a Hamiltonian principle of extremal variation of action.

There is complete statical interdetermination in a freely jointed frame of rigid bars without redundant members. Another instance of complete statical interdetermination is given by the newtonian potential, which is determined throughout free space or so long as no attracting matter is traversed, by its values in any region. This latter interdetermination holds for any isotropic uniform elastic medium, whether material solid or ether. Complete static interdetermination can take place only in real spaces; freedom demands that at least one dimension shall be imaginary. Radiation is the principal, if not the sole method of production of change of physical systems out of direct contact. If radiation it to take place in waves and if the medium is isotropic, time restricted to a single dimension, it can be shown that space must have three dimensions. The variation of the total intrinsic action of a physical system is postulated to be extremal. The space is thus considered to exist apart from the physical processes that take place in it. In special relativity, then, a space is characterized by its potentials. The moving sources or masses are attached to a framework, whose displacement presents an extremal variation of the action. The tensor thus introduced gives, by integration over a barrier surface, the mechanical effects on each source or mass. The action involves deformation of the space itself rather than disturbance of the medium, as in the ether theory. The principle of a space distribution of an action function, whose integral is postulated to be of extremal variation (centers of activity being cut out by barrier surfaces), is the only known system assigned by local characteristics which permits the deduction of relations between distant parts of the medium. This system is at variance with the Einstein prediction of displacement of the lines of the spectrum. The system also leads to a complete self-consistent theory before all atoms of the same substance are identified. A test of the validity of the theory is thus the existence of identical atoms, whose existence can be detected by radiation to a distance.

Dickson: *Some relations between the theory of numbers and other branches of mathematics.*

The methods of one branch of mathematics are a frequent aid in the development of another branch. Such theories as are developed by the use of methods and by analogy with the results of a different field are then of particular interest. As an illustration, Dickson shows how geometric methods may be used to find the rational solutions of certain Diophantine equations. Dickson then presents a new method, involving algebraic numbers and integral quaternions, for finding formulas for the integral solutions of Diophantine equations while the parameters take only integral values. The remainder of the lecture is devoted to an illustration of the nature of modular invariants, as developed by Dickson and his pupils, and to the extension of an application by Mordell of the theory of algebraic invariants.

De la Vallée-Poussin: *Sur les fonctions à variation bornée.*

Modern work in real variables may be divided roughly into two classes, the first the theory of linear partial differential equations and the second the theory developed in the present century by Baire, Borel, Lebesgue, and their followers. The theories are intimately connected in various ways; for example, by the theory of the development of arbitrary functions in terms of solutions of ordinary differential equations. The purpose of the entire lecture is to point out the central position occupied by functions of bounded variation in this second class of topics on real variables. Thus every indefinite integral in the sense of Lebesgue is a function of bounded variation. If an integrable function $f(x)$ is defined in an interval $a \leq x \leq b$, the function

$$F(E) = \int_E f(x)\,dx$$

is an example of a *fonction d'ensemble*, where E is any measurable point set (ensemble) contained in the interval; the function $F(E)$ has also the additive property, namely that

$$F(E_1)+F(E_2)+\cdots+F(E_n)+\cdots=F(E),$$

if E is the sum of the non-overlapping sets $E_1, E_2, \ldots, E_n, \ldots$ The fonctions d'ensemble are of the highest importance in the study of measure and of integration. It can be shown that every additive fonction d'ensemble is also a point function of bounded variation and conversely.* It seems probable that in future developments functions of bounded variation are destined to play a central role, not merely as a class of functions to be developed in terms of given sets of functions, but also in connection with the theory of the differentiation of functionals.

Volterra: *Sur l'enseignement de la physique mathématique.*

Senator Volterra, as one who has not only spent a lifetime in teaching mathematical physics but has made many diverse and fundamental contributions to the field, is eminently fitted to discuss methods of presentation of the general subject. Volterra commences his lecture by outlining the history of his subject. The analytical physics to be considered is not that developed in the heroic age, up to the time of Lagrange, nor is it the most recent age which has just begun, of the great modern work in relativity and the constitution of matter. The theory of the most importance at present and yet that has reached a certain degree of finality in its methods and results is that due to Lagrange, Laplace, Green, Fourier, Cauchy, Maxwell, Riemann, in short, the applications to physics of partial differential equations. The model of exposition of a physical theory is Lagrange's masterpiece, where the writer commences with a historical introduction, arrives at a mathematical formulation of his problems, and finally proceeds with the solution of the purely mathematical problems. This plan is followed directly by Volterra. The unification of the theory of anaytical physics is to be secured by means of partial differential equations, and that this can be done is made clear by the first part of the course which Volterra suggests, the derivation of the partial differential equations and the formulation of the precise mathematical problems involved. The second part of the course naturally consists in the classification of differential equations according to their characteristics, and the solution of the boundary value problems which arise. These problems my be solved by the method of Green (fundamental solutions), that of characteristics, or that of simple solutions as in Fourier's series. The third and last part of the course is the treatment of many problems which could not be solved or could be only partially solved by the former methods, and of some additional problems. These problems are unified by the notion of functions of lines, and lead us to integral equations, integro-differential equations, equations in functional derivatives, etc. The course thus outlined by Volterra has the advantage of unifying and systematizing analytical physics, and also of economy and of separating distinctly the physical notions and the mathematical processes. Every reader will hope that Volterra will be able actually to prepare and publish the course he suggests, not only a work of science but also a work of art.

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* In this connection and elsewhere de la Vallée-Poussin refers to his memoir of 1915 in the AMERICAN JOURNAL, a reference which should read to the 1915 volume of the TRANSACTIONS OF THIS SOCIETY.

Norlund: *Sur les équations aux différences finies.*

This lecture is an exposition of some of the author's striking results on difference equations. In the equation

$$F(x+1)-F(x)=\varphi(x), \tag{1}$$

$\varphi(x)$ is a given function of the complex variable x, where $F(x)$ is to be determined. Equation (1) has been studied by various writers, including Euler, Plana, Abel, and Cauchy. Guichard showed that, if $\varphi(x)$ is an entire function, then (1) always has as a solution an entire function which can be expressed as a definite integral. Appell likewise proved this result, obtaining the solution as an infinite series. The solution of (1) is not uniquely defined, even when restricted to be an entire function. For to a solution may be added any function of period unity, and the sum will be an entire function, a solution of (1). Norlund introduces the notion of *principal solution*, and studies its properties by means of divergent series.

The form and typography of the volume are excellent throughout, for the volume is uniform with and by the same publishers as the ANNALES DE LA FALCULTÉ DES SCIENCES DE TOULOUSE.

J. L. WALSH

SHORTER NOTICES

An Introduction to the Theory of Statistics. By G. Udney Yule. 6th Edition. London, C. Griffin, 1922. xv + 415 pp.

A noteworthy development of recent years has been an increasing use in various fields—economics, education, public health work, biometry, etc.—of a varied assortment of methods which are called statistical. Those general habits of statisticians which have a merely empirical foundation are not of interest to mathematicians; but certain principles and methods have been brought within the sphere of mathematics by theoretical development and discussion. The book under review was originally prepared (published, 1910) to meet the demand of those who possess only a limited knowledge of mathematics for a systematic elementary exposition of such statistical methods, especially of those developed by Galton, Pearson, Yule, and their collaborators. It has proved so serviceable in carrying out this purpose that five editions after the first have been called for, and the book has won general recognition as in a sense *the* book on statistical method, excluding the technique of gathering data and of graphic presentation.

The general experience of statisticians with earlier editions has been that it has been very difficult for a beginner to get the point of view and that even a person with considerable practical statistical experience found it necessary to read with great care, in order to get at the meaning of the unfamiliar terms and forbidding notation. One difficulty appears to be that the significance of the general principle back of a method or formula is not discussed from the standpoint of the uninitiated. The chapter on dispersion, for instance, begins: "The simplest measure of the dispersion of a series is the actual range" without discussing why any measure of dispersion is desirable or useful. But those who are willing to read and reflect, to come back again and again to the book as their statistical knowledge increases, have found it a veritable gold mine of careful and useful statistical thought.

The book, aside from introduction and appendices, is divided into three parts, the theory of attributes (68 pages), the theory of variables (179 pages), and the theory of sampling (103 pages). The first part, which deals with the methods of determining consistence and association of characteristics of individuals which are discrete in nature, has not entered vitally into current statistical thought, and is of minor interest to mathematicians because it gives no opportunity for the use of continuous variables. The second part discusses frequency distributions, averages, measures of dispersion, and correlation, both single and multiple. Yule's careful and detailed treatment of these subjects has been an important factor in the increasing use of more advanced statistical methods in many fields. His exposition of multiple correlation especially has been the recognized source of information on this topic. In the opinion of the reviewer, however, the treatment of this topic could be made more comprehensible by a full discussion of three-variable correlation before the general case is considered. More attention should be paid also to the limitations of a method which assumes that the functional relationship of n variables is linear. The third part discusses simple sampling, the binomial distribution and the normal curve, the normal correlation surface, and the derivation of probable errors. This part of the work is most hampered by the attempt to keep the difficulties within range of those with limited mathematical equipment.

The most noticeable omissions of the book as a whole are a full treatment of index numbers and the equations of skew frequency curves. The former topic has often been considered as important almost entirely for the economist, and the latter could hardly be discussed without more mathematics than the author assumes.

For the sixth edition, and for the fifth, published in 1919, no extensive revision of the text has been made. Aside from the corrections of minor errors, the changes consist of the addition of supplements and additional references. The supplements deal with I: *Direct deduction of the formulae for regressions* (by means of calculus rather than the Δ method, as in the text). II: *The Poisson law of small chances*, and III: *Goodness of fit*, with discussion and tables for the Pearsonian function χ used in testing goodness of fit of a theoretical to an actual distribution. These supplements appeal especially to the mathematical statistician, and the third one, which covers twenty pages, is of special value as a summary of several highly technical articles. The additional references strengthen the book in what was already one of its strongest points, as the lists of references are very full and well classified, and brief comments often indicate the general nature of the article.

In the opinion of the reviewer, mathematicians who desire to secure familiarity with the notions of statistics should surely include this book in their reading. It is true that many of the discussions could be much abbreviated by the use of higher mathematics, and that some points have to be passed over completely, but Mr. Yule has given a semi-mathematical version of the argument at many points where some other textbooks merely say "the formula is" so and so or "it may be shown that . . .". Until we have a series of statistical textbooks which cover statistical method from all the possible different points of view, Yule will continue to be an essential part of every statistician's library.

R. W. BURGESS

Théorie des Nombres. By M. Kraitchik. Paris, Gauthier-Villars et Cie. 1922. ix + 229 pp.

This book is underwritten by a brief preface by M. d'Ocagne in which a few historical statements are made to give the book its setting. There are six chapters followed by sixty pages of tables. The entire book is built around primes and probably might properly be called theory of rational primes. Possibly algebraic numbers in general, quadratic and other realms, ideals, and other topics which might be expected in a *Théorie des Nombres* are to be taken up in a later volume to which allusion is made (pp. ix, 97, 107, etc.).

Chapter I is a general introduction presenting the classical theory of the rational prime including the *indicateur*. In Chapter II on the linear congruence the author takes up the solving of a system of such congruences in one unknown by a method which is a sort of extension of Eratosthenes, sieve. He then (pp. 43, 44) explains how a calculating machine could be made which would solve such problems mechanically. This and graphical methods employed or suggested by the author (pp. 95, 159 where "troisième Chapitre" probably should be "deuxième Chapitre") indicate his ingenuity.

About a third of the theory in the book is taken up by Chapter IV on congruences of the second degree. This chapter is well done. The law of reciprocity for quadratic residues, Gauss' "gem of the higher arithmetic", is neatly put forth. Possibly for a book of this size some of the applications are a little extreme; as here, Lucas' problem of finding the integers having the same final ten digits as their squares (p. 90).

The Fermat-Euler and Wilson theorems are taken up in the theory contained in Chapter III on congruences of higher degrees. Indices are taken up in Chapter V, and application of a table of indices is made to the solving of binomial congruences of first and second degrees.

Chapter VI is on factorization and discusses methods employing the forms $x^2 \pm Dy^2$. The use of moduli in factoring is also introduced. Six lists of tables are given of which the longer are: II, Linear Divisors of $x^2 \pm Dy^2$ for $D < 200$, IV, A Table of Residues, V, A Table of Indices, VI, Decomposition of $2^n \pm 1$. In the book mention is made to unpublished tables by the author (pp. 119, 126, etc.).

The reviewer has made no attempt to check the tables given; he has found very few typographical errors in the book. M. Kraïtchik is fortunate in being both brief and lucid in his exposition. The following observations may, however, suggest limitations to the use of this *Théorie des Nombres* as a textbook: 1. There are numerous examples throughout the book but no exercises. 2. An appropriate number of mathematicians of this field are mentioned (cf. pp. 9, 22 and the index); but very few references are given.

L. C. Mathewson

Higher Geometry. An Introduction to Advanced Methods in Analytic Geometry. By Frederick S. Woods. Boston, Ginn and Co., 1922. x + 423 pp.

This book makes available in one volume material collected from many books and memoirs. The wide scope of the book can only be indicated by noting the topics treated. Its plan is to treat in succession geometries of one, two, three, and four or more dimensions, the later treatments serving, although at the expense of a little repetition, to go over the general method of treatment again and again.

Part I introduces the ideas of coordinates, duality, imaginaries, infinity, transformations, and groups in a total of seven pages, after which cartesian and projective coordinates of a one-dimensional form, linear transformations, crossratio, harmonic sets, projection, etc. follow rapidly.

Part II, on two-dimensional geometry, introduces homogeneous cartesian point coordinates, trilinear point coordinates, line coordinates, curves in point and line coördinates, linear transformations including their classification by means of geometric properties, projective measurement with the resulting classification of geometries as hyperbolic, elliptic, or parabolic, contact transformations, tetracyclical coordinates, etc.

Part III, on three-dimensional geometry, starts with circle geometry on a plane and continues later with the usual point and plane coordinates in three-space, with treatment of surfaces, geometric classification of collineations and pentaspherical coordinates.

Part IV, on geometry of four and higher dimensions, starts with Plücker's coordinates for lines in ordinary spaces. Sphere coordinates are then treated in detail, with the dualistic relation between line-space and sphere-space. The ordinary four-dimensional point coordinates and a brief consideration of n-dimensional geometry conclude the book.

While it is stated in the preface that the prerequisites are only the elementary courses in analytic geometry and calculus with a slight knowledge of determinants, the reviewer believes that, on account of the large number of new ideas involved, it would have to be diluted and supplemented, even for the ordinary class of seniors and beginning graduates. To understand it, some maturity and some familiarity with the abstract point of view would seem to be necessary.

The book has no definite references to original sources, and no effort is made to connect up with the various books on foundations of mathematics. Also more explicit statements of hypotheses would be helpful to the immature reader who is in danger of making erroneous assumptions.

In spite of the above criticisms, the book is the most stimulating work in geometry that the reviewer has read in years, and it forms a most important addition to the literature of the subject.

F. W. OWENS

Les Méthodes Actuelles de la Balistique Extérieure. By Dufrenois, Risser, and Rousier, with a preface by P. Appell. Paris, Gauthier-Villars, 1921. vii + 237 pp.

In striking contrast to the books on ballistics written by Vahlen and by Hoar, the book under review on present-day methods in exterior ballistics will be a joy to the ordnance engineer or the artillery officer who is interested not only in the theoretical aspects of firing large guns but in making a comparative study of the magnitudes of the quantities involved. The authors are content to confine themselves to the movement of the center of gravity of the projectile; the movement of the projectile about its center of gravity is regarded as such a special and advanced phase of ballistics that it does not belong properly in a manual on present-day methods.

The following features of the book will be of special interest to American readers: (1) a very thorough mathematical treatment of the analytic form of the law of resistance, which has been based upon the work of Langevin; (2) the use of the fact, which is clearly established, that, after the projectile reaches the point on its trajectory where the radius of curvature is a minimum, the path of the projectile will lie between the velocity parabola and the osculating circle, (3) the presence of numerical tables.

Two methods of calculating trajectories by short arcs are described in detail; the method of Gavre, and the method of Rousier-Dufrenois. These methods are similar and may be called with propriety "short arc" methods. In the same sense it would be better, in the reviewer's opinion, to call the method developed in this country during the World War the "short interval" method. The fictitious-velocity method, developed by Sugot, is quite interesting; it consists in calculating the trajectory by a successive approximation of velocities.

A table of comparison of the elements of a large number of trajectories, calculated by the Rousier-Dufrenois and by the Sugot methods, is given. It would be interesting to compare this table with results obtained from our ballistic tables when the same initial conditions are given. Probably this should not be done without the consent or the knowledge of the proper military authorities as such information is regarded as confidential.

The methods employed in correcting for non-standard conditions are about as efficacious as any corrections of this kind can be; until guns and projectiles can be manufactured with greater uniformity such corrections do not deserve the emphasis that usually has been given to them. To anyone who is interested in ballistics the first five chapters of the book will prove to be especially worth-while reading.

J. E. Rowe

Substance and Function, and Einstein's Theory of Relativity. By Ernst Cassirer (Translated by W. C. and M. C. Swabey). Chicago, Open Court, 1923. xii + 465 pp.

Such mathematicians as have sympathy with the philosophy of Immanuel Kant will find in the first part of this volume a valuable discussion of the foundations of mathematics from their special point of view.

Turning to the author's treatment of the philosophical aspects of the theory of relativity, we find (p. 430) the following remarkable statement: "Physics now proves not only the possibility, but the reality of non-Euclidean geometry; it shows that we can only understand and represent theoretically the relations which hold in 'real' space by reproducing them in the language of a four-dimensional non-euclidean manifold." Since, however, any non-euclidean m-space may always be embedded in a Euclidean space of not more than $\frac{1}{2}m(m+1)$ dimensions, it follows that the question whether the physical world is euclidean is meaningless.

Furthermore, if we admit discontinuous mappings, any continuum of m dimensions may always be mapped upon a continuum of n dimensions $(m \neq n)$. Therefore the question "Is the physical world four-dimensional?" is quite meaningless. In the sentence quoted above we have a good example of the fallacy involved when a philosopher takes a science, such as physics, at a given stage of development and concludes that its methods constitute final evidence as to the alleged restraints which the structure of the mind of man imposes upon so much of the world as may be intelligible.

Mathematicians create tools of considerable variety which the physicist may use whenever they suit his purpose. If, for example, he were to find that the consideration of the world line of a particle as the integral curve in a euclidean ten-space of an ordinary homogeneous linear differential equation of the eleventh order yielded significant results, then he would be at liberty to use that method. Accordingly, many of those who do not accept Bertrand Russell's exclusion of metaphysics and ontology from the domain of significant philosophy would doubt the existence of such metaphysical or ontological implications of the physicist's choices between mathematical tools as are drawn by Cassirer. Those who do accept Russel's restriction of philosophy to the field of logic would find Cassirer's discussion of relativity of no interest.

C. N. REYNOLDS, JR.

NOTES

The opening number of volume 46 (1924) of the AMERICAN JOURNAL OF MATHEMATICS contains the following papers: *On the theory of numbers and generalized quaternions*, by L. E. Dickson; *A study of the rational involutorial transformations in space which leave a web of sextic surfaces invariant*, by J. O. Osborn; *On the reduction of differential parameters in terms of finite sets, with remarks concerning differential invariants of analytic transformations*, by O. E. Glenn; *On the isodyadic septimic equations*, by J. C. Glashan.

The concluding number of volume 24, series 2, of the ANNALS OF MATHEMATICS contains: *An introduction to the theory of elliptic functions*, by G. Mittag-Leffler; *Cyclotomic quinquisection for all primes of the form* $10n+1$ *between* 1900 *and* 2100, by P. O. Upadhyaya; *Geodesic lines in Riemann space*, by R. Henderson; *A functional equation from the theory of the Riemann* $\zeta(s)$*-function*, by A. Arwin; *The geometry of paths and general relativity*, by L. P. Eisenhart.

The PUBLICATIONS OF THE UNIVERSITY OF JERUSALEM (SCRIPTA UNIVERSITATIS ATQUE BIBLIOTHECAE HIEROSOLYMITANARUM) will consist of four series, of which two, Mathematica et Physica, curavit A. Einstein, and Orientalia et Judaica, curavit Collegium Eruditorum, began publication in 1923. Articles will be printed in English, French, German or Italian, and will be accompanied by complete translations into Hebrew. Volume 1 (1923) of the mathematical-physical series contains papers by E. Landau, H. Bohr, G. Loria, J. Hadamard, A. Loewy, A. Fraenkel, A. Einstein and J. Grommer, L. S. Ornstein, T. Levi-Civita, T. v. Karman, S. Brodetsky, J. Popper-Lynkeus. Martinus Nijhoff, the Hague, is the publisher.

The Report of the National Committee on Mathematical Requirements has been reprinted, and may be obtained at the cost of the postage by addressing The Dartmouth Press, Hanover, N. H.

The following university courses are announced:

COLUMBIA UNIVERSITY.—By Professor T. S. Fiske: Theory of functions.—By Professor F. N. Cole: Algebra.—By Professor D. E. Smith: History of mathematics; Practicum in the history of mathematics; Seminar in the history of mathematics.—By Professor C. J. Keyser: Introduction to mathematical philosophy; Logical foundations of mathematics.—By Professor Edward Kasner: Seminar in differential geometry.—By Professor W. B. Fite: Differential equations.—By Professor J. F. Ritt: Theory of numbers.—By Professor G. A. Pfeiffer: The theory of sets of points.—By Dr. J. Douglas: Differential geometry.

CORNELL UNIVERSITY.—By Professor J. I. Hutchinson: Calculus of residues.—By Professor Virgil Snyder: Advanced analytic geometry;

Algebraic geometry.—By Professor F. R. Sharpe: Fourier series and potential functions.—By Professor A. Ranum: Differential geometry.—By Professor W. B. Carver: Projective geometry; Metric geometry in the plane.—By Professor W. A. Hurwitz: Theory of groups; Vector analysis.—By Professor C. F. Craig: Theory of ordinary differential equations.—By Professor F. W. Owens: Theory of probabilities.—By Professor H. C. M. Morse: Elementary differential equations; Calculus of variations.—By Professor W. L. G. Williams: Linear algebras; Galois fields.—By Mr. H. S. Vandiver: Theory of equations.—By Dr. D. S. Morse: Advanced calculus; Infinite series.

HARVARD UNIVERSITY.—By Professor W. F. Osgood: Advanced calculus (second semester); Theory of functions.—By Professor J. L. Coolidge: Elementary theory of equations; Probability; Geometry of the circle.—By Professor O. D. Kellogg: Advanced calculus (first semester); Theory of potential functions; Dynamics (second course).—By Professor G. D. Birkhoff: Analytical theory of heat; Elastic vibrations; Linear differential equations of the second order, real variables.—By Professor W. C. Graustein: Introduction to modern geometry; Differential geometry of curves and surfaces.—By Professor J. L. Walsh: Functions of real variables, part II; Integral equations.—By Mr. H. W. Brinkmann: Galois theory of equations.—By Mr. J. L. Holley: Functions of real variables, part I.—Professor Osgood and Mr. Brinkmann will conduct a fortnightly seminar in analysis. Courses of research are also offered by Professor Osgood in analysis, by Professor Coolidge in geometry, by Professor Kellogg in potential theory, by Professor Birkhoff in the theory of differential equations, by Professor Graustein in geometry, and by Professor Walsh in analysis.

UNIVERSITY OF ILLINOIS.—By Professor E. J. Townsend: Real variables; Differential equations and advanced calculus.—By Professor G. A. Miller: Theory of groups (introductory course); Theory of equations.—By Professor J. B. Shaw: Linear algebra.—By Professor A. B. Coble: Cremona transformations.—By Professor R. D. Carmichael: Linear differential equations.—By Professor Arnold Emch: Geometric transformations.—By Professor A. R. Crathorne: Theory of statistics.—By Professor G. E. Wahlin: Modern algebra.—By Professor A. J. Kempner: Theory of numbers.—By Professor H. Blumberg: Introduction to modern mathematics.—By Professor E. B. Lytle: History of mathematics.

The Société Mathématique de France celebrated its fiftieth anniversary May 22, 1924.

The Académie Royale de Belgique has received from an anonymous donor the sum of 3000 francs to found a prize for a contribution to the theory of the diurnal nutation and a confirmation of its existence.

Professor A. Einstein has been awarded by the assembly of the University of Amsterdam the gold medal of the Holland Society for the Progress of Natural Science.

Cambridge University has awarded Smith's prizes to T. M. Cherry, of Trinity College, for an essay *On the differential equations of dynamics,* and to W. J. Webber, of Trinity College, for an essay on *Some applications of the theory of integration.*

Professor A. S. Eddington has been awarded the Bruce gold medal of the Astronomical Society of the Pacific. The award was for his work in relativity and on the internal physical constitution of the stars.

The Leopoldinisch-Carolinische Akademie of Halle has elected the following members: Professors L. Bieberbach, O. Blumenthal, L. E. J. Brouwer, G. Castelnuovo, F. Enriques, T. Hayashi, R. von Mises, E. Neumann, F. Severi, and H. Weyl. The election of Professor W. F. Osgood to membership in this Academy was announced in the May-June issues of this BULLETIN.

Professor M. Pasch, of Giessen, has received honorary doctorates from the Universities of Frankfurt a. M. and Freiburg i. Br.

Professor E. Landau has been elected corresponding member of the Prussian Academy of Sciences.

M. G. Bigourdan has been elected president of the Paris Academy of Sciences. M. Maurice de Broglie has been elected académicien libre, in succession to the late M. Charles de Freycinet.

The University of Paris has decided to confer an honorary doctorate on Professor H. A. Lorentz, of the University of Leiden.

Professor Emile Borel delivered a series of lectures at the University of Copenhagen in March, 1924, on the Rask-Oersted foundation. On this occasion the University conferred on him the degree of doctor *honoris causa.*

Professor Max Planck has been elected an honorary member of the Physical Society of London.

Professor T. A. Brown, of University College, Exeter, and Mr. E. T. Copson and Professor C. G. Darwin, of Edinburgh, have been elected fellows of the Royal Society of Edinburgh.

Professor L. J. Mordell, of the University of Manchester, has been elected fellow of the Royal Society of London.

Professor G. H. Hardy, of Oxford, has been elected president of the British National Union of Scientific Workers.

Dr. W. D. Lambert, of the United States Coast und Geodetic Survey, has been elected recording secretary of the Washington Academy of Sciences.

Dr. F. Bohm has been appointed associate professor of mathematics at the University of Munich.

Professor H. L. Hamburger, of the University of Berlin, has been appointed professor of mathematics at the University of Cologne.

Dr. E. Kruppa, of the Vienna Technical School, has been promoted to a full professorship of mathematics.

Dr. F. Levi, of the University of Leipzig, has been promoted to an associate professorship of mathematics.

Dr. M. Winkelmann, of the University of Jena, has been appointed full professor of applied mathematics.

The following German professors of mathematics have recently retired from active teaching: F. von Lindemann, A. Schleiermacher, F. Schur, and P. Volkmann.

Professor E. Bompiani, of the Milan Technical School, has been transferred to the chair of projective and descriptive geometry at the University of Bologna, and Professor M. Picone, of the University of Catania, to the chair of higher analysis at the University of Pisa.

The following 25 doctorates with mathematics as major subject were conferred by American universities in the calendar year 1923; the university, month in which degree was conferred, minor subject if any outside of mathematics, and the title of the dissertation are given with each name.

E. F. Allen, Missouri, April, Physics, *A revision of certain topics of the Lie theory.*

Constance R. Ballantine, Chicago, June, *Modular invariants of a binary group with composite modulus.*

J. P. Ballantine, Chicago, June, *A postulational introduction to the four-color problem.*

A. D. Campbell, Cornell, June, Philosophy, *The classification of linear systems of conics in various domains.*

W. E. Clelland, Princeton, October, *Permutable transformations F and transformations W.*

M. M. Feldstein, Chicago, March, *Invariants of the linear group, modulo* p^r.

C. A. Garabedian, Harvard, June, *A method of series in elasticity, with applications* (1) *to circular plates of constant or variable thickness, and* (2) *to rods of constant or variable circular cross-section.*

R. E. Gleason, Princeton, October, *On a calculus of average-value functions.*

B. Z. Linfield, Harvard, June, *On the theory of discrete variates.*

N. B. MacLean, Chicago, August, *On certain surfaces related covariantly to a given ruled surface.*

D. S. Morse, Cornell, June, Physics, *Relative inclusiveness of certain definitions of summability.*

F. H. Murray, Harvard, June, *The real solutions of certain systems of differential equations.*

H. L. Olson, Chicago, August, *Congruences with constant absolute invariants.*

J. O. Osborn, Cornell, June, Education, *A study of the rational involutorial transformations in space which leave a web of sextic surfaces invariant.*

G. E. Raynor, Princeton, June, *Dirichlet's problem.*

Emeterio Roa, Michigan, June, *A number of new generating functions, with applications to statistics.*

G. E. F. Sherwood, Chicago, August, *Equivalence of triples of bilinear forms.*

J. M. Thomas, Pennsylvania, June, *Congruences of circles studied with reference to the surface of centers.*

T. Y. Thomas, Princeton, June, *Geometries of paths admitting first integrals.*

C. E. Van Horn, Chicago, March, *A system of relative existential propositions connected with the relation of class-membership.*

F. M. Weida, Iowa, July, Physics, *The valuation of life annuities with refund of an arbitrarily assigned part of the purchase price.*

Louis Weisner, Columbia, February, *Groups whose maximum cyclic subgroups are independent.*

R. L. Wilder, Texas, June, *Concerning continuous curves.*

R. E. Wilson, Chicago, March, *Representations of certain functions of two variables by Stieltjes integrals.*

B. F. Yanney, Chicago, March, Astronomy, *Modular invariants of the binary quartic.*

Professor G. D. Birkhoff, of Harvard University will be exchange professor at Pomona College, Colorado College, and Grinnell College during the first half of the academic year 1924-25. During his residence at Pomona College, he will deliver a series of eight lectures on *Relativity* at the Southern Branch of the University of California.

Professor C. de la Vallée-Poussin, who was president of the International Mathematical Union at the International Congress in Toronto, will remain in this country this Fall and will lecture under the auspices of the Educational Foundation established by the Commission for the Relief of Belgium at several universities, on the two topics:

(1) *Les fonctions d'ensemble et les fonctionelles,*

(2) *L'approximation des fonctions de variables reélles et les fonctions quasi-analytiques.*

A partial list of the institutions at which he will lecture, approximately in the order of time, follows: (August) The University of Chicago; (September) The University of California at Berkeley, The Southern Branch of the University of California at Los Angeles, California

Institute of Technology; (October) The University of Wisconsin, The University of Minnesota; The University of Michigan; Cornell University; Columbia University; (November) Massachusetts Institute of Technology, Brown University, Yale University; (December) Princeton University, Johns Hopkins University.

At Harvard University, Mr. H. W. Brinkmann and Mr. J. L. Holley have been appointed Benjamin Peirce instructors, and Mr. P. D. Edwards and Mr. Malcolm MacLaren, Jr. instructors, for the year 1924–25. Dr. J. L. Walsh has been promoted to an assistant professorship of mathematics.

Professor L. L. Silverman, who has just returned from a year's leave of absence in Germany and Russia, has been promoted to a full professorship at Dartmouth College.

Professor E. R. Hedrick, of the University of Missouri, has been appointed professor of mathematics at the University of California, and will be assigned to the department of mathematics at the Southern Branch at Los Angeles.

The following appointments and promotions in the department of mathematics at Hunter College of the City of New York have been announced: to succeed Professor Emma M. Requa, retired, as head of the department, Professor Tomlinson Fort, of the University of Alabama; to be an associate professor, Miss Lao G. Simons, who has been acting head of the department for the year 1923–24; to be an instructor, Miss Helen Kunte.

Professor A. N. Whitehead, of the Imperial College of Science and Technology, London, has been appointed professor of philosophy at Harvard University.

Associate Professor Clara E. Smith has been promoted to a full professorship of mathematics at Wellesley College.

Dr. G. A. Pfeiffer, of Columbia University, has been promoted to an assistant professorship of mathematics.

Professor H. Frischauf, of the University of Graz, died January 7, 1924, at the age of eighty-six years.

The death is announced of Professor G. Helm, of the Dresden Technical School, at the age of seventy-two years.

Professor J. Majcen, of the University of Zagreb (Agram) died February 1, 1924, at the age of forty-nine years.

Professor H. Rothe, of the Vienna Technical School, died December 18, 1923.

Dr. R. E. Froude, C. B., F. R. S., known for his work in naval architecture, died March 19, 1924.

Dr. William Jack, formerly professor of mathematics at the University of Glasgow, died March 20, 1924.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

APOLLONIUS. See VER EECKE (P.).

BERDIALES (B.). Estudio de la inversión en el plano. (Publicaciones del Circulo Matemático del Instituto Nacional del Profesorado Secundario, No. 2.) Buenos Aires, 1924, 39 pp.

BLASCHKE (W.). Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie. I: Elementare Differentialgeometrie. 2te verbesserte Auflage. Berlin, Springer, 1924. 12 + 242 pp.

BORTOLOTTI (E.). Lezioni di geometria analytica. Bologna, Zanichelli, 1923. Volume 1: 39 + 382 pp. Volume 2: 229 pp.

BRANDENBERGER (C.). Abgekurztes Rechnen. Zürich, Füssli, 1922.

COOLIDGE (J. L.). The geometry of the complex domain. Oxford, Clarendon Press, and New York, Oxford University Press, 1924. 242 pp. $6.00

DOEHLEMANN (K.). Projektive Geometrie in synthetischer Behandlung. II. 5te Auflage. (Sammlung Goschen.) Berlin, de Gruyter, 1924. 138 pp.

DROSTE (J.) en DE GROOT (W. F.). Functies. Deel 1: Graphische voorstellingen. Groningen, Wolters, 1923. 163 pp.

ENRIQUES (F.). Per la storia della logica. I principii e l'ordine della scienza nel concetto dei pensatori matematici. Bologna, Zanichelli, 1922. 300 pp.

GAMBIOLI (D.). See YOUNG (J. W. A.).

GEIGER (M.). Systematische Axiomatik der euklidischen Geometrie. Augsburg, Dr. Benno Filser Verlag, 1924. 24 + 271 pp.

VAN DE GRIEND (J.). Voortgezette planimetrie. Hilversum, J. Schaafstal, 1923. 102 pp.

DE GROOT (W. F). See DROSTE (J.).

GROSGURIN (L.). Enseignement de l'arithmétique. Méthodologie. Lausanne, Payot, 1922. 300 pp.

LAMB (H.). An elementary course of infinitesimal calculus. Revised edition. Cambridge, University Press, 1924. 540 pp.

LARSSON (L.). See NÖRLUND (N. E.).

LECAT (M.). Bibliographie de la relativité suivie d'un appendice sur les déterminants à plus de deux dimensions, le calcul des variations, les séries trigonométriques et l'azéotropisme. Bruxelles, Lamertin, 1924. 292 + 47 pp.

LEFSCHETZ (S.). L'analysis situs et la géométrie algébrique. Paris, Gauthier-Villars, 1924. 154 pp.

Lietzmann (W.) und Trier (V.). Wo steckt der Fehler? 3te Auflage. (Mathematisch-physikalische Bibliothek, Band 52.) Leipzig, Teubner, 1923.

Mach (E.). Populär-wissenschaftliche Vorlesungen. 5te vermehrte und durchgesehene Auflage. Leipzig, Barth, 1923. 12 + 628 pp.

Miller (B. I.). Romance in science. Boston, Stratford Company, 1924. 87 pp. $1.00

Norlund (N. E.). Bland siffror och tal. Populara uppsatser. Til svenska av Robert Larsson. Stockholm, Hugo Gebers Forlag, 1923. 113 pp.

d'Ocagne (M.). Calcul graphique et nomographie. Paris, Doin, 1924.

Poincaré (H.). Calcul des probabilités. Rédaction d'A. Quiquet. 2e édizione, revue et augmentée. Paris, Gauthier-Villars, 1923. 336 pp.

Quiquet (A.). See Poincaré (H.).

Schneider (E.). Mathematische Schwingungslehre. Theorie der gewohnlichen Differentialgleichungen mit konstanten Koeffizienten sowie einiges über partielle Differentialgleichungen und Differenzengleichungen. Berlin, Springer, 1924. 6 + 194 pp.

Schuh (F.). Vraagstukken over differentiaal- en integraalrekening en over analytische en beschrijvende meetkunde. Tweede deel. Groningen, Noordhoff, 1923. 198 pp.

Schluter (H.). Die hohere Mathematik. Eine gemeinverstandliche Darstellung der Elemente. 2te verbesserte Auflage. Berlin, Meusser, 1922.

Siceloff (L. P.) and Smith (D. E.). College algebra. Boston, Ginn, 1924. 6 + 258 pp.

Smith (D. E.). History of mathematics. Volume 1: General survey of the history of elementary mathematics. Boston, Ginn, 1923. 22 + 596 pp.

———. See Siceloff (L. P.).

Thorndyke (E. L.) and others. The psychology of algebra. New York, Macmillan, 1923. 11 + 483 pp.

Trier (V.). See Lietzmann (W.).

Tropfke (J.). Geschichte der Elementar-Mathematik in systematischer Darstellung. Band 6: Analysis, analytische Geometrie. 2te, verbesserte Auflage. Berlin, de Gruyter, 1924. 169 pp.

Valeiras (A.). Estudio de una correspondencia geométrica. (Publicaciones del Circulo Matemático del Instituto Nacional del Profesorado Secundario, No. 1.) Buenos Aires, 1923. 45 pp.

Ver Eecke (P.). Les coniques d'Apollonius de Perge. Traduites du grec en français, avec une introduction et des notes par Paul Ver Eecke. Bruges, Desclée, de Brouwer et Cie., 1924.

Young (J. W. A.). L'insegnamento delle matematiche. Traduzione di D. Gambioli. Palermo, Sandron, 1924. 28 + 526 pp.

PART II. APPLIED MATHEMATICS

ADAMS (H.). The mechanics of building construction. New edition. New York and London, Longmans, 1923. 251 pp.

ALLIATA (G.). Die Planetenanomalien im Weltbild der Aethermechanik. Leipzig, Hillmann, 1922. 16 pp.

BALDIT (A.). Etudes élémentaires de météorologie pratique. 2e édition. Paris, Gauthier-Villars, 1923. 13 + 428 pp.

BOHR (N.). Om Atomernes Bygning. København, Gjellerup, 1923. 44 pp.

BOREL (E.). See PAINLEVÉ (P.).

BOUASSE (H.). Gyroscopes et projectiles. Paris, Delagrave, 1923. 200 pp.

BOUNY (F.). Leçons de mécanique rationnelle. Tome 1: Calcul vectoriel, cinématiqué, potentiel. Paris, Blanchard, et Mons, Librairie Leich, 1924. 8 + 600 pp.

BROSE (H. L.). See FREUNDLICH (E.).

BÜCHLER (R.). Lehrsätze über das Weltall mit Beweis in Form eines offenen Briefes an Professor Einstein. Aachen, Aachener Verlags- und Druckereigesellschaft, 1923. 11 pp.

BUNET (P.). Les transformateurs. Paris, J. B. Baillière, 1923. 632 pp.

CASKEY (L. D.). Geometry of Greek vases. Boston, Museum of Fine Arts, 1922. 4to. 11 + 235 pp. $6.00

CASSIRER (E.). Substance and function and Einstein's theory of relativity. Authorized translation by W. C. Swabey and M. C. Swabey. Chicago and London, Open Court, 1923. 12 + 465 pp. $3.75

COPPERNICUS. Über die Umdrehungen der Himmelskorper. Aus seinen Schriften und Briefen. Posen, Verlag der Deutschen Bücherei, 1923. 77 pp.

DAMIENS (A.). Les Isotopes. Paris, Gauthier-Villars, 1923. 9 + 118 pp.

DINGLER (H.). Die Grundlagen der Physik. Synthetische Prinzipien der mathematischen Naturphilosophie. 2te, völlig neubearbeitete Auflage. Berlin, de Gruyter, 1923. 14 + 336 pp.

EINSTEIN (A.), LORENTZ (H. A.), MINKOWSKI (H.), SOMMERFELD (A.), and WEYL (H.). The principles of relativity. Translated by G. B. Jeffery and H. Weyl. London, Methuen, 1923. 8 + 216 pp.

ERIKSSON (K. V.). Astronomiens grunder. Nya upptackter. Goteborg, Josef Bergendahls Boktryckeri, 1923. 60 pp.

EWALD (P. P.). Kristalle und Röntgenstrahlen. Berlin, Springer, 1923. 9 + 327 pp.

FOPPL (O.). Grundzuge der technischen Schwingungslehre. Berlin, Springer, 1923. 6 + 151 pp.

FREUNDLICH (E.). The theory of relativity. Three lectures for chemists. Translated by H. L. Brose. London, Methuen, 1924. 12 + 98 pp.

HART (I. B.) and LAIDLER (W.). Elementary aeronautical science. Oxford, Clarendon Press, 1923. 6 + 288 pp.

HART (W. L.). The mathematics of investment. Boston, Heath, 1924. 12 + 220 + 88 pp.

HEWES (L. I.) and SEWARD (H. L.). The design of diagrams for engineering formulas and the theory of nomography. London and New York, McGraw-Hill, 1923. 13 + 111 pp.

HUEBOTTER (H. A.). Mechanics of the gasoline engine. London, McGraw-Hill, 1923. 9 + 313 pp.

INTERNATIONAL RESEARCH COUNCIL. Second assembly held at Brussels, July 25 to July 29, 1922. Report of proceedings edited by Arthur Schuster. London, Harrison, 1923. 145 pp.

JEANS (J. H.). Report on radiation and the quantum theory. 2d edition. London, Fleetway Press, 1924.

JEFFERY (G. B.). See EINSTEIN (A.).

LAIDLER (W.). See HART (I. B.).

VON LAUE (M.). Die Relativitätstheorie. 2te umgearbeitete Auflage. Band 2. Braunschweig, Vieweg, 1923. 13 + 290 pp.

LEDUC (A.). Thermodynamique. Energétique. Théorie cinématique des gaz. Paris, Doin, 1923. 333 pp.

LEGGE (W. S.). See NORRIS (P. W.).

LENARD (P.). Über Äther und Urather. 2te, vermehrte Auflage mit einem Mahnwort an deutsche Naturforscher. Leipzig, Hirzel, 1922. 66 pp.

LEWITT (E. H.). Hydraulics. London, Pitman, 1923. 8 + 264 pp.

LORENTZ (H. A.). See EINSTEIN (A.).

MARTENS (F.). Das Wesen der elektrischen Erscheinungen. Elberfeld, Hofbauersche Buchhandlung, 1922. 32 pp.

MAURAIN (C.). Physique du globe. Paris, Armand Colin, 1923. 204 pp.

———. See PAINLEVÉ (P.).

MINKOWSKI (H.). See EINSTEIN (A.).

MOHOROVIČIĆ (S.). Die Einsteinsche Relativitatstheorie und ihr mathematischer, physikalischer und philosophischer Charakter. Berlin, de Gruyter, 1923.

NENNING (A.). Kosmische Dynamik. München-Zurich, J. A. Mahr, 1923. 64 pp.

NORRIS (P. W.) and LEGGE (W. S.). Mechanics via the calculus. New York, Longmans, 1923. 375 pp.

PAINLEVÉ (P.), BOREL (E.) et MAURAIN (C.). L'aviation. Nouvelle édition. Paris, Alcan, 1923. 308 pp.

SCHUSTER (A.). See INTERNATIONAL RESEARCH COUNCIL.

SOMMERFELD (A.). See EINSTEIN (A.).

SWABEY (M. C.). See CASSIRER (E.).

SWABEY (W. C.). See CASSIRER (E.).

WEYL (H.). See EINSTEIN (A.).

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THE MAY MEETING IN NEW YORK

The two hundred thirty-sixth regular meeting of the Society was held at Columbia University, on Saturday, May 3, 1924, extending through the usual morning and afternoon sessions. The attendance included the following forty-nine members of the Society.

Archibald, Babb, C. R. Ballantine, J. P. Ballantine, Beal, Bernstein, W. J. Berry, Herman Betz, Borden, Bowden, Daniel Buchanan, Abraham Cohen, Philip Franklin, Gafafer, Gehman, Gill, Glenn, Hazeltine, Hazlett, E. R. Hedrick, Himwich, Huntington, Joffe, Kasner, Kline, Lamson, Langman, Harry Levy, Locke, Longley, MacDuffee, Meder, C. L. E. Moore, Northcott, Pfeiffer, Rainich, Raynor, R. G. D. Richardson, Ritt, L. B. Robinson, Rowe, Seely, Sosnow, Vandiver, Veblen, H. S. White, Whittemore, Norbert Wiener, M. M. Young.

No meeting of the Council was held. At a meeting of the Board of Trustees, the finances of the Society were reviewed.

The three amendments to the By-Laws recommended by the Council at the April meeting were unanimously approved by the Society. The first concerns the elimination of ephemeral regulations governing the business of the Society during the period of incorporation and the first year thereafter. The second permits the Council to transact business by correspondence provided there is a favorable vote of at least fifteen, the favorable vote to be at least three-quarters of the votes transmitted to the Secretary. The third introduces a new class of membership to be known as Sustaining Membership, for which the annual dues shall be not less than one hundred dollars. The revised By-Laws will be printed in full in the List of Officers and Members to be issued late in 1924.

President Veblen presided at the sessions of the Society, relieved by Professors White, Whittemore, and C. L. E. Moore. At the beginning of the afternoon session, a paper was read by Professor J. F. Ritt, at the request of the Program Committee, on *Rational substitutions*.

Titles and abstracts of the papers presented at this meeting follow below. Mr. Kormes was introduced by Dr. Pfeiffer, Mr. Widder by Professor Birkhoff, and Mr. Sheffer and Mr. Roos by Professor Evans. The papers of Bray, Foster, Maria, Michal, Moore, Roos, Sheffer, Taylor, Wedderburn, Widder, and Wilder were read by title.

1. Dr. Norbert Wiener: *The Dirichlet problem.*

The author develops a necessary and sufficient condition for the solvability of the Dirichlet problem in any number of dimensions, and applies this criterion to the deduction and extension of a large number of existing criteria for the solvability of the Dirichlet problem.

2. Dr. Philip Franklin: *Analytic transformations of everywhere dense point sets.*

It is well known that any two enumerable, everywhere dense point sets on a straight line may be mapped on one another by a one to one transformation which preserves order. We show here that we may set up an analytic function, analytic in the interval covered by the first point set, its inverse being analytic in the interval covered by the second point set, which effects the mapping. Some extensions to point sets everywhere dense in a two-dimensional region are given.

3. Dr. Philip Franklin: *Functions with an isolated essential singularity.*

In this note we prove some properties of functions with an isolated essential singularity, derived from Picard's theorem, of which the following are typical: If an entire function has no zeros, any function obtained from it by changing a finite number of coefficients in its Taylor's expansion about any point has an infinite number of zeros. The function formed by adding a polynomial, not a constant, to a periodic function, not a constant, possessing no poles, always has an infinite number of zeros.

4. Professor J. R. Kline: *Concerning the sum of two bounded continua irreducible between the same pair of points.*

In the present paper, the author obtains necessary and sufficient conditions that two distinct bounded continua,

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each irreducible between the same pair of points, may have a sum irreducible between two of its points.

5. Mr. H. M. Gehman: *On extending a continuous* (1—1) *correspondence of two plane continuous curves to a correspondence of their planes.*

In this paper, it is proved that if two plane continuous curves that contain no simple closed curve are in continuous (1—1) correspondence in such a way that sides of arcs are preserved under the correspondence, a continuous (1—1) correspondence of their planes can be defined so that the correspondence of the two continuous curves is preserved.

6. Mr. Mark Kormes: *Treatise on basis sets.*

Algebraic (rational, linear) fields are defined, and their properties discussed. By the use of these fields, a definition of algebraic (linear) basis sets is given. This definition includes the definitions of Hamel, Zermelo, and E. Noether as special cases. The following theorems are proved: There exist algebraic (linear) basis sets for each set M. The basis set of a non-denumerable set M has the same potency as the set M, and there exist $2^{\aleph_\mu}$ different basis sets of M, if $\aleph_\mu$ is the potency of M. There exist basis sets that are measurable in Lebesgue's sense, but they have the measure zero. There exist also basis sets that are not measurable (L). A theorem of Sierpinski is generalized: Every algebraic basis set of the continuum is non-measurable in Borel's sense. The basis sets of a linear space are considered. It is shown, in consequence of a general theorem, that the linear basis set of all continuous functions has the potency of the continuum. New sets S^M are defined and considered as to their potency and measurability properties. An example of a perfect set which is a basis set is given.

7. Professor O. E. Glenn: *Differential combinants.* Preliminary report.

The subject of this paper is the algorisms and fundamental theorems of a complete theory of differential combinants and associated forms. Combinants of n forms $\alpha^m_{dx}, \beta^m_{dx}, \cdots$ are found to be functions of determinants of order n of a matrix whose elements are derivatives of the coefficients of the forms. A translation principle leading to ternary parameters is explained.

8. Professor H. S. White: *A set of invariants connected with seven points on a gauche cubic.*

While 7 points on one gauche cubic curve are known to determine 7 osculating planes of a second, the algebraic nature of the discovery of the parameters of those 7 planes has not been known in detail. In this paper it is shown that this work leads inevitably to a system of 28 relations, of similar form in the two sets of parameters, *invariant under the well known group of order* 168. These 28 relations are equivalent to 4, and are *solvable rationally* for either set of parameters, except for the 3 residual arbitraries inherent in a collineation.

9. Professor R. L. Moore: *A characterization of a continuous curve.*

If A and B are points belonging to a connected point set M, a point set K is said to separate A from B in M if K is a proper subset of M and $M-K$ is the sum of two mutually separated point sets which contain A and B respectively. It is shown that in order that a plane continuum M shall be a continuous curve it is necessary and sufficient that, for every two points A and B that belong to M, there exist a finite number of continua whose sum separates A from B in M. In order that a bounded plane continuum M shall be a continuous curve which neither contains a domain nor separates the plane, it is necessary and sufficient that, for every two distinct points A and B which belong to M, there exist a point which separates A from B in M.

10. Dr. R. L. Wilder: *A theorem on continua.*

In the above paper by Professor Moore, the following property of continuous curves is established: Of two concentric circles, C_1 and C_2, let C_1 be the smaller. Denote by H the point set which is the sum of C_1 and C_2, and the annular domain bounded by C_1 and C_2. Let M be a continuous curve which contains a point A interior to C_1 and a point B exterior to C_2. If N is any connected subset of M containing A and B, N will contain at least one point of some continuum which is a subset of M and H, and which has at least one point in common with each of the circles C_1 and C_2. It is shown in the present paper that this is a property of *any* closed and connected point set M.

11. Mr. D. V. Widder: *A general mean-value theorem.*

This paper deals with the following problem. Given a linear differential expression, $L[u(x)]$, of order n, the coefficients being continuous in an interval (a, b). Under what conditions will $n+1$ conditions on the zeros of a function $f(x)$ and of its derivatives, $f^{(h_i)}(x_i) = 0$ $(i = 0, 1, 2, \ldots, n)$, insure the change of sign of $L[u(x)]$ in (a, b)? A necessary and sufficient condition is obtained. Special cases of the problem have been treated by G. D. Birkhoff and by G. Pólya. The present paper establishes connections with their results.

12. Mr. A. D. Michal: *Integro-differential invariants of one-parameter groups of projective transformations of function space.* Preliminary report.

In this paper the author studies functionals of a function y and its derivatives admitting a given arbitrary one-parameter group of projective functional transformations defined by $\delta y(x) = [\alpha(x) y(x) + \int_0^1 H(x, s) y(s)\, ds]\, \delta a$. In particular, it is shown that a necessary and sufficient condition that an "analytic" functional $F\left[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}\right]$ admit this group of functional transformations is that it satisfy a certain completely integrable equation in partial functional derivatives (cf. an article by the author in a forthcoming issue of the ANNALS OF MATHEMATICS). The expansion of the solution of this equation is obtained. The above discussion assumes $\alpha(x)$ and $H(x, s)$ to be such as to insure the non-vanishing of a certain Fredholm determinant. The author also considers the invariants when the above general method breaks down.

13. Mr. J. M. Sheffer: *Systems of equations in an infinite number of unknowns whose solutions involve an arbitrary parameter.*

The author considers equations of the form

$$x_i + \sum_{j=i+1}^{\infty} (\lambda_j + b_{ij}) x_j = c_i, \qquad (i = 1, 2, \cdots),$$

which are not of a canonical type, and imposes sufficient conditions that the equations have a solution. He finds a set of solutions depending on an arbitrary parameter appearing in the general solution of incidental difference equations. The system is characterized by the hypothesis that the limit of λ_j with j infinite is not zero.

14. Mr. C. F. Roos: *Mathematical theory of competition.*

The author considers a question in economic dynamics by applying the postulates of competition to the formula for total profit over an interval of time. The question, unlike the case of monopoly, does not reduce to a straight problem in the calculus of variations. Moreover it is related to the theory of integral equations and their application to phenomena of hysteresis.

15. Dr. J. S. Taylor: *The theory of testimony. Second paper: n witnesses.*

In another paper (JOURNAL OF MATHEMATICS AND PHYSICS, vol. 3, No. 3) the author presented certain theorems concerning the probability of the occurrence of an event when it is attested to by a single witness. In the present paper the investigation is extended to the case of n witnesses, with special attention to the case of two witnesses, the traditional treatment of which has been severely criticised by J. M. Keynes. Methods are discussed of determining approximately both the "credibility" and the "personal accuracy" of an observer by means of the testimonies of n observers.

16. Professor J. F. Ritt: *Rational substitutions.*

This address dealt with the work performed during the last few years by Julia, Fatou, and the author on the iteration of rational functions and on related problems.

17. Professor J. F. Ritt: *Analytic functions and periodicity.*

This paper appeared in full in the October number of this BULLETIN.

18. Mr. G. Y. Rainich: *The curvature of space-time, the electromagnetic tensor, and radiation.*

In an earlier paper (PROCEEDINGS OF THE NATIONAL ACADEMY, April, 1924) the author found that the presence of the electromagnetic field imposes on the contracted curvature tensor F_{ij} certain conditions which he gave in geometric form; in the present paper these conditions are given analytically as follows: $F_{\sigma\sigma}=0$, $F_{\sigma i}F_{\sigma j}=\delta_j^i\omega^2$ and $q_{ij\sigma,\sigma}=0$ with $q_{ijk}\cdot\omega^2=F_{\sigma i,j}F_{\sigma k}+F_{\sigma j,k}F_{\sigma i}+F_{\sigma k,i}F_{\sigma j}$. These conditions being satisfied, the curvature tensor determines the electromagnetic tensor as soon as the value of a certain scalar quantity is given at one point. This quantity plays in the simplest case the role of the phase of radiation.

19. Dr. Malcolm Foster: *Note on a special congruence.*

This paper appears in full in the present number of this BULLETIN.

20. Mr. Harry Levy: *Ricci's coefficients of rotation.*

This paper will appear in full in an early number of this BULLETIN.

21. Professor Edward Kasner: *The addition of quadratic differential forms.*

In this paper the author considers quadratic differential forms in n variables which can be regarded as the sum of two or more forms in separated variables. For example, if $n = 4$, the possible types of separation are (1, 1, 1, 1), (1, 1, 2), (1, 3), and (2, 2). In special cases, the given form may belong to more than one type, so that the problem of overlapping types must be studied. The type (1, 1, 1, 1) of course defines flat space. It is proved that if the Einstein equations are to be satisfied the only separated type possible is (2, 2), and then the result is the sum of two binary forms representing equal spheres. The immersing of separated spaces in higher flat spaces is also studied.

22. Professor Edward Kasner: *Einstein solutions in space of seven dimensions.*

In his second paper, the author considers the summation of forms involving common independent variables. This is applied to the decompositions of solutions of the cosmological field equations whose coefficients are functions of a single variable, say x_1. These may be broken up into the sum of three binary forms belonging to three surfaces of revolution. The final four-dimensional manifold may then be constructed synthetically and immersed in a seven-flat. The rotating curves involve abelian integrals.

23. Professor J. H. M. Wedderburn: *Algebras which do not possess a finite basis.*

In the classification of algebras which do not necessarily possess a finite basis difficulties arise which are not found in the corresponding theory when a finite basis is present. In this paper no basis is assumed to exist, but it is found that the principal results of the ordinary theory can be deduced with the aid of some simple assumptions regarding idempotent elements. Several examples are given.

24. Professor J. H. M. Wedderburn: *A theorem on simple algebras.*

This paper will appear in full in an early number of this Bulletin.

25. Dr. H. E. Bray: *Green's formula.*

In this paper a new proof of Green's formula, $\int\int_S(\partial Q/\partial x)\,dxdy = \int_C Qdy$, is given, where the boundary C is a rectifiable closed curve without multiple point and the quantities in the two members are respectively Lebesgue and Stieltjes integrals. The theorem is briefly proved in the case where Q is continuous in (x,y) and absolutely continuous in x for every y. Next the condition that Q be continuous is removed; Q is assumed to be bounded and equal to the symmetric superficial derivative of its own Lebesgue integral, and the line integral is treated as a Lebesgue-Stieltjes integral. Finally the Stieltjes potential function $\int\int_E \log(1/r)\,df(e)$ is discussed, and the equation

$$\int\int_S dxdy \int\int_E \frac{\cos xr}{r}\,df(e) = \int_C dy \int\int \log(1/r)\,df(e)$$

is shown to be valid for nearly every position into which the curve C may be brought be translation.

26. Mr. A. J. Maria: *Functions of plurisegments.*

In the present paper the analysis of Vitali (Rendiconti di Palermo, vol. 46 (1922), p. 388) is applied to the two-dimensional case of functions of plurisegments. To determine the structure of functions of limited variation of this type we associate with every such function an additive function, called a discard. This function measures the quantity by which the function falls short of being absolutely continuous when there are no point values. It is shown that the property which characterizes a discard is that the discard coincides with its own discard. The theorem is proved that every such function can be decomposed into the sum of a function of point values, an absolutely continuous function, and a finite sum or infinite series, each term of which is an elementary discard multiplied by a constant.

R. G. D. Richardson,
Secretary.

ON SIMPLE GROUPS OF LOW ORDER*

BY F. N COLE

1. *Introduction.* The known simple groups of composite order were tabulated by Dickson.† All the groups there enumerated as far as order 7920 belong to well known infinite systems. The exhaustive determination of the orders of simple groups was carried by Siceloff‡ and his predecessors as far as order 3640. Recently G. A. Miller§ has shown that there is only one type of simple group of order 2520, and it is easily proved that no other order below 5616 affords more than a single type of simple group.

In what follows, the exhaustive enumeration of orders is carried as far as 6232. The only orders found are those of Dickson's table, viz., 4080, 5616, 6048, 6072. There is only one simple group of each of the orders 4080 and 6072; whether there is more than one in the other two cases remains to be decided.

Elementary considerations exclude all orders but the following:

3648	4080	5472	6048
3744	4320	5616	6072
4032	5040	5760	.

A bare epitome of the reduction process is here given,—just sufficient to enable the reader to retrace the essential steps. Primitive substitution groups are completely known up to degree 20; an unknown simple group cannot have a set of less than 21 conjugate subgroups. In the text the letters G, H, I, s represent entire group, subgroup, invariant subgroup and element.

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* Presented to the Society, October 25, 1924.

† See this BULLETIN, vol. 5, p. 474; and *Linear Groups*, p. 309.

‡ AMERICAN JOURNAL, vol. 34, p. 361.

§ See this BULLETIN, vol. 28, p. 98.

2. *Order* $3648 = 64 \cdot 3 \cdot 19$. In a simple group of this order the 96 H_{38} would supply 1728 s_{19} and 456 s_2. The group has then 2432 elements whose orders divide $19 \cdot 64$, and only 247 whose orders divide 64 remain to be located. But each of the 456 s_2 above is invariant in an H_8 of type (1 1 1) and containing 4 of the s_2. Each H_{64} contains 2 of these H_8; they generate an H_{16} which has another H_8 with 2 cyclical H_4, whose common H_2 is invariant in H_{64} and has 57 conjugates in the whole group; the 114 conjugate H_4 supply $57 \cdot 5 = 285$ elements where only 247 were permissible.

3. *Order* $3744 = 32 \cdot 9 \cdot 13$. The 144 H_{26} supply 1728 s_{13} and 468, 234 or 156 s_2. The H_{32} are non-abelian. There is not room for 208 independent H_9; an H_3 occurs in 4 H_9 and is invariant in an H_{36}, H_{72} or H_{144}. The first of these cases is quickly rejected. In the second the invariant H_3 of the H_{72} might have 1 or 13 conjugates in the H_{72}, but the latter choice would result in the entire group having only 105 elements whose orders divide 9; there are 208 H_9 and each has 3 H_3 common with other H_9 and 1 H_3 not in any other H_9; the total number of elements of order 3 is then 728, and the number of order 6 is at least $3 \cdot 52 \cdot 6 = 936$. The H_{72} has an invariant H_4; this has conjugates in the H_{72}, which is found to have 9 dihedral H_8 containing 21 s_2 conjugate in the entire group and each common to 3 cyclical H_4; these supply 147 s_2 and s_4. These s_2 are not those of the H_{26}; there must be at least 156 of the latter, but this would lead to at least 312 new s_6 and a total in excess of the order of G.

Hence an H_3 common to two H_9 is invariant in an H_{144} and G is expressible in 26 letters. There are 52 H_9 and each has 2 H_3 in 21 letters and 2 H_3 in 24 letters. The H_{144} has transitive sets of $9 + 12 + 4$ letters; the construction does not succeed, there is no simple group of order 3744.

4. *Order* $4032 = 64 \cdot 9 \cdot 7$. A simple group of this order would be expressible in 21 letters, but not in 64 letters. There are 288 H_7. An H_3 common to two H_9 is common

to four H_9 and is invariant in an H_{36}, H_{72} or H_{144}, all of which cases are successively rejected. There are then 28 independent, non-cyclical H_9, and their 112 H_3 are all conjugate. In the expression of G in 28 letters the H_{144} that leaves one letter fixed has a transitive set of 9 letters; it is one-to-one isomorphic with the unique (doubly) transitive H_{144} in 9 letters; it has 36 s_8 in 26 letters, 54 s_4 in 24 letters, 21 s_2 in 24 letters, 8 s_3 and 24 s_6 each in 27 letters. All but 378 elements of G are now located.

When G is expressed in 21 letters, each H_{192} in 20 letters contains 16 H_3, each in 18 letters, giving 32 s_3 and 32 s_6; there are 3 H_{64}, and the 20 letters divide into transitive sets of $8+12$. The H_{192} has an invariant I_4 leaving 9 letters fixed and having 9 conjugates in its I_{32}, whose elements other than identity are all of order 2. The construction fails; there is no simple group of order 4032.

5. *Order* $4080 = 16 \cdot 3 \cdot 5 \cdot 17$. There are 120 H_{34}, supplying 1920 s_{17} and 255 or 510 s_2. If there are 51 H_{80}, each of them has 5 H_{16}; there are 85 or 340 H_3, each with a train of s_6; the case does not succeed. There are therefore 136 H_{30}; their H_{15} furnish $136 \cdot 14 = 1904$ elements; the group has exactly 255 s_2, which make up 17 independent H_{16}. The only simple group of order 4080 is the well known $LF(2, 2^4)$.

6. *Order* $4320 = 32 \cdot 27 \cdot 5$. Here there must be an H_9 invariant in an H_{108} with 4 H_{27} and 9 H_4 each invariant in an H_{12} of the H_{108}. This H_{12} has a single H_4 whose s_2 are conjugate; the 9 H_4 are independent, the elements of the H_{108} are all accounted for and they do not include any s_6. The construction does not succeed. there is no simple group of this order.

7. *Order* $5040 = 16 \cdot 9 \cdot 5 \cdot 7$. The 120 H_{42} are of the seven-letter type. There are 126 H_{40}, each with 5 H_8 with invariant I_2. The case where I_2 has 21 conjugates in G does not work through; I_2 has 126 conjugates and each H_{10} contains 6 or 26 of them and in fact just 6 of them.

The H_8 of the H_{40} are of type (2 1) or (1 1 1), and each of them contains just one s_2 of the H_{42} above. The number of these s_2 in the entire group is 105; each is invariant in an H_{48} containing 6 of the I_2 above and 3 of the H_8 above. A detailed study of the H_{48} shows that a simple group of order 5040 cannot be fitted together.

8. *Order* $5472 = 32 \cdot 9 \cdot 19$. The 96 non-cyclical H_{57} contain $16 \cdot 19$ or $32 \cdot 19\, H_3$. A different H_3 is common to $4\, H_9$ and invariant in an H_{72}; the group being expressed in 76 letters, the invariant H_3 of the H_{72} affects 75 letters, there are in all $304\, H_9$, each has a second H_3 common to $4\, H_9$ and affecting 72 letters; the remaining H_3 affect 75 letters. The H_{72} has an invariant I_4, with conjugates in H_{72}; hence its H_8 are dihedral or of type (1 1 1), and it turns out that there are 3 of them and their type is (1 1 1). An s_2 of their common I_4 must have just 12 conjugates in H_{72}, but this proves impracticable and there is no simple group of order 5472.

9. *Order* $5760 = 128 \cdot 9 \cdot 5$. Here an H_{32} is common to $3\, H_{64}$ and invariant in an H_{192}; G is expressible in 30 letters. The 576 non-cyclic H_{10} supply $2304\, s_5$. There are 640, 160 or $40\, H_9$; in any case an H_3 is common to $4\, H_9$ and invariant in an H_{72} or H_{144}. The H_{72} must have $3\, H_8$, dihedral or (1 1 1), neither case works through. The group is then expressible in 40 letters; its H_3 are not all conjugate, this case fails on comparing with the expression in 30 letters. The H_{144} in 39 letters must have an invariant H_3 in 39 letters, 4 conjugate H_3 in 36 letters, and a residue of $8\, H_3$ in 39 letters. Its transitive systems are $9 + 12 + 18$ or $9 + 12 + 9 + 9$. An examination of the transitive set of 12 letters discloses the impossibility of a simple group of this order.

10. *Order* $6072 = 8 \cdot 3 \cdot 11 \cdot 23$. The $24\, H_{253}$ furnish $276\, H_{11}$ each invariant in a non-cyclical H_{22} whose s_2 affect 24 letters. Only one construction is possible; there is only one type of simple group of this order.

COLUMBIA UNIVERSITY

COMPLETE CLASS NUMBER EXPANSIONS FOR CERTAIN ELLIPTIC THETA CONSTANTS OF THE THIRD DEGREE*

BY E T. BELL

1. *Introduction and Summary.* The expansions of the nine possible theta constants of the third degree in which the parameter of each theta is q,

$$\vartheta_0^3,\ \vartheta_2^3,\ \vartheta_3^3,\ \vartheta_0\vartheta_2^2,\ \vartheta_0\vartheta_3^2,\ \vartheta_2\vartheta_3^2,\ \vartheta_0^2\vartheta_2,\ \vartheta_0^2\vartheta_3,\ \vartheta_2^2\vartheta_3,$$

are of fundamental importance for the arithmetic of class number relations. Two of these. (6), (7) below, have not previously been stated. The expansions of this pair depend upon those of the remaining seven and upon some additional facts concerning representations in a certain ternary quadratic form. The present paper therefore contains the complete set of expansions.

In a recent note† the author has shown that the classical series of Kronecker and Hermite, in which all summations are with respect to $n = 0$ to ∞,

$$(1)\qquad \begin{aligned} \vartheta_2(q^4)\vartheta_3^2(q^4) &= 4\sum q^{4n+1}F(4n+1), \\ \vartheta_2^2(q^4)\vartheta_3(q^4) &= 4\sum q^{4n+2}F(4n+2), \end{aligned}$$

are immediate consequences of the algebraic equivalent

$$(2)\qquad \vartheta_3^3 = 12\sum q^n E(n), \quad E(n) \equiv F(n) - F_1(n),$$

of the theorem of Gauss on representations as sums of three squares.

It was remarked also that their series

$$(3)\qquad \vartheta_2^3(q^4) = 8\sum q^{8n+3}F(8n+3)$$

is also implied by another result due to Gauss. In the foregoing, $F(n)$, $F_1(n)$ denote, respectively, the numbers of odd, even classes, with all the usual conventions, of binary quadratic forms of negative determinant $-n$, so that $E(0) = 1/12$.

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* Presented to the Society, April 5, 1924.

† This BULLETIN, vol. 30, p. 236.

From (1) on replacing q by $\sqrt[4]{q}$, changing q into $-q$ in the result, and then replacing q by q^4, we get

$$(4)\qquad \begin{aligned} \vartheta_0^2(q^4)\vartheta_2(q^4) &= 4\sum q^{4n+1}(-1)^n F(4n+1), \\ \vartheta_0(q^4)\vartheta_2^2(q^4) &= 4\sum q^{4n+2}(-1)^n F(4n+2), \end{aligned}$$

and from (2) by changing the sign of q,

$$(5)\qquad \vartheta_0^3 = 12\sum q^n(-1)^n E(n).$$

There thus remain to be expanded only $\vartheta_0\vartheta_3^2$ and $\vartheta_0^2\vartheta_3$. These expansions are of equal importance with the rest. For example, the series for these two constants are needed more frequently than those for any of the others when we attempt to elicit from Jacobi's theta formula (or from the three-term equations of Weierstrass, Kronecker, or Briot and Bouquet) the numerous class number relations containing arbitrary odd or even functions which such formulas imply. The expansions do not seem to have been obtained hitherto. We shall show that

$$(6)\qquad \vartheta_0\vartheta_3^2 = 4\sum q^n\Big[(-1)^n + \{1+(-1)^n\}\cos n\frac{\pi}{2} + \{1-(-1)^n\}\sin n\frac{\pi}{2}\Big]E(n),$$

and therefore, on changing q into $-q$ and at the same time indicating the corresponding alternative form for (6), we have

$$(7)\qquad \vartheta_0^2\vartheta_3 = 4\sum q^n[1+i^n(1+i)\{1-(-1)^n i\}]E(n),\quad i=\sqrt{-1}.$$

The structure of the coefficient $C(n)$ of q^n in (6) may be noted. It is readily seen that

$$(8.1)\qquad C(4n) = 12E(4n) = 12E(n),$$

$$(8.2)\qquad C(4n+1) = 4E(4n+1) = 4F(4n+1),$$

$$(8.3)\qquad C(4n+2) = -4E(4n+2) = -4F(4n+2),$$

$$(8.4)\qquad C(8n+3) = -12E(8n+3) = -8F(8n+3),$$

$$(8.5)\qquad C(8n+7) = -12E(8n+7) = 0,$$

the third members of which follow from the well known elementary reduction formulas for $E(n)$.

To check (6), (7) we shall write down for $n > 0$ an interesting consequence of (6),

$$(9)\quad 2(-1)^n\epsilon(n) + 4\xi(n) + 8\sum(-1)^a\xi(n-a^2) = C(n),$$

in which $\epsilon(n) = 1$ or 0 according as n is or is not the square of a positive integer, $\xi(n) =$ the excess of the number of divisors of n that are $\equiv 1 \bmod 4$ over the number of those $\equiv -1 \bmod 4$, and the summation refers to all $a = 1, 2, 3, \ldots,$ such that $n - a^2 > 0$. I have verified (9) numerically, hence checking (6), (7).

2. *Expansion of* $\vartheta_0\vartheta_3^2$. Write $N(n = f)$ for the number of representations of the integer $n \geqq 0$ in the form f. Then

$$(10)\quad \vartheta_0\vartheta_3^2 = \sum q^n[N(n = 4n_1^2 + n_2^2 + n_3^2) \\ - N(n = m_1^2 + n_2^2 + n_3^2)],$$

in which n_1, n_2, n_3 are integers $\gtreqless 0$, and $m_1 \gtrless 0$ is an odd integer. From (1), (2) we have

$$N(4n+1 = m_1^2 + 4n_2^2 + 4n_3^2) = 4F(4n+1) = 4E(4n+1),$$
$$N(4n+2 = m_1^2 + m_2^2 + 4n_3^2) = 4F(4n+2) = 4E(4n+2),$$
$$N(4n+3 = m_1^2 + m_2^2 + m_3^2) = 12E(8n+3),$$

and we transcribe from a former paper,*

$$N(4n = 4n_1^2 + n_2^2 + n_3^2) = 12E(n) = 12E(4n),$$
$$N(4n+1 = 4n_1^2 + n_2^2 + n_3^2) = 8F(4n+1) = 8E(4n+1),$$
$$N(4n+2 = 4n_1^2 + n_2^2 + n_3^2) = 4F(4n+2) = 4E(4n+2),$$
$$N(4n+3 = 4n_1^2 + n_2^2 + n_3^2) = 0.$$

From these and (10), since in (10) either of n_2, n_3 may be odd or even, it follows on referring to (8.1)-(8.5) that

$$\vartheta_0\vartheta_3^2 = \sum q^n C(n),$$

which is equivalent to (6).

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* AMERICAN MATHEMATICAL MONTHLY, vol. 21 (1924), p. 126.

The relation (9) follows by comparing (6) with the identity

$$\vartheta_0\vartheta_3^2 = \left[1+2\sum_{a=1}^{\infty} q^{a^2}(-1)^a\right]\left[1+4\sum_{n=1}^{\infty} q^n\xi(n)\right],$$

the second factor on the right being the algebraic expression of the well known theorem which gives the number of representations of n as a sum of two integer squares.

THE UNIVERSITY OF WASHINGTON

NOTE ON A SPECIAL CONGRUENCE*

BY MALCOLM FOSTER

1. *Introduction.* Let S be any surface referred to its lines of curvature. With every point M of S we associate the trihedral of the surface, taking the x-axis of the trihedral tangent to the curve $v = \text{const.}$ We consider the congruence of lines l parallel to the z-axis, the normal to S, which pierce the xy-plane at the point $(\xi, \eta_1, 0)$.† The equations of l are $x = \xi$, $y = \eta_1$, and the coordinates of any point on l are

$$x = \xi, \quad y = \eta_1, \quad z = t, \tag{1}$$

where t is the distance on l measured from the point $(\xi, \eta_1, 0)$.

2. *Condition for a Normal Congruence.* If there be a surface normal to the congruence we must have

$$\delta z = dz + p_1 y dv - q x du = 0,\text{‡} \tag{2}$$

for all values of $\frac{dv}{du}$. Using (1) equation (2) becomes

$$\frac{\partial t}{\partial u}du + \frac{\partial t}{\partial v}dv + p_1\eta_1 dv - q\xi du \equiv 0; \tag{3}$$

hence

$$\frac{\partial t}{\partial u} - q\xi = 0, \qquad \frac{\partial t}{\partial v} + p_1\eta_1 = 0. \tag{4}$$

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* Presented to the Society, May 3, 1924.

† The notation used in this paper is the same as in Eisenhart's *Differential Geometry of Curves and Surfaces*; see pp. 166–176.

‡ Eisenhart, p. 170

The condition of integrability is

$$\frac{\partial}{\partial v}\left(\frac{\partial t}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial t}{\partial v}\right).$$

Hence from (4) we must have

$$q\frac{\partial \xi}{\partial v} + \xi\frac{\partial q}{\partial v} = -\left(p_1\frac{\partial \eta_1}{\partial u} + \eta_1\frac{\partial p_1}{\partial u}\right),$$

which reduces to* $q\eta_1 r - p_1\xi r - p_1\xi r_1 + q\eta_1 r_1 = 0$. This may be written in the form

$$(5) \qquad (q\eta_1 - p_1\xi)(r + r_1) = 0.$$

Conversely, when (5) is satisfied, the function t as given by (4) satisfies (3), and hence there exists a single infinity of parallel surfaces normal to the lines l. If $q\eta_1 - p_1\xi = 0$, the surface S is minimal.† Hence we have the following theorem.

THEOREM 1. *A necessary and sufficient condition that the congruence of lines l be normal is that S be a minimal surface, or else that $r + r_1 = 0$.*

3. *Equation of the Curves defining the Developables.* As M, the vertex of the trihedral, is displaced along some curve C on S the locus of l is a ruled surface of the congruence; we seek the equation of the curves C for which the locus of l is developable. For l to generate a developable surface the displacement of some point on l must be in the direction of the line; hence for that point

$$(6) \qquad \delta x = \delta y = 0.$$

By means of (1), equations (6) take the forms‡

$$(7) \quad \begin{cases} \dfrac{\partial \xi}{\partial u}du - \eta_1 r dv + \xi du + tq du - \eta_1(r du + r_1 dv) = 0, \\ \xi r_1 du + \dfrac{\partial \eta_1}{\partial v}dv + \eta_1 dv - tp_1 dv + \xi(r du + r_1 dv) = 0. \end{cases}$$

The elimination of t between these two equations gives the following equation of the curves on S defining the developable surfaces of the congruence:

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* Eisenhart, p. 168, formulas (48), and p. 170, formulas (55).

† Eisenhart, p. 174, formulas (73).

‡ Eisenhart, p. 170.

$$(8)\qquad q\xi(r+r_1)\,du^2+\left[p_1\left(\frac{\partial\xi}{\partial u}+\xi-\eta_1 r\right)\right.$$
$$\left.+q\left(\frac{\partial\eta_1}{\partial v}+\eta_1+\xi r_1\right)\right]du\,dv-p_1\eta_1(r+r_1)\,dv^2=0.$$

The elimination of the ratio $\frac{dv}{du}$ between equations (7) gives the following equation for the distances along l to the focal points:

$$(9)\qquad p_1 q t^2+\left[p_1\left(\frac{\partial\xi}{\partial u}+\xi-\eta_1 r\right)-q\left(\frac{\partial\eta_1}{\partial v}+\eta_1+\xi r_1\right)\right]t$$
$$-\left(\frac{\partial\xi}{\partial u}+\xi-\eta_1 r\right)\left(\frac{\partial\eta_1}{\partial v}+\eta_1+\xi r_1\right)-\xi\eta_1(r+r_1)^2=0.$$

The condition that equation (8) define an orthogonal system is*

$$\xi^2 p_1\eta_1(r+r_1)-\eta_1^2 q\xi(r+r_1)=0,$$

which may be written

$$(10)\qquad (q\eta_1-p_1\xi)(r+r_1)=0,$$

since ξ, $\eta_1 \neq 0$. Since (10) is identical with (5) we have the following theorem.

THEOREM II. *If the congruence of lines l be normal the developables are represented on S by an orthogonal system.*

We note from (8) that if the congruence be normal by virtue of the relation $r+r_1=0$ the curves defining the developables are the lines of curvature.

4. *Asymptotic Lines on the Normal Surfaces.* Let C be any curve on S through M, the vertex of the trihedral, and let l and l_1 be the lines of the congruence corresponding to M and M_1, a neighboring point on C. As M_1 approaches M along C the foot on l of the common perpendicular to l and l_1 approaches a certain limiting position called the central point of the generator l. The locus of the central points is the line of striction of the ruled surface

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* Eisenhart, p. 80.

defined by C. We wish to determine the distance along l to the line of striction of this ruled surface.

To this end we consider a second trihedral T_0* with vertex at some fixed point in space, whose x_0-, y_0-, and z_0-axes are in every position parallel to the x-, y-, and z-axes of the moving trihedral. Relative to the trihedral T_0 the coordinates of the point on the unit sphere corresponding to l are (0, 0, 1). As the vertex of the moving trihedral is displaced along C the absolute displacements of the point (0, 0, 1) in the directions of the axes of the trihedral T_0 will be the variations experienced by the direction-cosines of l. If these variations are denoted by $\delta\alpha$, $\delta\beta$, $\delta\gamma$, we have† $\delta\alpha = q\,du$, $\delta\beta = -p_1 dv$, $\delta\gamma = 0$, since for the motion of the trihedral T_0 the translations ξ and η_1 are zero. The direction-cosines of l_1 relative to the trihedral at M are therefore $q\,du$, $-p_1 dv$, 1. The displacement of the central point on l must be orthogonal to both l and l_1. Hence for that point we must have $\delta z = 0$, $q\,du\,\delta x - p_1 dv\,\delta y + \delta z = 0$. Combining these equations we get $q\,du\,\delta x - p_1 dv\,\delta y = 0$, which becomes

$$q\,du\left[\frac{\partial \xi}{\partial u}du - \eta_1 r\,dv + \xi\,du + qt\,du - \eta_1 r\,du - \eta_1 r_1 dv\right]$$
$$-p_1 dv\left[\xi r_1 du + \frac{\partial \eta_1}{\partial v}dv + \eta_1 dv + \xi r\,du + \xi r_1 dv - tp_1 dv\right] = 0.$$

This may be written in the form

$$(11)\quad q\left(\frac{\partial \xi}{\partial u} + \xi + qt - \eta_1 r\right)du^2 - (q\eta_1 + p_1\xi)(r + r_1)\,du\,dv$$
$$-p_1\left(\frac{\partial \eta_1}{\partial v} + \eta_1 + \xi r_1 - tp_1\right)dv^2 = 0.$$

When the value of dv/du which determines the curve C is put in this equation we have an equation in t which determines the distance along l to the line of striction of the ruled surface defined by C. Conversely, given t a function of u

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* Eisenhart, p. 168.

† Eisenhart, p. 170.

and v, the equation (11) determines two curves on S, though not necessarily real, which define two ruled surfaces of the congruence for which t is the distance to their lines of striction.

We suppose that the congruence is normal by virtue of the relation $r + r_1 = 0$. Hence equation (11) is of the form

$$L(u, v)\, du^2 - M(u, v)\, dv^2 = 0,$$

and consequently represents a system of curves symmetrically placed relative to the lines of curvature. Now the normals to a surface along the asymptotic lines form ruled surfaces for which the asymptotic lines are the lines of striction.[*] Hence we have the following theorem.

THEOREM III. *If the congruence of lines l be normal by virtue of the relation $r + r_1 = 0$, the curves on S which represent the asymptotic lines on the normal surfaces form a system which is symmetrically placed relative to the lines of curvature.*

5. *Minimal Surfaces.* We now suppose that S is a minimal surface with the parameters of the lines of curvature so chosen that the linear element of the surface has the form[†]

$$ds^2 = \varrho (du^2 + dv^2), \tag{12}$$

where ϱ is the absolute value of each principal radius. Hence we have[‡]

$$\left\{\begin{aligned} &\xi = \eta_1 = \sqrt{\varrho}, \quad q = -\frac{D}{\sqrt{E}} = -\frac{1}{\sqrt{\varrho}}, \\ &p_1 = \frac{D''}{\sqrt{G}} = -\frac{1}{\sqrt{\varrho}}, \\ &r = -\frac{1}{\sqrt{G}} \frac{\partial}{\partial v} \sqrt{E} = -\frac{\frac{\partial \varrho}{\partial v}}{2\varrho}, \\ &r_1 = \frac{1}{\sqrt{E}} \frac{\partial}{\partial u} \sqrt{G} = \frac{\frac{\partial \varrho}{\partial u}}{2\varrho}. \end{aligned}\right. \tag{13}$$

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[*] Eisenhart, p. 268, Ex. 14.

[†] Eisenhart, p. 253.

[‡] Eisenhart, p. 174.

When the values of $\xi, \eta_1, q, p_1, r, r_1$, as given in (13) are substituted in (9), it is readily seen that the coefficient of t vanishes. Hence we have the following theorem.

THEOREM IV. *If S be a minimal surface with the parameters of the lines of curvature so chosen that the linear element has the form* (12), *the congruence of lines l is normal and has for its middle envelope the given minimal surface.*

The minimal surface S is therefore the mean evolute of each of the normal surfaces.

6. *Envelope of a Two-Parameter Family of Surfaces.* Let S be any surface referred to any parametric system. With every point M of S we associate the trihedral of the surface, giving the x-axis its most general orientation relative to the curve $v =$ const. Let

$$F(x, y, z, u, v) = 0 \tag{14}$$

be the equation relative to the trihedral at M of some surface $\sum$. We consider the envelope of such a two-parameter family of surfaces.

The characteristic is defined by (14) and the equations

$$\left\{\begin{aligned} &\frac{\partial F}{\partial x}\frac{dx}{du}+\frac{\partial F}{\partial y}\frac{dy}{du}+\frac{\partial F}{\partial z}\frac{dz}{du}+\frac{\partial F}{\partial u}=0,\\ &\frac{\partial F}{\partial x}\frac{dx}{dv}+\frac{\partial F}{\partial y}\frac{dy}{dv}+\frac{\partial F}{\partial z}\frac{dz}{dv}+\frac{\partial F}{\partial v}=0.\end{aligned}\right. \tag{15}$$

Since the characteristic is fixed in space, we must have

$$\begin{aligned} \delta x &= dx+\xi du+\xi_1 dv+(q du+q_1 dv)z-(r du+r_1 dv)y=0,\\ \delta y &= dy+\eta du+\eta_1 dv+(r du+r_1 dv)x-(p du+p_1 dv)z=0,\\ \delta z &= dz \qquad\qquad\quad +(p du+p_1 dv)y-(q du+q_1 dv)x=0,\end{aligned}$$

for all values of $\frac{dv}{du}$. Hence

$$\left\{\begin{aligned} &\frac{dx}{du}=ry-qz-\xi,\ \frac{dy}{du}=pz-rx-\eta,\ \frac{dz}{du}=qx-py,\\ &\frac{dx}{dv}=r_1y-q_1z-\xi_1,\ \frac{dy}{dv}=p_1z-r_1x-\eta_1,\ \frac{dz}{dv}=q_1x-p_1y.\end{aligned}\right. \tag{16}$$

By (16), equations (15) become

$$(17)\quad \begin{cases} \dfrac{\partial F}{\partial x}(ry-qz-\xi)+\dfrac{\partial F}{\partial y}(pz-rx-\eta) \\ \qquad\qquad +\dfrac{\partial F}{\partial z}(qx-py)+\dfrac{\partial F}{\partial u}=0, \\ \dfrac{\partial F}{\partial x}(r_1y-q_1z-\xi_1)+\dfrac{\partial F}{\partial y}(p_1z-r_1x-\eta_1) \\ \qquad\qquad +\dfrac{\partial F}{\partial z}(q_1x-p_1y)+\dfrac{\partial F}{\partial v}=0. \end{cases}$$

The coordinates $(x, y, z,)$ of the characteristic are therefore given by equations (14) and (17).

7. *Applications.* As an application, consider the envelope of certain two-parameter families of planes. We choose S as any surface referred to its lines of curvature, and choose the x-axis of the trihedral tangent to the curve $v=\text{const.}$ Consider the two two-parameter families of planes

$$(18)\qquad x=\xi,$$

and

$$(19)\qquad y=\eta_1.$$

For the family of planes (18), equations (17) become

$$(20)\qquad ry-qz-\xi-\frac{\partial\xi}{\partial u}=0,\quad r_1y+\eta_1r=0.$$

Hence solving equations (18) and (20) the coordinates of the characteristic of the planes (18) are

$$(21)\quad x_1=\xi,\quad y_1=-\frac{\eta_1r}{r_1},\quad z_1=-\frac{r_1\left(\xi+\dfrac{\partial\xi}{\partial u}\right)+\eta_1r^2}{qr_1}.$$

For the family of planes (19), equations (17) become

$$(22)\qquad rx+\xi r_1=0,\quad p_1z-r_1x-\eta_1-\frac{\partial\eta_1}{\partial v}=0.$$

The solution of equations (19) and (22) gives the following for the coordinates of the characteristic of the planes (19):

$$(23)\quad x_2=-\frac{\xi r_1}{r},\quad y_2=\eta_1,\quad z_2=\frac{r\left(\eta_1+\dfrac{\partial\eta_1}{\partial v}\right)-\xi r_1^2}{rp_1}.$$

Let us now assume the relation

$$r+r_1=0. \tag{24}$$

Then, from (21) and (23), we have

$$\left\{\begin{aligned} & x_1=x_2=\xi, \qquad y_1=y_2=\eta_1, \\ & z_1=\frac{\eta_1 r-\xi-\dfrac{\partial\xi}{\partial u}}{q}, \qquad z_2=\frac{\xi r_1+\eta_1+\dfrac{\partial\eta_1}{\partial v}}{p_1}. \end{aligned}\right. \tag{25}$$

Hence when the relation (24) holds, the characteristics of both families of planes lie on the line l. Moreover, when the relation (24) holds the roots of equation (9) are

$$t_1=\frac{\eta_1 r-\xi-\dfrac{\partial\xi}{\partial u}}{q}, \qquad t_2=\frac{\xi r_1+\eta_1+\dfrac{\partial\eta_1}{\partial v}}{p_1}. \tag{26}$$

Hence, since t_1 and t_2 are identical with z_1 and z_2 in (25) we have the following theorem.

THEOREM V. *If S be a surface referred to its lines of curvature for which the relation $r+r_1=0$ holds, the envelopes of the two families of planes $x=\xi$ and $y=\eta_1$ are the two focal sheets of the normal congruence of lines l.*

The planes (18) and (19) are therefore the focal planes of the congruence. From (8), the curves defining the developables are the lines of curvature. Hence from (7),

$$t=t_1=\frac{\eta_1 r-\xi-\dfrac{\partial\xi}{\partial u}}{q},$$

for $v=$ const., and

$$t=t_2=\frac{\xi r_1+\eta_1+\dfrac{\partial\eta_1}{\partial v}}{p_1},$$

for $u=$ const. Consequently the planes $x=\xi$ and $y=\eta_1$ envelope the focal sheets determined by the developables $v=$ const. and $u=$ const., respectively.

YALE UNIVERSITY

CONCERNING RELATIVELY UNIFORM CONVERGENCE*

BY R. L. MOORE

According to E. H. Moore, a sequence of functions $f_1(p)$, $f_2(p), f_3(p), \cdots$, defined on a range K, is said to converge, to a function $f(p)$, relatively uniformly with respect to the scale function $s(p)$ if, for every positive number e, there exists a positive number δ_e such that if $n > \delta_e$ then, for every p which belongs to K, $|f_n(p)-f(p)| < e|s(p)|$†.

In this note I will establish the following theorem.

THEOREM. *If S is a convergent sequence of measurable functions $f_1(x), f_2(x), f_3(x), \cdots$ defined on a measurable point set E and S converges for each x belonging to E, then E contains a subset E_0 of measure zero such that the sequence S converges relatively uniformly for all values of x on the range $E-E_0$.*

PROOF. Suppose that S converges on E to the limit function $f(x)$. By a theorem due to Egoroff‡, E contains a subset E_1 of measure less than 1 such that S converges to $f(x)$ uniformly on $E-E_1$. Similarly E_1 contains a subset E_2 of measure less than $1/2$ such that S converges to $f(x)$ uniformly on E_1-E_2. Continue this process thus obtaining a sequence of point sets E_1, E_2, $E_3, \cdots$ such that, for each n, (1) the measure of E_n is less than $1/n$, (2) E_{n+1} is a subset of E_n, (3) S converges uniformly on E_n-E_{n+1}. Let E_0 denote the set of points common to the sets $E_1, E_2, E_3, \cdots$. The set E_0 is either vacuous or of measure 0. Furthermore

$$E = E_0 + (E-E_1) + (E_1-E_2) + \cdots.$$

Since S converges uniformly on each point set of the countable collection $E-E_1$, E_1-E_2, $E_2-E_3, \cdots$, it follows, by a theorem due to E. W. Chittenden, that[*] S converges relatively uniformly on the sum of all the point sets of this collection. But this sum is $E-E_0$.

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* Presented to the Society, April 14, 1922.

† See E. H. Moore, *Introduction to a Form of General Analysis*, The New Haven Mathematical Colloquium (Yale University Press, New Haven, 1910).

‡ COMPTES RENDUS, Jan. 30, 1911.

THE UNIVERSITY OF TEXAS

THE THEORY OF CLOSURE OF TCHEBYCHEFF POLYNOMIALS FOR AN INFINITE INTERVAL[†]

BY J. A. SHOHAT (J. CHOKHATE)

1. *The Theorem of Closure.* Suppose we have a function $p(x)$, not negative in a given interval (a, b), for which all the integrals

$$\int_a^b p(x)x^n dx, \qquad (n=0,1,2,\cdots)$$

exist. It is well known that we can form a normal and orthogonal system of polynomials

$$\varphi_n(x)=a_n x^n+\cdots, \quad a_n>0, \quad (n=0,1,2,\cdots),$$

uniquely determined by means of the relations

$$\int_a^b p(x)\varphi_m(x)\varphi_n(x)dx=\begin{cases}0, & m \neq n,\\ 1, & m=n.\end{cases}$$

We call these polynomials *Tchebycheff polynomials* corresponding to the interval (a, b) with the *characteristic function* $p(x)$. The simplest example is given by Legendre polynomials, corresponding to the interval $(-1, +1)$ with $p(x)=1$.

The most important application of Tchebycheff polynomials is their use in the development of functions into a series of polynomials whose coefficients are easily obtained as in Fourier series. Thus we have for every function, for which the integrals

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[*] E. W. Chittenden, *Relatively uniform convergence of sequences of functions*, TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), pp. 197–201. As Chittenden observes, this is an extension of a theorem given by E. H. Moore on page 87 of his *Introduction to a Form of General Analysis*, loc. cit.

[†] Presented to the Society, December 29, 1923.

$$\int_a^b p(x)f(x)x^n dx, \qquad (n=0,1,2,\cdots),$$

exist, the formal development into a series

$$(1) \qquad f(x) \sim \sum_0^\infty{}_n A_n \varphi_n(x), \quad A_n = \int_a^b p(x)f(x)\varphi_n(x)dx.$$

For the case of a *finite interval* (a, b) there always exists the following remarkable equation, which is called the *closure equation*:

$$(2) \qquad \int_a^b p(x)f^2(x)dx = \sum_0^\infty{}_n A_n^2,$$

for every function $f(x)$, for which the integral in the left-hand member exists, even if the development (1) diverges.

The existence of (2) was proved by W. Stekloff.* He showed also the great importance of the "closure equation" in the general theory of Tchebycheff polynomials, in particular for investigating the convergence of the series (1). These results suggest that it might be of interest to investigate the case of an *infinite interval.*

Stekloff† considered only two special cases involving infinite intervals:

(a) polynomials of Laguerre-Tchebycheff
$(a=0, b=\infty; p(x)=e^{-x}x^{\beta-1}, \beta>0)$;

(b) polynomials of Laplace-Hermite-Tchebycheff
$(a=-\infty, b=+\infty; p(x)=e^{-x^2})$.

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* W. Stekloff, *Sur la théorie de fermeture des systèmes des fonctions orthogonales* ..., MÉMOIRES DE L'ACADÉMIE DES SCIENCES, Petrograd, (7), vol. 30 (1911).

† W. Stekloff, *Théorème de fermeture pour les polynomes de Laplace-Hermite-Tchebycheff*, BULLETIN DE L'ACADÉMIE DES SCIENCES, Petrograd, 1916, pp. 403-16.

——— *Théorème de fermeture pour les polynomes de Tchebycheff-Laguerre,* ibid., 1916, pp. 633-42.

He proved that the closure equation holds in those cases. I have investigated the general case of an *infinite interval* (a, ∞). The results obtained can be formulated in the following theorem.

THEOREM. 1°. *The closure equation holds if, for x sufficiently large,*

$$p(x) < e^{-k|x|^{\lambda}}$$

with $k > 0$ *and with* $\lambda > \frac{1}{2}$ *for* $a = 0$, $\lambda > 1$ *for* $a = -\infty$;
2°. *The closure equation is not true in general, if*

$$p(x) > e^{-k|x|^{\lambda}}$$

with $k > 0$ *and with* $\lambda < \frac{1}{2}$ *for* $a = 0$, $\lambda < 1$ *for* $a = -\infty$.

The assertion (1°) contains the two cases considered by Stekloff. The proof of the first part (1°) of the above statement is based on the paper *Theory of functions deviating the least from zero in an infinite interval,* developed by myself in 1921.* The principal results obtained there are as follows.

Consider two function $f(x)$ and $p(x)$ defined in the infinite interval $(-\infty, +\infty)$ and subject to the following conditions:

(a) $f(x)$ and $p(x)$ are single-valued and finite for every x;

(b) for $x = \pm\infty$ $\lim f(x) = \lim p(x)\,x^n = 0$, $\quad (n = 0, 1, 2, \cdots)$.

I proved that *there exists at least one polynomial* $Q_n(x)$ *of degree n such that the function*

$$\varphi(x) \equiv f(x) - p(x)\,Q_n(x)$$

deviates the least from zero in $(-\infty, \infty)$, i. e., $Q_n(x)$ minimizes

$$E_n = \text{max.}\,|\varphi(x)| \text{ in } (-\infty, \infty) \tag{3}$$

among all polynomials of degree n. Furthermore,

$$\lim_{n \to \infty} E_n = E \geqq 0$$

exists. In general we have $E > 0$. But under certain additional conditions imposed on $f(x)$ and $p(x)$ we have

$$\lim_{n \to \infty} E_n = 0$$

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* ANNALS OF THE URAL-UNIVERSITY, No. 1, 1921, Ekaterinburg.

as in the case of a finite interval with $p(x)=1$. The results hold for any infinite interval (a, ∞).

As an application of the results above to Tchebycheff polynomials I gave in the same paper the following theorem.

THEOREM. *The closure equation holds for any orthogonal and normal system of Tchebycheff polynomials corresponding to the interval* (a, ∞) *with the characteristic function*

$$p(x) = e^{-k|x|^{\lambda}}q(x), \qquad (k>0,\ q(x)\geqq 0),$$

if the integral

$$\int_a^{\infty} e^{-(k-\varepsilon)|x|^{\lambda}}q(x)dx$$

exists for a certain positive ε, *with* $\lambda>\frac{1}{2}$ *for* $a=0$, $\lambda>1$ *for* $a=-\infty$.

From this theorem our statement (1°) follows immediately; for, if we set

$$p(x) = e^{-k|x|^{\lambda}}q(x),$$

we get $q(x)<1$ for x sufficiently large, and consequently the integral

$$\int_a^{\infty} e^{-(k-\varepsilon)|x|^{\lambda}}q(x)dx$$

exists for every positive $\varepsilon<k$.

In order to prove the statement (2°) concerning the non-existence of the closure equation I consider first the interval $(0, \infty)$, and give an example of an orthogonal and normal system of Tchebycheff polynomials for which the closure equation is not true.

Let us set

$$p(x) = e^{-x^{\lambda}},$$
$$f(x) = \sin(x^{\lambda}\tan\lambda\pi),$$

with $\lambda<\frac{1}{2}$. Then, by a theorem of Adamoff,*

$$\int_0^{\infty} p(x)f(x)x^n dx = 0, \qquad (n=0,1,2,\ldots),$$

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* A. Adamoff, *Proof of a theorem due to Stieltjes*, PROCEEDINGS OF THE MATHEMATICAL SOCIETY AT KAZAN, 1901, (in Russian).

from which we deduce immediately that *all the coefficients in the development* (1) *vanish*:

$$A_n = 0, \qquad (n = 0, 1, 2, \ldots).$$

If we assume the existence of the closure equation in this case we get, from (2),

$$\int_0^\infty p(x) f^2(x)\, dx = 0,$$

which is evidently impossible.

In the case of the interval $(-\infty, \infty)$ we set

$$p(x) = e^{-x^\beta} \quad \left(\beta = \frac{2s}{2s+1}, \quad s \text{ a positive integer}\right),$$

$$f(x) = \cos\left(x^\beta \tan\frac{\beta\pi}{2}\right);$$

then, from a result of W. Stekloff,*

$$\int_{-\infty}^\infty p(x) f(x) x^n\, dx = 0, \quad (n = 0, 1, 2 \ldots).$$

The rest of the proof is as before.

The case

$$p(x) = e^{-|x|^\lambda}$$

with $\lambda = \frac{1}{2}$ for $(0, \infty)$, $\lambda = 1$ for $(-\infty, \infty)$ requires a more delicate analysis.

2. *Application to the Theory of Functions Deviating the Least from Zero in an Infinite Interval.* In my paper mentioned above I showed that

$$\lim_{n\to\infty} E_n = E > 0,$$

(see (3)), if the closure equation does not hold for the system of Tchebycheff polynomials with the characteristic function $p(x)$. Thus we conclude

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* W. Stekloff, *Application de la théorie de fermeture à la solution de certaines problèmes des moments*, Mémoires de l'Académie des Sciences, Petrograd, (7), vol. 33 (1915).

In the case of

$$p(x) > e^{-k|x|^{\lambda}}, \qquad (k > 0)$$

with $\lambda < \frac{1}{2}$ *for* $(0, \infty)$, $\lambda < 1$ *for* $(-\infty, \infty)$ *we have in general*

$$\lim_{n \to \infty} E_n > 0.$$

3. *Connection with the Theory of Continued Fractions.* In conclusion I want to point out the intimate connection between the theory of closure of Tchebycheff polynomials corresponding to the interval (a, b) with the characteristic function $p(x)$, and the theory of Stieltjes' continued fraction

$$\cfrac{1}{b_1 x + \cfrac{1}{b_2 x + \cfrac{1}{b_3 x + \dots}}},$$

which arises from the integral

$$\int_a^b \frac{p(y)}{x - y}\, dy, \qquad (a \geqq 0,\ b > a).$$

Indeed, if (a, b) is a finite interval, then the continued fraction for every $p(y)$ converges, and in this case the closure equation always holds. On the other hand, if the interval considered is infinite, say $(0, \infty)$, and if

$$p(y) = e^{-k|y|^{\lambda}}, \qquad (k > 0),$$

we know, according to Stieltjes,[*] that the continued fraction converges for $\lambda \geqq \frac{1}{2}$ and diverges for $\lambda < \frac{1}{2}$. This is in accordance with the statements (1^0, 2^0) given above.

The closure theory is also closely connected with so called *problem of moments*, due also to Stieltjes.[†] I hope to show this in another paper.

THE UNIVERSITY OF CHICAGO

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[*] Stieltjes, *Recherches sur les fractions continues*, ANNALES DE LA TOULOUSE, vol. 8 (1894), pp. J. 1–J. 122, vol. 9, pp. A. 1–A. 47.

[†] Loc. cit., vol. 9, pp. A. 1–A. 47.

NUCLEAR AND HYPER-NUCLEAR POINTS IN THE THEORY OF ABSTRACT SETS*

BY E W. CHITTENDEN

1. *Introduction.* In his note, *Le théorème de* Borel *dans la theorie des ensembles abstraits,*† Fréchet considers the problem: determine the most general class (L) for which the theorem of Borel holds true. This class is found to be a class (S), that is, a class (L) in which the derived set of every set is closed. At the end of the note he calls attention to the fact that the stronger theorem of Borel-Lebesgue may not hold in a given class (S) and proposes the question: what is the most general class (L) for which we may state the theorem of Borel-Lebesgue? That such a class (L) be a class (S) is necessary but not sufficient.

This attracted the attention of R. L. Moore,‡ who showed by the aid of the theory of transfinite ordinals that the most general class (L) which admits the theorem of Borel-Lebesgue is a class (S) with the further property "every compact set is perfectly compact". The property *perfectly compact,* so named by Fréchet,§ is defined as follows. A set E is perfectly compact if every monotone sequence of subsets of E determines an element which is common to all the sets of the sequence or to their derived sets. A sequence of sets is monotone if of any two sets of the sequence one contains the other.

Later Fréchet,|| developing the theory of classes (V)

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* Presented to the Society, December 1, 1923.

† Bulletin de la Société de France, vol. 45 (1917), pp. 1–8. Called Fréchet, I hereafter.

‡ *On the most general class* (L) *of* Fréchet *in which the Heine-Borel-Lebesgue theorem holds true,* Proceedings of the National Academy of Sciences, vol. 5 (1919), pp. 196–210.

§ *Sur les ensembles abstraits,* Annales de l'Ecole Normale (3), vol. 38 (1921), p. 342. Called Fréchet, II hereafter.

|| *Sur la notion de voisinage dans les ensembles abstraits,* Bulletin des Sciences Mathématiques, (2), vol. 42 (1918), pp. 138–156. Called Fréchet, III hereafter.

more general than classes (L), resumed the study of the theorem of Borel. This theorem was found for classes (V) to be the equivalent of the property of Hedrick; that is, for every set E in the class (V), element p interior to E, and set F which has p for a limit element, there is a subset F_1 of F such that every element of F_1 is interior to E.* This property implies the closure of derived classes. The theorem of R. L. Moore was extended by Fréchet to these more general classes (V) with the following results. The property of Borel-Lebesgue implies the property perfectly compact in the most general class (V). Under the hypothesis of the property of Hedrick, every set which is perfectly compact in itself possesses the property of Borel-Lebesgue.†

Kuratowski and Sierpinski‡ presented another solution of the problem. An element or point p of a set E is of power μ relative to E if there is a subset of E of power μ *interior* to every neighborhood of p, but not every neighborhood of p contains a subset of E of power greater than μ. It is convenient to say that a point p of a set E of power μ is a *hyper-nuclear* point of E if p is of power μ relative to E. A necessary and sufficient condition that the theorem of Borel-Lebesgue hold in a class (L) is that every infinite compact set whose derived set is also compact determine at least one hyper-nuclear point. Since in a class (L) the properties perfectly compact, and derived sets are closed, together form a necessary and sufficient condition for the theorem of Borel-Lebesgue it follows that in a class (L) they are equivalent to the condition of Kuratowski and Sierpinski. Section 4 of the present paper contains a generalization of the theorem of Borel-Lebesgue which holds in the most general class (V) and reduces to the theorem of Kuratowski and Sierpinski when the class (V) is equivalent to a class (L). This new theorem is shown by the example of § 5 to be independent of the closure of derived sets.

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* Fréchet, III, p. 155.

† Fréchet, II, pp. 346–49, §§ 8–10.

‡ *Le théorème de Borel-Lebesgue dans la théorie des ensembles abstraits*, FUNDAMENTA MATHEMATICAE, vol. 2 (1921), pp. 172–78.

Fréchet[*] has shown that in a class (D), that is a class admitting a generalization of distance, every compact set is perfectly compact, and proposes the problem: determine the most general class (V) with this property. Section 4 below presents in terms of the concept *nuclear* point[†] a necessary and sufficient condition that every compact subset of a class (V) be perfectly compact.

2. *On Terminology.* As a basis for the following discussion we postulate a space P of points p and a system of families of subsets V of P called neighborhoods. To each point p is assigned in a definite way a family of neighborhoods V_p. There is no loss of generality in assuming that V_p contains p, and for convenience we shall make that assumption. A point p is a limit point of a set E if every neighborhood of p contains a point of E distinct from p. The relation limit point so defined has the following properties.

1) Every limit point of a set E is a limit point of every set containing E.

2) Whether p is a limit point of a set E or not depends only on the elements of E other than p[‡].

A point p is interior to a set G if p is an element of G and if G contains a point of every set which has p for a limit point. Then p is interior to every neighborhood V_p. Furthermore if p is interior to G then some neighborhood of p, V_p, is a subset of G.

A series S of sets G is called monotone if of any two sets G, G' of the series one is a subset of the other. A monotone series will be called *closed* if the sets G or their derived sets have a common element; otherwise *open*. If each such common element belongs to a set E the series will be said to be closed in E.

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[*] Fréchet, II, p. 346.

[†] A point p is a nuclear point of a set E of power μ if every neighborhood of p contains a subset of E of power μ.

[‡] Fréchet, III, p. 140.

A set E is perfectly compact if every monotone series S of subsets G of E is closed; perfectly self-compact, if every such series S is closed in E. Fréchet has shown that every perfectly self-compact set is compact and that every perfectly self-compact set is self-compact*.

3. *Procedure under the Zermelo Axiom.* The following procedure based upon the assumption of the Zermelo axiom and the well known properties of the transfinite ordinal numbers was employed by Kuratowski and Sierpinski† and is the basis for several of the following proofs. Let Q be an aggregate of power μ and of elements q. Of the transfinite ordinals Ω for which the aggregate of all ordinals $\alpha < \Omega$ has the power μ there is a least, Ω_0. Let

$$(1) \quad q_1, q_2, q_3, \ldots, q_n, \ldots, q_\omega, \ldots, q_\alpha, \ldots \; (\alpha < \Omega_0)$$

represent a 1—1 correspondence between the aggregate Q and the aggregate of all ordinal numbers $\alpha < \Omega_0$. If a sequence of ordinals $\beta < \Omega_0$ is determined so that for every α there is a $\beta > \alpha$ then the β's form a series of the ordinal type Ω_0 and the aggregate of all such ordinals β is of power μ. Several of the proofs in the sequel depend upon a correspondence of the type (1) and the further fact that the elements q_β form an aggregate of power μ.

4. *Nuclear Points.* A point p will be said to be a *nuclear point* of a set E in case every neighborhood V_p contains a subset H of E equivalent to E; that is, of the same power or cardinal number as E.

THEOREM 1. *If an infinite set E is perfectly compact E determines at least one nuclear point.*

Let E be an infinite set of points p and denote the cardinal number of E by μ. Then as in § 3, we have an ordinal Ω_0 and a 1—1 correspondence

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* Fréchet, II, p. 343. A set E is compact if every infinite subset of E has a limit point; self-compact if it has a limit point in E.

† Loc. cit., p. 175

(2) $p_1, p_2, p_3, \ldots, p_n, \ldots, p_\omega, \ldots, p_\alpha, \ldots \quad (\alpha < \Omega_0)$

between the ordinal numbers α and elements p of E. Let

$$G_\alpha = \sum_{\beta \geqq \alpha}^{\Omega_0} p_\beta.$$

Then $G_1 = E$. The G_α form a monotonic sequence of subsets of E with no common element. But E is perfectly compact, so there must exist an element q common to all the G'_α.

Let V be a neighborhood of the point q. If q is not in E then it is a limit point of every G_α and therefore V contains a point q_α of G_α distinct from q. If q is in E there is an α such that G_α does not contain q. For every $\alpha' \geqq \alpha$ the point q is a limit point of $G_{\alpha'}$. Again q is a limit point of every G_α, and consequently V contains a point q_α of G_α. Let Q be the set of all distinct q_α, and for a given element q_α of Q let β denote the index such that q_α is in G_β but not in $G_{\beta+1}$. The index β is in fact that index α which corresponds to q_α regarded as an element of E and determined by the correspondence (2). These indices β are such that for every $\alpha < \Omega_0$ there is a $\beta > \alpha$. It follows at once that the β's and therefore the points of Q form an aggregate of power μ. Therefore q is a nuclear point of E.

COROLLARY. *Every set E which is perfectly self-compact contains a nuclear point.*

For the point q of the preceding proof may be assumed to be a point of the set E.

THEOREM 2. *If every infinite subset of a set E of points of the space P determines a nuclear point then E is perfectly compact.*

Let S be an open monotonic sequence of subsets G of a set E, satisfying the hypothesis of Theorem 2. Let H be a set of points p such that every G contains a point of H.

Let H be of power μ and let Ω_0 be the least ordinal of that power. We may assume a $1-1$ correspondence of the type (2) between the elements of H and the ordinals $\alpha < \Omega_0$. For each element p_α we may select a set G_α which does not contain p_α and such that $G_{\alpha+1}$ is contained in G_α. Since every set G of the series S contains some element of H it follows that every G contains some G_α. Now by hypothesis H has a nuclear point q. Let V be a neighborhood of q and let Q denote the subset of H of power μ in V. Then if β is any ordinal, $\beta < \Omega_0$, there is an element p_α of H in Q for which $\beta < \alpha$. Otherwise, because of the hypothesis on Ω_0, Q would be of power less than μ. It follows at once that V contains an element of every G_α and therefore of every G. Consequently q is common to all the G', contrary to the hypothesis that S is an open sequence.

Theorems 1 and 2 imply the following theorem.

THEOREM 3. *A necessary and sufficient condition that a set E be perfectly compact is that every infinite subset of E determine at least one nuclear point.*

COROLLARY. *A necessary and sufficient condition that a class (V) be a class (M); that is, that every compact set E be perfectly compact, is that every infinite compact set E possess at least one nuclear point.*

For if a compact set is finite it is perfectly compact. If it is infinite the result follows from Theorem 3.

5. *Hyper-Nuclear Points.* The concept *hyper-nuclear point* is helpful in generalizing the theorem of Kuratowski-Sierpinski. A point p is a *hyper-nuclear point* of a set E of power μ in case there is a subset H of E of power μ interior to every neighborhood of p.

The theorem to be generalized may be stated as follows. A necessary and sufficient condition that the theorem of Borel-Lebesgue apply to a class (L) is that every infinite compact set E whose derived set E' is also compact possess at least one hyper-nuclear point.

The extension of this result to classes (V) in general is a consequence of the two following theorems.

THEOREM 4. *If a self-compact set H contains a hyper-nuclear point of every infinite subset of H, then H admits the property of Borel-Lebesgue.*[*]

Let us suppose that H does not admit the property of Borel-Lebesgue. Then there is a family F of sets I which covers H and contains no finite subfamily with the same property. Among such families F there is (according to the theorem of Zermelo) at least one, say F_0, whose power μ is a minimum. Let Ω_0 be the first transfinite ordinal of power μ. There is a well ordered set

$$(3) \qquad I_1, I_2, I_3, \ldots, I_n, \ldots, I_\omega, \ldots, I_\alpha, \ldots \quad (\alpha < \Omega_0)$$

of order type Ω_0 comprising the totality of the sets I of F_0 and in one-one correspondence with the numbers $\alpha < \Omega_0$.

From the definition of F_0, every point of H is interior to at least one of the sets I_α but no segment of the sequence (3) has this property. We can also suppose that every set I_α contains in its interior an element of H, denoted by p_α, which is not interior to any set I_ξ for $\xi < \alpha$, since all the sets I_α which do not have this property could be suppressed in the sequence (3) without reducing the ordinal type (because there is no family covering H of power less than μ, and the reduced family would not fail to cover H).

The set Q of all the points p_α is evidently of the power μ (since $p_\alpha \neq p_\beta$ for $\alpha < \beta$; p_α being interior to I_α while p_β is not). But Q is a subset of H and so by hypothesis H contains a hyper-nuclear point q of Q. This point q is interior to some one of the sets I_α of the sequence (3), let it be I_γ. Since q is a hyper-nuclear point of Q there is a subset of Q of power μ interior to every neighborhood of q. But q is interior to I_γ which must contain a neighborhood V of q. It follows at once that there is a subset of

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[*] The proofs of Theorems 4 and 5 differ from the corresponding proofs of Kuratowski and Sierpinski (loc. cit., pp. 174–75) only with respect to those details which are involved in the generalization.

Q of power μ interior to I_γ. But the totality of the points p_ξ for $\xi \leqq \gamma$ is of cardinal number less than μ. Therefore I_γ contains in its interior an element p_ν of Q whose index ν exceeds γ. But by definition p_ν is not interior to any set $I_\xi (\xi < \gamma)$. This is a contradiction, and the assumption that H does not possess the property of Borel-Lebesgue fails.

THEOREM 5. *If an aggregate H admits the property of Borel-Lebesgue, H is self-compact and contains a hyper-nuclear point of every infinite subset of H.*

That the aggregate H of the theorem is self-compact is well known.* Let Q be an infinite subset of H of power μ and suppose that to each point p of H there is a neighborhood V_p such that the subset of Q in V_p is of power less than μ. The totality of such neighborhoods V_p covers H, and, by the Borel-Lebesgue property, it can be replaced by a finite family with the same property. Then Q can be represented as the sum of a finite number of sets each of power less than μ, contrary to the hypothesis that Q is of power μ.

From Theorems 4 and 5 we have the following theorem.

THEOREM 6. *A necessary and sufficient condition that an aggregate H possess the property of Borel-Lebesgue is that H be self-compact and that every infinite subset of H be hyper-nuclear in H.*

It should be noted that the hypothesis "derived sets are closed" does not enter this theorem directly. That for classes (V) in general the theorem above is independent of the closure of derived classes is shown by the example of the following section.

In a class (L) every self-compact class is compact and closed. From the theorem of Kuratowski and Sierpinski and Theorem 6 above we have the following theorem.

THEOREM 7. *In a class* (L) *a necessary and sufficient condition that every infinite compact set E whose derived set is compact determine a hyper-nuclear point is that every closed compact set H have the property "every infinite subset of H has a hyper-nuclear point in H".*

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* Fréchet, III, p. 152, § 19.

Since Fréchet has shown that every set with the property of Borel-Lebesgue is perfectly self-compact it follows from Theorem 6 that we have the following theorem.

THEOREM 8. *If a set H is self-compact and contains a hyper-nuclear point of every infinite subset of H, then H is perfectly self-compact.*

This theorem should be compared with Theorem 2. It would be interesting to have an example showing that the conclusion does not hold if the word "hyper-nuclear" in Theorem 8 is replaced by the word "nuclear".

6. *An Example.* It has been shown that in a class (L) the theorem of Borel implies the closure of derived classes but that in the more general classes (V) the theorem of Borel and that of Borel-Lebesgue may be formulated without the use of the hypothesis of the closure of derived sets. The following example completes the proof of this independence.

Let P be the class of all number pairs $p = (n, m)$ where $n = 1, 2, 3, \cdots$; $m = 1, 2, 3$, together with the number pair $(0, 0)$. The neighborhoods V are defined as follows. The elements $p = (n, 1)$, $n = 1, 2, 3, \cdots$, have but one neighborhood consisting in each case of the element p alone. Each element of the form $p = (n, 2)$ has a family of neighborhoods V_{kp}, consisting for given k of the point p and all points $q = (n, 1)$ for which $n \geqq k$ $(k = 1, 2, 3, \cdots)$. The elements $p = (n, 3)$ have a family of neighborhoods V_{kp}, each comprising the point p and all points $(n, 1)$ and $(n, 2)$ for which $n \geqq k$ $(k = 1, 2, 3, \cdots)$. The neighborhoods V_{kp} of the point $p = (0, 0)$ contain all the points $(n, 2)$ and $(n, 3)$ for $n = k$ $(k = 1, 2, 3, \cdots)$, in addition to the point $p = (0, 0)$ itself. It is easy to see that P is compact, and that every infinite subset of P determines a hyper-nuclear point. It may be verified directly that although derived sets are not closed, the theorem of Borel holds. The theorem of Borel-Lebesgue is satisfied vacuously, since the class P is enumerable.

THE UNIVERSITY OF IOWA

FIVE AXIOMS FOR POINT AND TRANSLATION IN AFFINE GEOMETRY

BY A. A BENNETT

1. *Introduction.* Postulate systems for geometry are so numerous that the development of a new system is of itself of little interest. Before explaining the present system, some remarks about various lines of approach to geometry may be of value in estimating its significance.

(1) The introduction of a new form of geometry upon the basis of an assumed familiarity with other geometrical studies is a well known practice. This is what is done in the usual treatment of analytical geometry, of descriptive geometry, and frequently of projective geometry. There is even a tendency these days to teach euclidean geometry along semi-intuitional lines. This procedure appeals to elementary students as being "concrete", but it makes any appreciation of the logical structure of the subject difficult.

(2) A "synthetic" axiomatic treatment of geometric figures is the classical and still the orthodox line of approach. This is the method of Euclid, of the familiar non-euclidean studies, of Hilbert's *Foundations*, and of Veblen and Young's *Projective Geometry*, not to mention others. This is however subject to the disadvantage that the employment of powerful and economical analytical methods is necessarily delayed. It has also in the past been the innocent cover for much inaccurate reasoning. While a wide acquaintance with actual synthetic theorems is essential to the neat handling of complicated relations, it is difficult to justify an avoidance of analytical tools when these would simplify the discussion. The utility of quasi-analytical notions is admitted even by the extreme euclidean purists, not only in the simpler relations but when these appear under such titles as "method of similar figures", "method of translation", "methods of rotation and reflexion", "method of

inversion". Of course algebraic methods can be proved justifiable on this basis, addition and multiplication being matters of definition, as Euclid did to some extent with his theory of proportion, and has been done completely by modern mathematicians. But there is at least a long, and to many students, painful delay, in the introduction of these algebraic operations.

(3) An axiomatic treatment based upon point and transformation may be followed, where the transformations are of the most general type leaving the propositions invariant. This is in keeping with Klein's famous *Programm*, is suggested by the non-axiomatic treatment of Lie, and has been carried out by Pieri, Hilbert, R. L. Moore, Gaba, and others. Here analytic machinery early suggests itself, but is not at once available. The transformation in its full generality impresses most students as neither an elementary nor an obvious notion, and so in a sense psychologically undesirable for a basic undefinable. The more familiar special transformations, such as translations, are found to be difficult of definition. Even in the case of such an elementary figure as a straight line, an extensive and intricate study is required before a definition is available.

(4) The number field may be first constructed, and geometry identified with the study of a hypercomplex algebra, addition and multiplication in the number field being developed in parallel. This is the characteristically analytical method, and that usual in vector analysis. Objections to the customary treatment are many. For instance there may be much duplication incident to a repetition for the hypercomplex numbers of the rules laid down for the original number field. Despite a possible formal independence, multiplication cannot be regarded as largely arbitrary. A thorough study of addition among points conditions in advance many of the properties that must be assumed for any useful system of multiplication, so that the extent to which multiplication may be defined in any given case is a matter of interest. Another frequent criticism of the

hypercomplex treatment is due to its usual dependence upon coordinate systems, and the absence of any early test of purely geometrical, that is, of invariantive properties

2. *Proposed Conditions.* It would seem not without interest to approach geometry on an axiomatic basis with the following conditions in mind.

(1) Analysis shall be available early.

(2) Save that congruence conditions are postponed, the system shall be essentially euclidean from the start.

(3) The analysis shall be of the simplest and most obvious sort, even at the possible sacrifice of symmetry.

(4) The one-dimensional case shall not be featured with special emphasis at the start.

(5) No special frame of reference shall be made fundamental.

A word as to these conditions. As to (1), no discussion should be required provided that it is not inconsistent with the retention of other ideals. As to (2), by postponing congruence conditions, all theorems become available not merely for the euclidean case but also for affine geometry, for which, as for projective geometry, there is no universal invariant for a point pair. Condition (3) gracefully excludes projective geometry with its extraneous factor of proportionality. Conditions (4) and (5) suggest that addition and multiplication be considered throughout as applied to hypercomplex numbers, and that further only their geometrically invariant features be discussed.

Beside the undefined element, *point*, it is natural from this line of attack to take as undefined the operation *translation*. Thus vector addition and subtraction may be expected to appear early in the discussion. The general notion of an affine transformation is eventually required. The question arises as to how far this may be secured by mere definition. From this stage on, the situation agrees in its essentials with that arising in general projective geometry. Within a single net of rationality no new undefined notion is required. For the general concept of straight line in its relation to affine transformations, a "fundamental" postulate is needed, and for the euclidean theory, not only must congruence be specialized, but order relations also must be established. One might refer for such questions to Veblen and Young's *Projective Geometry.*

3. *The Five Axioms.* The present brief study is confined for brevity and simplicity to the particularly elementary case of a finite geometry based on a Galois field where the prime modulus appears only to the first power, so that the field is modular in the primitive sense. It will be noted that three distinct "spaces" are considered, that of "points", that of "translations", and that of "automorphisms". The set of axioms is not categorical. Modifications are necessary in extension to cases which will admit the euclidean real or complex geometry as special instances, along the lines indicated above. Since the purpose of this set is to point out the extensive deductions inferrable from so few and apparently such mild conditions, and is not the study of this special and trivial type of space, questions of independence and completeness will not be discussed. The axioms together with some familiar and perhaps unnecessary definitions follow. "Point" and "translation" only are undefined.

Ax. I. *P is a finite set of elements.*

The elements of P may be called *points* and designated by the use of the letter p, with or without subscripts or other diacritical marks. By a permutation of P is meant an operation based upon a biuniform correspondence among the elements of P, by which each element of P is replaced by its uniquely corresponding element of P.

Ax. II 1. *A is a set of permutations of P.*

The elements of A may be called *translations* and designated by the use of the letter a, with or without subscripts or other diacritical marks. With respect to a given permutation, an element of P is said to be *fixed*, if the permutation replaces this element by itself. The *identical* permutation is that which leaves each element fixed. We have not required that A be a group or even a semi-group, but there will be a group of permutations of P generated by the elements of A and their inverses, or as we may say, by the set A.

Ax. II 2. *The identical permutation is the only permutation in the group generated by the set A which leaves as much as a single element of P fixed.*

Ax. II 3. *For the given set, P, it is not possible to extend the set A, by adjoining further permutations, so that the extended set shall also satisfy* II 1, *and* II 2.

From these axioms it follows readily that A is itself a group. With a little discussion of the consequences of A being possibly intransitive in P, it is seen that if A were intransitive, a new permutation, b, could be adjoined, commutative with each a, and interchanging subsets of P within each of which A is transitive. Furthermore the set A together with b would satisfy II 1 and II 2, by making explicit use of the commutative character of b, so that II 3 would fail to hold if A were not transitive. Zermelo's axiom in the finite case is assumed as a principle of logic.

For an unknown set P, the axioms II place no restrictions upon A other than that A shall be a group. For a given group A, the set P is found to be in biuniform correspondence with A and in as many ways as one chooses an element of P to correspond to the identical permutation. No points of P are specialized, while in A there is at least one element not to be confused with any other, namely, the identical permutation. Two permutations, which are elements of A, are said to be *equivalent*, if and only if there is an automorphism of A (hence preserving the multiplication table of A), which replaces the first by the second, and also an automorphism of A replacing the second by the first. Since an automorphism may replace distinct elements by a single one, this double condition is needed.

Ax. III. *Any two elements of A, neither of which is the identical element, are equivalent.*

We may now conclude that with the exception of the identical element, all elements of A are of the same order. Since any cyclic group of composite order has proper cyclic subgroups, we conclude that the common order of all non-identical elements is a prime, say p. The total order of the group can contain no prime factor other than this p, so that since the set is of finite order, this total order may be designated by p^n. We may call this n the dimensionality of P and of A. Any set of p points, each of which is obtainable from each other by the operations of a cyclic subgroup of A, may be called a *line*. Thus each line has p distinct points.

It is a well known theorem* in the study of finite groups that every prime power group contains elements other than the identity commutative with every other element of the group. Applying this to A, and using the fact that the elements, a, other than the identity are equivalent, we infer that A is commutative. We may now identify the operation of obtaining the result of two successive permutations, with the operation of adding these elements, a.

The *totality of automorphisms* of A is called the set M. Thus if m is an element of M, and a_1 and a_2 are two elements of A, it follows that $m(a_1+a_2)=m(a_1)+m(a_2)$. Given two automorphisms, m_1 and m_2, and denoting by a an unrestricted variable in A, we may define a' and a'' by the conditions $m_1(a)=a'$, $m_2(a)=a''$. Now there is an automorphism replacing a by the sum $a'+a''$, since there is obviously such a uniform correspondence and it satisfies the distributive condition essential for an automorphism, since in fact $(a_1' + a_2') + (a_1'' + a_2'') = (a_1' + a_1'') + (a_2' + a_2'')$. Since M contains each automorphism of A, it contains this also. We shall naturally designate it by $m_1 + m_2$. Thus M is completely distributive with respect to the addition of elements of A.

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* Cf. Hilton, *Finite Groups*, p. 142.

The result of the successive application of two elements of M is again in M, and M is found to constitute a semigroup, which furthermore contains an identical element for the semigroup. The terms *multiplication* and *product* may be used in a strict sense for this M. While M is a semigroup with respect to multiplication, it may be shown to form with respect to addition, as already used, a group. Its identical element with respect to addition is the *null-automorphism*, that which replaces each element of A by O, the identical element of A. The element of M identical with respect to multiplication is of course that which replaces each a by itself. Thus M taken with its rules of operation constitutes a linear algebra. This M includes singular automorphisms, that is automorphisms replacing distinct elements of A by a common element. It is therefore no group. Whether approached from the side of the modular theory of numbers, or from the side of specializations in the theory of general projective geometry, it is found that M is simply isomorphic with the semigroup of all affine projectivities among translations in a modular space, modulo p, of n dimensions, leaving the null translation invariant, and with the semigroup of all linear homogeneous substitutions in n variables, taken modulo p, no factor of proportionality being suppressed.

In the one-dimensional case, multiplication is commutative in M and the number of elements in M is the same as in P and A. For cases of higher dimensionality, there are p^{n^2} elements in M and multiplication is not commutative, although within any cycle multiplication is of course always commutative.

THE UNIVERSITY OF TEXAS

SOME PROBLEMS OF CLOSURE CONNECTED WITH THE GEISER TRANSFORMATION*

BY ARNOLD EMCH

1. *Introduction.* Problems of closure may be defined as series of geometrical operations of the same type performed on a given figure with the property that the series closes after a finite number of steps and that the closure in one instance has as a consequence the closure of an infinite number of series with the same number of steps performed on the same given figure. As examples of such problems may be mentioned the well known Steiner series of circles attached to two given non-intersecting circles, the Poncelet polygons, the Steiner polygons inscribed in a cubic, etc. There are various methods of treating problems of this kind. One very effective method for a certain class of problems is by means of elliptic functions, as inaugurated by Jacobi† and Clebsch.‡

Another method, distinguished by its simplicity and directness, has been established by A. Hurwitz§ and is based on the correspondence principle in a one-parameter algebraic domain. For example, if the correspondence between the elements is (m, n) on a rational curve, there are $m+n$ coincidences. Now it is possible that in certain cases the correspondence may be such that there are more than $m+n$ coincidences. If this happens, then there are an infinite number of such coincidences and we have a problem of closure. Algebraically the problem may be stated thus: If the solution of a geometrical problem leads to an equation of degree n in one unknown (parameter), and if under certain imposed conditions this equation admits of more than n roots, then the equation has an infinite number of roots, and the problem admits of an infinite number of solutions.

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* Presented to the Society, April 18, 1924.

† *Über die Anwendung der elliptischen Transcendenten auf ein bekanntes Problem der Elementargeometrie,* Crelle's Journal, vol. 3, p. 376.

‡ *Über einen Satz von Steiner und einige Punkte der Theorie der Curven dritter Ordnung,* Crelle's Journal, vol. 63 (1864), pp. 94–121.

§ *Über unendlich-vieldeutige geometrische Aufgaben, insbesondere über die Schliessungsprobleme,* Mathematische Annalen, vol. 15 (1879), pp. 8–15.

Recently Professor A. B. Coble has worked out a method of procedure of an invariantive character by which he has been able to solve problems of closure (porisms) in a very elegant manner.* The purpose of this paper is to establish some new problems of closure by a certain mapping process applied to previously known problems of closure. It was also by a mapping process that the writer found the problems of closure stated in the recently published paper on the geometry of the symmetric group.†

2. *The Geiser Transformation.*‡ For a better understanding of what follows it is perhaps well to state the principal properties of this well known involutory Cremona transformation in a plane in agreement with our notation. The seven base-points $A_1, A_2, \cdots, A_7$ in a general position determine a net of cubics so that any two cubics of the net intersect in two points P, P' which as a pair of corresponding points define the Geiser transformation. The base-curves C_i are nodal cubics through the base-points, with their nodes at the A_i's respectively. The Φ_i's are octics with triple points at each of the A_i's, and determine a net of octics. The transformation has the pointwise invariant sextic C_6^2 with double points at each of the A_i's,

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* *Multiple binary forms with the closure property,* AMERICAN JOURNAL, vol. 43 (1921), pp. 1–19.

† AMERICAN JOURNAL, vol. 45 (1923), pp. 192–207. See also the author's monograph, *Applications of elliptic functions to problems of closure,* THE UNIVERSITY OF COLORADO STUDIES, vol. 1 (1902), pp. 81–133.

‡ Geiser, *Uber zwei geometrische Probleme,* CRELLE'S JOURNAL, vol. 67 (1867), pp. 78–89; Sturm, *Die Lehre von den Geometrischen Verwandtschaften,* vol. 4, 1909, pp. 96–103.

which is consequently of genus 3, and which is the Jacobian of the net of cubics. It has the same nodal tangents at the A_i's as the nodal base-cubics. The transformation is of class 1. The invariant isologue curve C_S attached to a point S is an elliptic cubic which outside of the A_i's cuts the C_6^2 in four points P_1, P_2, P_3, P_4 which form a Steinerian quadruple on the cubic C_S; i. e., the tangents to the C_S at the P_i's meet in S which is on the C_S. From this follows that corresponding pairs P, P' of the Geiser transformation on the C_S are also corresponding pairs in an involutory quadratic transformation with P_1, P_2, P_3, P_4 as the quadrangle of invariant points. This property of the C_S makes it easy to establish the relation between an invariant C_μ of the Geiser transformation and the corresponding class-curve K_ν from which it is generated.

The order of the transformed C_μ is in general 8μ. Hence in order that this number reduce to μ it is necessary that the C_μ have multiplicities j_i at the A_i's, such that $\sum 3j_i = 8\mu - \mu = 7\mu$, which shows that μ must be a multiple of three. The class ν of the K_ν is equal to the number of lines joining couples of corresponding points P, P' on the C_μ through any given points S. All such couples also lie on the attached C_s. The intersections of the C_μ and the C_s at the A_i's absorb $\sum j_i$ points, so that outside of the A_i's there are $3\mu - \sum j_i$ intersections which arrange themselves into half that many couples aligned through S. Hence $\nu = (1/2)(3\mu - \sum j_i)$. In case of an invariant C_3, $\nu = 1$, which corroborates the fact that the net of cubics through the A_i's is identical with the net of isologue cubics.

Let us consider an invariant C_{3m} in the Geiser transformation. As for a general C_{3m} the corresponding curve is of order $24m$, a curve of order $21m$ must split off in order to have a proper corresponding curve of order $3m$. Assuming the simple case in which the C_{3m} has equal multiplicities at the A_i's, these multiplicities are necessarily of order m, since whenever a branch of the C_{3m} passes through an A_i, the corresponding base-cubic splits off as a part of the C'_{3m}. Now the isologue of a point S cuts the C_{3m} in $3\cdot 3m - 7\cdot m = 2m$ points outside of the A_i's. Hence for an invariant C_{3m} these arrange themselves into m couples of corresponding points on lines through S. Hence the invariant C_{3m} may be generated from a K_m.

As the K_m is determined by $m(m+3)/2$ conditions, the manifold of invariant C_{3m}'s of this type is equal to this number. The number of conditions of the class of C_{3m}'s with multiplicities of order m at each A_i is $k = (1/2)\,3m(3m+3) - 7(1/2)m(m+1) = m(m+1)$. The question is, how must the k points be chosen to insure an invariant C_{3m}. It is obvious that since k is even, a sufficient condition is that the k points form $k/2$ couples. But this is not always necessary. From the k points choose γ couples of corresponding points, then the C_{3m} and C'_{3m} have at least $7m^2 + 2\gamma + 4m$ points in common, since the C_{3m} cuts the pointwise invariant sextic in $6\cdot 3m - 7\cdot 2\cdot m = 4m$ points. If $7m^2 + 2\gamma + 4m \geqq 9m^2$, then $C_{3m} \equiv C'_{3m}$. This condition reduces to $2\gamma \geqq 2m^2 - 4m$. On the other hand $2\gamma \leqq m(m+1)$. For $m = 5$, this gives $30 \leqq 2\gamma \leqq 30$. Thus in case of an invariant C_{15}, 15 couples determine such a curve uniquely. As $2m^2 - 4m - (m^2 + m) = m(m-5)$ is positive for all values of $m > 5$, in case of C_{3m}'s for $m > 5$, $(1/2)m(m+1)$ couples determine the invariant C_{3m} uniquely.

As an example of particular interest may be mentioned the class of invariant sextics with double points at each of the base-points. In the first place a sextic with double points at each of the A_i's depends on six free constants and is transformed into a sextic of the same type since $6\cdot 8 - 2\cdot 7\cdot 3 = 6$. Such a non-invariant sextic C^*_6 cuts the C^i_6 in $36 - 7\cdot 4 = 8$ points J outside of the A_i's. The C^*_6 and the C^i_6 determine a pencil of sextics with the same multiplicities at the A_i's and all passing through the 8 points J. Through every point of C^i_6 there is a selfcorresponding direction, i. e., a line on which lie two corresponding points of the Geiser transformation infinitely close to the C^i_6. Consider now any of the 8 points J and the corresponding line-element through J. There is just one sextic C_6^* of the pencil through the A_i's and the J's which will have this element as a tangent at J. The transformed $C_6^{*\prime}$ belongs to the pencil, since the J's are invariant, and along the element through J cuts the C_6^* in two consecutive points. Hence the two curves intersect in 37 points and are therefore identical. This establishes the existence of invariant sextics. As every invariant sextic C_6 determines a pencil of non-invariant curves and as all non-invariant curves are contained among such pencils, the system of invariant sextics with double points at the A_i's depends on five effective constants. This may be verified as follows: An invariant cubic C_3 cuts an invariant C_6 in four points outside of the A_i's. These lie in couples of corresponding points on two lines through S. Consequently the lines joining corresponding points on an invariant C_6 with double points at the A_i's envelope a conic K_2. Conversely any invariant sextic of this sort may be generated from a conic K_2 of class two. There are therefore ∞^5 such sextics. If we denote by ψ_1, ψ_2, ψ_3 three linearly independent invariant cubics, any invariant sextic of the system may be represented in the form $\sum_{i,k=1}^{3} a_{ik}\psi_i\psi_k = 0$. To sum up we have the following theorem.

THEOREM I. *The entire class of* ∞^5 *invariant sextics with double points at the base-points may be generated from the class of conics* K_2. *An invariant sextic* C_6 *cuts the pointwise invariant sextic* C_6^i *in* 8 *points, so that the tangents to the* C_6 *at these points touch a conic* (K_2).

The pointwise invariant curve C_6^i connected with seven generic points in a plane is sometimes called the *Aronhold* curve. It cannot be represented as a polynomial of the second degree of three linearly independent cubics through the seven points. However its square may be expressed as a certain polynomial of the fourth degree in the three cubics $C_3^{(1)}$, $C_3^{(2)}$, $C_3^{(3)}$,, so that $(C_6^i)^2 = F(C_3^{(1)}, C_3^{(2)}, C_3^{(3)})$, where F is a general quartic with the C_3's as projective coordinates. Hence the general quartic can be resolved by three linearly independent cubics through seven points.

It is easy to study higher systems of invariant curves in the Geiser transformation by the properties of the transformation and by the curves K_ν from which they are generated. A K_ν generates a reducible curve of order 9ν with an equation of the form $(C_6^i)^\nu H(\psi_1, \psi_2, \psi_3) = 0$, in which the H is of degree 3ν in the x's. The pointwise invariant curve C_6^i splits off ν times and leaves an invariant curve H of order 3ν with ν-fold points at the base-points.

3. *Mapping by the Geiser Transformation on a General Cubic Surface.* Consider the cubic Cremona transformation between two quaternary spaces $\sum$ and $\sum'$ in which to the planes p of $\sum$ correspond cubic surfaces C' through the base-curve S' in $\sum'$ and conversely to the planes p' of $\sum'$ correspond cubic surfaces C through the base-curve S in $\sum$. The curves S and S' are sextics of genus three and the base-surfaces of the transformation are octic surfaces formed by the trisecants of S and S'. To a point of S corresponds a trisecant of S', and conversely to a point of S' a trisecant of S.

Choose a definite plane p in $\sum$ as the plane of a Geiser transformation which by the cubic transformation is mapped into a definite cubic C' of $\sum'$. The sextic S cuts p in six points $A_1, \ldots, A_6$ which in general do not lie on a conic. These points we take as six points of the Geiser transformation, while A_7 may be chosen in some fixed position independent of the six other points. To $A_1, \ldots, A_6$ correspond on C' six lines $a'_1, \ldots, a'_6$, while the image of A_7 is some point A'_7 on C'. To a plane section p' through A'_7 with C' corresponds in p a plane cubic C_3^p through $A_1, \ldots, A_7$. Likewise to another plane section q' through A'_7 corresponds in p another plane cubic C_3^q through the base-points. C_3^p and C_3^q intersect, outside of the A's, in two points P and Q which are corresponding points in the Geiser transformation. To them correspond on C' two points P' and Q' which lie in p' and q', and which are therefore the points in which the line of intersection of p' and q' cuts the cubic surface C'. Thus to pairs of corresponding points of the Geiser transformation correspond on the cubic C' pairs of corresponding points cut out by secants u' through A_7'. Every line in p carries one pair of corresponding points, and to it corresponds in $\sum'$ uniquely a line u' through A_7'.

THEOREM II. *The correspondence between* u *and* u' *is* (1,1) *and involutory. Hence the points* P *and lines* u *of* p *are in involutory reciprocity with the planes* p' *and lines* u' *through* A_7'. *In this reciprocity, to a curve* K_m *of class* m *in* p *corresponds in* $\sum'$ *a cone* K_m' *of order* m *with* A_7' *as a vertex. The locus of pairs of corresponding points on tangents* u' *of* K_m *is an invariant* C_{3m} *of the Geiser transformation, as has been proved in* § 2. *The cone* K_m' *cuts* C' *in a curve* C_{3m}' *of order* $3m$ *which is invariant in the involution* (P', Q') *on* C'. *Thus* C_{3m} *and* C_{3m}' *correspond to each other in the cubic transformation between* $\sum$ *and* $\sum'$.

To the plane sections through A_7' and a_i' correspond in p the base-curves of the Geiser transformation. To the plane section of the tangent-plane to C' at A_7' corresponds the base-curve with A_7 as a node. To the tact-sextic of the tangent cone from A_7' to C' corresponds in p the Aronhold curve or the pointwise invariant sextic or Jacobian of the net of plane cubics through the A's.

4. *Problems of Closure.* A cone K_m' with A_7' as a vertex cuts the cubic C' in a curve C_{3m}' of order $3m$. As K_m' cuts each of the lines a_i' in m points, the corresponding C_{3m} in p, generated from the class-curve K_m, has $A_1, \ldots, A_6$ as multiple points of order m. But the C_{3m}' has an m-fold point at the point A_7', so that also A_7 has the multiplicity m on C_{3m}. A tangent-plane to the cone K_m' along an element u' cuts C' in a cubic C_3' which touches C_{3m}' in the two points P' and Q' cut out by u'. Thus, to C_3' corresponds in p a cubic C_3 which touches C_{3m} in corresponding points of the Geiser transformation.

As a particular case, choose two quadric cones K_2' and L_2' with common vertex at A_7', but not intersecting in real generators (u'). They intersect C' in two sextics T' and S'. Every tangent plane of K_2' cuts C' in a cubic C_3' which touches T' in two points on a u' and cuts S' in two pairs of points whose joins also pass through A_7', since they lie on generators of L_2'. If we now construct a pyramid with vertex at A_7' such that its faces are tangent to K_2' and its edges are generators of L_2', and if once such a pyramid closes for two fixed cones K_2' and L_2', then there are an infinite number of such pyramids, circumscribed to K_2' and inscribed to L_2'. If by means of the cubic transformation we go back to the Geiser plane p, we derive immediately the following theorem.

THEOREM III. *Let T and S be two invariant sextics of the Geiser transformation and C_1 an invariant cubic touching T in two points T_1 and T_1' and cutting S in two couples of corresponding points S_1, S_1' and S_2, S_2'. Through S_2, S_2' pass an invariant cubic C_2 touching T in a couple of corresponding points T_2, T_2' and cutting S in a couple S_3, S_3'. Through S_3, S_3' pass a cubic C_3 touching T in T_3, T_3' and cutting S in S_4, S_4', and so forth. Suppose this process continued n times, so that the last cubic C_n of the series cuts S in S_{n+1}, S_{n+1}'. If once S_{n+1}, S_{n+1}' coincides with S_1, S_1, i. e., if the process closes, then there is always closure after n operations, no matter what initial cubic C_1 touching the sextic T in a couple T_1, T_1' is chosen.*

The joins of all couples T_i, T_i' envelope a conic K_2, those of S_i, S_i' a conic L_2, such that K_2 and L_2 are correlative to K_2' and L_2'.

A general cubic cone K' with vertex at A_7' cuts the cubic surface C' in a curve S_9' of order 9. To it corresponds in p an invariant curve of the same order, generated from a correlative plane cubic of class 3. Through a generator u' of K' pass two tangent planes α_g and α_h touching K' along the generators g and h. α_g and α_h cut

C' in plane cubics C'_g and C'_h which pass through the intersection U and U' of u' with C', and which touch the C'_9 in couples G, G' and H, H' of corresponding points in the involution on the cubic C'. The closure property of the quadrilateral pyramid with vertex at A'_7 whose six edges are generators of K', all in analogy with the closure theory of Steiner polygons inscribed in a plane cubic, transferred to the Geiser plane leads to the following theorem.

THEOREM IV. *Given an invariant S_9 in the Geiser transformation and on it a couple (U, U') of corresponding points. Through (U, U') draw two invariant cubics touching S_9 in two couples (G, G') and (H, H') respectively. Through (G, G') pass any invariant cubic C_1 cutting S_9 in two couples (A, A') and (B, B'); through (H, H') and (B, B') pass an invariant cubic C_2 cutting S_9 in (C, C'); through (G, G') and (C, C') pass the invariant cubic C_3 cutting S_9 in (D, D'); through (H, H') and (D, D') pass the invariant cubic C_4; then C_4 will always pass through (A, A'), no matter what initial first couple (A, A') or initial cubic C_1 is chosen.*

By the mapping process explained above a number of problems of closure may be obtained without difficulty. Thus we may state the following theorem.

THEOREM V. *Given an invariant curve C_9 of order* 9 *with triple points at the base-points of the Geiser transformation. Through a couple of corresponding points of C_9 draw the four invariant cubics touching the C_9, each touching the C_9 in a couple of corresponding points. Thus four couples are obtained which we define as a Steinerian octuple on the C_9. Consider a second octuple of this sort on the C_9. Any couple of the first octuple together with any couple of the second octuple determine an invariant cubic uniquely. In this manner are determined* 16 *invariant cubics which by fours intersect in four couples of a third octuple on the C_9.*

THE UNIVERSITY OF ILLINOIS

A SYMMETRIC COEFFICIENT OF CORRELATION FOR SEVERAL VARIABLES*

BY DUNHAM JACKSON

Let $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ be two sets of n real numbers each, neither set consisting entirely of zeros. In order that the ensuing use of technical terms may be in accordance with established practice, let it be supposed that

$$\sum x_k = \sum y_k = 0,$$

though the mathematical relations to be considered are in themselves not dependent on this hypothesis. The coefficient of correlation between the x's and the y's, defined by the formula

$$r = \frac{\sum x_k y_k}{\sqrt{\sum x_k^2 \sum y_k^2}},$$

may be regarded as a measure of the degree of resemblance of the two given sets of numbers, and may be interpreted as the cosine of the angle $POQ = \theta$, if O, P, Q represent respectively the origin, the point with coordinates $(x_1, x_2, \cdots, x_n)$, and the point $(y_1, y_2, \cdots, y_n)$, in space of n dimensions.†

Let a third set of real numbers $(z_1, z_2, \cdots, z_n)$ be given, not all zero (and subject, let us say, to the condition $\sum z_k = 0$). There may sometimes be occasion to consider a measure of the degree of resemblance of the three sets (x_k), (y_k), and (z_k), not an asymmetric measure of the dependence of a specified set on the other two, as in the case of a coefficient of double correlation, but a formula treating all three sets alike.* Such a formula will be discussed in the following pages. Some theoretical advantages belonging to it will appear in the course of the discussion; its practical utility must naturally await the test of experience. If it has already been treated elsewhere, as is not unlikely, the writer hopes that the present exposition may nevertheless serve a useful purpose in giving it wider publicity.

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† For an elementary exposition of this idea, and for bibliographical references, see a paper by the author entitled *The trigonometry of correlation*, AMERICAN MATHEMATICAL MONTHLY, vol. 31 (1924), pp. 275-280.

Without loss of generality, it may be assumed that the given sets of numbers are normalized, so that

$$\sum x_k^2 = \sum y_k^2 = \sum z_k^2 = 1.$$

Then the points P and Q, and the point R with coordinates $(z_1, z_2, \cdots, z_n)$, are on the surface of a sphere of unit radius about the origin. The discussion is concerned with the degree of propinquity of these three points. As the correlation of the first two sets of numbers is measured by means of the angle POQ, there would be a certain analogy in making use, at the next stage, of the solid angle $O—PQR$, or, in other words, of passing from the length of the arc PQ to the area of the spherical triangle PQR as a quantitative indication. But the area of the triangle is zero whenever the three points lie on the same great circle, whether the points themselves are close together or widely separated, so that it seems necessary to look further for a satisfactory definition.

Let S be the middle point of the chord PQ. The value of the coefficient of correlation r can of course be expressed in terms of the angle $\Phi = \frac{1}{2}\theta = POS$, the relation being

$$r = \cos\theta = \cos 2\Phi = 2\cos^2\Phi - 1.$$

The extreme limiting values of $\cos\Phi$, for all possible positions of P and Q, are not -1 and 1, as in the case of $\cos\theta$, but 0 and 1. As $\cos\Phi = OS$ is the perpendicular distance of the chord PQ from O, an analogous quantity in some respects would be the distance from O to the plane of P, Q, and R. But a knowledge of this quantity would scarcely give the kind of information that is desired, since the distance is zero whenever the three points are on the same great circle, and may be either zero or arbitrarily near unity for three points arbitrarily close to each other, becoming wholly indeterminate between these limits in the one case for which an unmistakable characterization is most essential, the case of actual coincidence of P, Q, and R.

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* This problem was suggested to the writer by a study of the distribution of freshman grades, in which Professor W. H. Bussey was engaged.

It seems more instructive to regard OS as the distance from O to the *center of gravity* of the points P and Q, regarded as material particles of equal weight. Let T be defined as the center of gravity of P, Q, and R, and let $\bar{r}_1 = OT$. This $\bar{r}_1$ is 1 if and only if the three points are coincident; it is 0 if and only if they are situated on a great circle at angular distances of 120° from each other; and it has a value intermediate between 0 and 1 in all other cases. A formal expression for it is readily calculated. The coordinates $(w_1, w_2, \cdots, w_n)$ of T are given by

$$w_k = \frac{1}{3}(x_k + y_k + z_k),$$

so that $\bar{r}_1 = \sqrt{w_1^2 + w_2^2 + \cdots + w_n^2}$

$$= \frac{1}{3}\sqrt{\sum x_k^2 + \sum y_k^2 + \sum z_k^2 + 2\sum x_k y_k + 2\sum x_k z_k + 2\sum y_k z_k}.$$

In consequence of the hypothesis that the x's, y's, and z's are normalized, the sums of squares under the last radical are each equal to 1, and the product sums are the respective coefficients of correlation r_{12}, r_{13}, r_{23} of the given sets of numbers, taken two sets at a time. That is,

$$\bar{r}_1 = \frac{1}{3}\sqrt{3 + 2r_{12} + 2r_{13} + 2r_{23}}.$$

Taken on its own merits, independently of any geometric interpretation, this formula defines a number which is manifestly subject to the inequalities $0 \leqq \bar{r}_1 \leqq 1$, since the radical is of course to be taken as positive or zero, and none of the numbers r_{12}, r_{13}, r_{23} can exceed 1; the quantity under the radical sign is equal to 9 if $r_{12} = r_{13} = r_{23} = 1$, is 0 if[*] $r_{12} = r_{13} = r_{23} = -\frac{1}{2}$, and can never be negative,[†] since it is equal to $9 \sum w_k^2$.

A measure somewhat more closely analogous to the ordinary coefficient of correlation, and slightly easier to compute, the measure which it is the main purpose of this paper to suggest, is related to $\bar{r}_1$ as $\cos\theta$ is related to $\cos\varphi$, and *is defined by the formula*

$$\bar{r} = 2\bar{r}_1^2 - 1 = \frac{4}{9}(r_{12} + r_{13} + r_{23}) - \frac{1}{3}.$$

From its dependence on $\bar{r}_1$, it ranges between the extreme values -1 and $+1$, the latter value signifying perfect correlation of all three of the given sets of quantities. It is simply a linear function of the "average coefficient of correlation" of the given sets, adjusted so as to have -1 and $+1$ for its extremes; this adjustment, however, appears from the above treatment not as a mere piece of algebraic formalism, but as a natural corollary of simple geometric relations.

When there are m variables instead of three, that is, m given (normalized) sets of n quantities each, the notation is to be changed by letting $(w_1, \cdots, w_n)$ stand for the co-

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[*] These relations, together with the conditions $\Sigma x_k = \Sigma y_k = \Sigma z_k = 0$, are satisfied if, for example, the x's, y's and z's are respectively

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \cdots, 0\right), \quad \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, \cdots, 0\right),$$
$$\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \cdots, 0\right),$$

all the entries after the first three being zero in each parenthesis.

[†] In particular, such a combination as $r_{12} = r_{13} = r_{23} = -1$ is impossible; it is readily seen that if $r_{12} = r_{13} = -1$, the third coefficient r_{23} must necessarily be $+1$.

ordinates of the center of gravity of the m corresponding points of the unit sphere, so that

$$w_k = \frac{1}{m}(x_k + y_k + z_k + \cdots), \qquad (k = 1, 2, \cdots, n);$$

$$\bar{r}_1 = \sqrt{\sum w_k^2} = \frac{1}{m}\sqrt{m + 2\sum r_{ij}},$$

$$(i = 1, 2, \cdots, m, j = i + 1, i + 2, \cdots, m);$$

$$\bar{r} = 2\bar{r}_1^2 - 1 = \frac{4}{m^2}\sum r_{ij} - \frac{m-2}{m}.$$

The quantities r_{ij} are once more the coefficients of correlation of two sets at a time of the given m sets of numbers, supposed subject to the conditions

$$\sum x_k = \sum y_k = \sum z_k = \cdots = 0.$$

It is readily seen, both algebraically and quasi-geometrically, that $\bar{r}$ varies from the value -1 at one extreme to the value $+1$, denoting perfect correlation, at the other.

It is well known that the center of gravity of a system of particles corresponds to the solution of a simple problem in least squares. If the particles are of equal weight, as is assumed to be the case here, the center of gravity is the point so situated that the sum of the squares of the distances of the various particles from it is a minimum. Something analogous to the idea of least squares can be made apparent in the present connection under a slightly different form.

Since $\cos\theta$ is approximately equal to $1 - \frac{1}{2}\theta^2$, when θ is small, the minimizing of a sum of squares may be looked upon as related to the problem of making a sum of cosines a maximum. Let $(\mu_1, \mu_2, \cdots, \mu_n)$ be the direction cosines of an arbitrary* line OM, that is, an arbitrary set of n numbers subject to the condition $\mu_1^2 + \mu_2^2 + \cdots + \mu_n^2 = 1$. If the point M is taken at unit distance from the origin, the numbers μ_k will be its coordinates. Let the angles POM, $QOM, \cdots$, be denoted by $\varphi_1, \varphi_2, \cdots$, and let the problem be proposed of determining the μ's so that

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* It is not assumed that $\sum \mu_k = 0$.

$\cos\varphi_1 + \cos\varphi_2 + \cdots + \cos\varphi_m$ shall be a maximum. Since

$$\cos\varphi_1 = x_1\mu_1 + x_2\mu_2 + \cdots + x_n\mu_n,$$

etc., the quantity s to be made a maximum is

$$\begin{aligned} s &= (x_1 + y_1 + \cdots)\mu_1 \\ &\qquad + (x_2 + y_2 + \cdots)\mu_2 + \cdots + (x_n + y_n + \cdots)\mu_n \\ &= m(w_1\mu_1 + w_2\mu_2 + \cdots + w_n\mu_n), \end{aligned}$$

or, if λ_k denotes the kth direction cosine of the line from O to the center of gravity* T,

$$\lambda_k = \frac{w_k}{\sqrt{w_1^2 + w_2^2 + \cdots + w_n^2}} = \frac{w_k}{\bar{r}_1},$$

then

$$s = m\bar{r}_1(\lambda_1\mu_1 + \lambda_2\mu_2 + \cdots + \lambda_n\mu_n).$$

The last parenthesis is the cosine of the angle TOM, and can not exceed 1, being equal to 1 when $\mu_k = \lambda_k$, that is, when the line OM is drawn through T. So OT is determined as the line for which the sum of cosines is greatest. The maximum value of the sum is seen to be $m\bar{r}_1$.

As $\cos\varphi_i$ is equal to

$$1 - 2\sin^2\left(\frac{1}{2}\varphi_i\right),$$

it can also be said that the sum of the squares of the sines of the half angles has been made a minimum, and hence that there has been literally a solution of a problem in least squares. Let L denote the point with coordinates $(\lambda_1, \lambda_2, \cdots, \lambda_n)$, where the line OT pierces the unit sphere, and let $\varphi_1, \varphi_2, \cdots$, now stand for the angles $POL, QOL, \cdots$. Then $2\sin(\frac{1}{2}\varphi_1)$ is the length of the chord LP, etc., and L is determined on the surface of the sphere as the point for which the sum of the squares of the rectilinear distances from $P, Q, \cdots$, is a minimum. If these distances from $P, Q, \cdots$, to L are denoted by $d_1, d_2, \cdots$,

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* The problem becomes indeterminate in the limiting case in which T falls at the origin.

$$\sum_{i=1}^{m} d_i^2 = 4\sum_{i=1}^{m}\sin^2\frac{\varphi_i}{2} = 2\sum_{i=1}^{m}(1-\cos\varphi_i) = 2(m-m\bar{r}_1),$$

and $\bar{r}_1$ has the value

$$\bar{r}_1 = 1-\frac{1}{2m}\sum_{i=1}^{m} d_i^2.$$

But this reckoning does not seem to lead to any particularly simple geometric interpretation for $\bar{r}$. It may turn out, in some circumstances at least, that $\bar{r}_1$ is the more significant measure after all.

THE UNIVERSITY OF MINNESOTA

METHODS FOR FINDING FACTORS OF LARGE INTEGERS*

BY H. S. VANDIVER

1. *Introduction.* We shall examine, in this paper, the problem of finding factors of integers beyond the range of Lehmer's factor tables, by methods shorter than that of dividing the integer by all the primes less than its square root.

Three methods will be proposed here. The first two depend on the representation of the integer as a definite quadratic form, and the third on the representation as an indefinite quadratic form. As I hope to devote another paper to the development of the last two methods, only outlines and a few examples will be given in connection with them.

The theory of quadratic forms has been applied in several different ways to the problem.†

In particular, Seelhoff‡ gave an expeditious method with the use of tables, which, however, is limited in application, since if the integer is composite, it does not always enable us to determine the factors.

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* Presented to the Society, September 7, 1923, under the title *A method of finding factors of integers of the form* $8n+1$. The author was enabled to carry out this investigation through a grant from the Heckscher Foundation for the Advancement of Research.

† Dickson, *History of the Theory of Numbers,* vol. 1, pp. 361–66.

‡ AMERICAN JOURNAL, vol. 7, p. 264; vol. 8, p. 26.

The first method explained in the present paper applies only to integers of the form $8n+1$ and $12n+1$ which are the product of two distinct primes. Although the process will always yield the factors in a finite number of steps, it is impracticable on account of its length if the integer is as large as 10^7. *Unlike other methods, however, the particular forms used are always the same.* Hence they lend themselves conveniently to the construction of tables with a view of lessening the amount of computation involved. The theory of indefinite forms has been applied by Tchebycheff* to our problem, but the method is quite different from the one given here which involves indefinite forms.

2. *First Method of Factorization.* Suppose that we have an integer m of the form $8n+1$ which is the product of two distinct prime factors. It is immediately seen that each factor has the same residue modulo 8, otherwise the product would not be of the form $8n+1$. If both are of the form $8n+1$ or both of the form $8n+5$ then m may be presented in more than one way in the form x^2+y^2, x and y prime to each other. Similarly if both are of the form $8n+3$ then m may be represented in more than one way in the form x^2+2y^2, x and y prime to each other. In the case where both are of the form $8n+7$ it will now be shown that m may be represented in at least two different ways in the form $2x^2-y^2$, where x and y are each $<\sqrt{m}$ and prime to each other.

In a paper which appeared in this BULLETIN (vol. 22 (1915), pp. 61–66) I proved that if p is a prime and a is a positive integer prime to p, then there is at least one and not more than two sets (x, y) such that

$$ay \equiv \pm x \pmod{p}$$

when x and y are positive integers prime to each other and $0<x<\sqrt{p}$, $0<y<\sqrt{p}$.

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* LIOUVILLE, (1), vol. 16, p. 257.

On page 64 of the same article, I remarked that the existence of at least one set (x, y) satisfying the conditions

$$ay \equiv \pm x \pmod{m} \tag{1}$$

where m is composite followed also from the reasoning used. If two sets $(x_1\, y_1)$ and $(x_2 y_2)$ are regarded as the same if and only if $x_1 y_2 = x_2 y_1$ then for m any integer there are not more than two sets, a fact which was pointed out to me by Dr. C. F. Gummer.

If

$$\begin{cases} kx_1 \equiv y_1 \pmod{m}, \\ kx_2 \equiv y_2 \pmod{m}, \end{cases} \tag{2}$$

where $|x_1|$, $|y_1|$, $|x_2|$ and $|y_2|$ are each $< \sqrt{p}$, then from the first congruence

$$kx_1x_2 \equiv x_2y_1 \pmod{m};$$

and from the second

$$y_2x_1 - x_2y_1 \equiv 0 \pmod{m}.$$

This gives, on account of the range of values for the x's and y's,

$$y_2x_1 - x_2y_1 = \pm m \quad \text{or} \quad y_2x_1 - x_2y_1 = 0,$$

and the last relation can exist only if we regard the sets (x_1, y_1) and (x_2, y_2) as the same. If the first relation holds and therefore the two sets exist, then a third set different from them cannot exist. To show this we note in (2) that we may take both x_1 and x_2 positive, and if this is done then it follows, using the relation $y_2x_1 - x_2y_1 = \pm m$, that y_1 and y_2 have opposite signs. Hence we may write

$$kx_1 \equiv y_1 \pmod{m},$$
$$kx_3 \equiv y_3 \pmod{m},$$

where x_1 and x_3 are positive and y_1 and y_3 have the same sign. It follows as before that

$$y_3x_1 - x_3y_1 = \pm m,$$

and this is impossible since the expression on the left is, in absolute value, less than m.

Now consider the solution of

$$u^2-2\equiv 0 \pmod{m}, \tag{3}$$

where m is the product of two distinct prime factors each of the form $8n+7$. There are four incongruent roots of the congruence which we will designate by $a, -a, b, -b$, where $a\not\equiv \pm b$. According to the theorem first proved we may write

$$av \equiv w \pmod{m},$$

$$bv_1\equiv w_1 \pmod{m},$$

where $|v|$, $|w|$, $|v_1|$, $|w_1|$ are each less than $\sqrt{m}$. Squaring and using (3) we have

$$\begin{cases} 2v^2-w^2\equiv 0 \pmod{m}, \\ 2v_1^2-w_1^2\equiv 0 \pmod{m}, \end{cases} \tag{4}$$

or,

$$2v^2w_1^2\equiv 2v_1^2w^2 \pmod{m},$$

and since m is odd,

$$(vw_1-v_1w)(vw_1+v_1w)\equiv 0 \pmod{m}. \tag{5}$$

Also from (4) we have

$$\begin{cases} 2v^2-w^2=m, \\ 2v_1^2-w_1^2=m. \end{cases} \tag{6}$$

We shall now show that in (5) one of the factors on the left is divisible by a factor of m but not by m. For if

$$vw_1+v_1w\equiv 0 \pmod{m},$$

then, since v, w, v_1, w_1 are all prime to m, we have, from (4), since m is not divisible by a square,

$$\frac{v}{w}\equiv\frac{-v_1}{w_1} \pmod{m},$$

or

$$a\equiv -b \pmod{m},$$

contrary to hypothesis; and if

$$vw_1-v_1w\equiv 0 \pmod{m},$$

then

$$a\equiv b \pmod{m},$$

which is also contrary to hypothesis.

Consequently, by taking the smallest positive factor in (5), and finding the greatest common divisor of it and m, we obtain a factor of m.

THEOREM. *If a number m of the form $8n+1$ is the product of two distinct primes then it is expressible in at least two different ways by one of the forms x^2+y^2, x^2+2y^2, and $2y^2-x^2$, where x and y are each positive and $< \sqrt{m}$. It is also possible to find two such representations which yield, by a direct process, a factor of m.*

This theorem gives the following scheme for finding a factor of an integer of the form $8n+1$.

(1.) Divide the integer by all primes less than its cube root. If it is divisible by a prime within this limit which is not of the form $8n+1$, the process cannot be carried further. Otherwise we may proceed as follows.

(2.) Extract the square root of the possibly new number m. If it is a perfect square, all the factors of m are known, as it cannot have more than two distinct prime factors other than unity. If m is not a perfect square we proceed to the third step.

(3.) Find by Gauss' method of exclusion all representations of m in the form x^2+y^2. If there is but one representation, then m is prime. If there is more than one, then the factors of m may be found from the several representations. If there are no representations then proceed to the next step.

(4.) Find by the method of exclusion all the representations of m in the form $2y^2-x^2$, where x and y are each $< \sqrt{m}$. It cannot have a unique representation, as it follows from this that m is prime, and this would have been detected in step 3. If there is more than one representation then the factors of m may be found from them. If there are no representations then we use the fifth step.

(5.) Find by the method of exclusion all the representations of m in the form x^2+2y^2. The factors of m are obtained from the several representations.

3. *Example.* Let us show how to factor $532481 = N$. Divide by all the primes < 82, since $[\sqrt[3]{N}] = 81$. It is found that N is not divisible by any of these primes. Also N is not a square. Hence it is prime, or is the product of two distinct primes. We seek first all the values x, y, such that

$$x^2 + y^2 = N.$$

Either x or y is odd* and each is < 730. Suppose x is odd. Also $N \equiv 2 \pmod 3$. This gives $x \equiv \pm 1 \pmod 3$, since if $x \equiv 0 \pmod 3$ we find $y^2 \equiv 2 \pmod 3$, which is impossible. Hence since x is odd, we have $x \equiv \pm 1 \pmod 6$. Now $N \equiv 1 \pmod 5$; and, in a similar way, we find $x \equiv 0, \pm 1 \pmod 5$. Therefore $x \equiv 1, 5, 11, 19, 25, 29 \pmod{30}$. We have $N \equiv 5 \pmod 7$, and $x \equiv \pm 1, \pm 2 \pmod 7$. Modulo 210, x has the possible residues 1, 5, 19, 29, 41, 55, 61, 65, 71, 79, 89, 121, 125, 131, 139, 145, 149, 155, 169, 181, 191, 205, 209. Similarly x has the possible residues $0, \pm 1, \pm 2, \pm 5 \pmod{11}$; and we then set down the least possible residues of x modulo $11 \cdot 210$, which are less than 730. We find 68 numbers. Using the modulus 16, we may exclude numbers of the form 3, 5, 11 and 13, modulo 16. Similarly we may exclude numbers of the form 1, 4, 6, 11, 14, 19, 21, 24, modulo 25, and 3, 4, 5, 10, 11, 12, modulo 13. This leaves 12 possible values for x which we may test directly, and we find none that will give a representation of N in the required form.

We now proceed to step 4 and find all representations of the form

$$2y^2 - x^2 = N,$$

where x and y are each $< \sqrt{N}$. In this relation $516 < y < 730$. Also $y \equiv 1 \pmod 2$. Possible values of y satisfy $y \equiv 0, \pm 1 \pmod 5$, $y \equiv 0, \pm 1 \pmod 7$. Modulo 70, y has therefore the possible residues 1, 15, 21, 29, 35, 49, 55, and 69. We find similarly $y \equiv 0, \pm 3, \pm 4 \pmod 9$. We then write down the integers between 516 and 730 having the above properties and find 519, 525, 545, 561, 581, 589, 609, 615, 645, 651, 679, 699, 715.

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* The work could have been made shorter here by taking x even and noting that $x \equiv 0 \pmod 4$.

Now y cannot be congruent to 0, 4, 5, 6 or 7 (mod 11), and we may exclude integers of this type from the above set, leaving 519, 525, 581, 615, 651, 679. Actual trial of all these gives only two representations,

$$2\cdot 525^2-137^2=2\cdot 519^2-79^2=N,$$

and

$$525\cdot 79-519\cdot 137=29628.$$

The greatest common divisor of N and 29628 is 823. Hence $N=823\cdot 647$, each of which is a prime.

A similar method may be applied to integers of the form $12n+1$. If N is of this form and the product of two distinct primes, then each has the same residue, modulo 12. If both are of the form $12n+1$, or $12n+5$, then N is expressible as the sum of two squares. If each is of the form $12n+7$ then $N=x^2+3y^2$. If each is of the form $12n+11$ then it may be shown that $2N=3y^2-x^2$, where x and y are each $<\sqrt{N}$. For in case the congruence $u^2-3\equiv 0$ (mod N) is solvable and if a is one of its roots then $ma\equiv k$ (mod N) where m and k are each $<\sqrt{N}$, and therefore $3m^2-k^2\equiv 0$ (mod N) and $3m^2-k^2=N$ or $2N$. But the first case is impossible, since $N\equiv 1$ (mod 3). We then have all the material necessary to carry through a method for factoring integers of the form $12n+1$ analogous to that described for the case $8n+1$.

4. *Second Method of Factorization.* The second method of factorization is briefly as follows. Find by known methods a negative integer $-a$ which is a quadratic residue of n, the integer to be factored (n not a perfect power). From a known result in the theory of quadratic forms it follows that, corresponding to every root u of the congruence $u^2\equiv -a$ (mod n) there is a set of integers x, y and k such that

$$x^2+ay^2=kn, \tag{1}$$

$k<2\sqrt{a/3}$, x and y prime to each other.* From (1) we have $(-a/k_1) = 1$ where k_1 is any divisor of k, and $(kn/a_1) = 1$ where a_1 is any divisor of a. We select the k's satisfying these conditions, and also $k < 2\sqrt{a/3}$ and we find the possible forms

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* This theorem was used by H. J. S. Smith, WORKS, vol. 1, p. 148 in the solution of the quadratic congruence.

$$x_s^2 + ay_s^2 = k_s n,$$

$s = 1, 2, \ldots, \imath$. Find by Gauss' method of exclusion or other schemes all actual representations of this type; then all factors of n may be found from them, by known methods.

It is best to select a, if possible, so that it contains several small primes as factors since this will diminish the number of possible values of k.

5. *Example* 1. $n = 532481$. In order to find values of a as described above, expand $\sqrt{n}$ as a continued fraction. We have the corresponding values given in the following table.

Denominator of comp. quotient	Terms in cont. fraction
1	729
$1040 = 5\cdot 13\cdot 4^2$	1
419	2
$608 = 4^2\cdot 2\cdot 19$	2
$95 = 5\cdot 19$	14
$1280 = 16^2\cdot 5$	1
97	14
$160 = 4^2\cdot 2\cdot 5$	9
79	18

Hence $-5\cdot 13$ is a residue of n, $-19\cdot 2$ is an R (residue), $5\cdot 19$ is an R, -5 is an R, -10 is an R, whence 2 and 13, -19 and -5 are residues. Also 79 is an R. Hence $-a = -2\cdot 5\cdot 13\cdot 79$ is a residue. Hence

$$x^2 + ay^2 = kn,$$

where $k < 118$. From the relations $p_r^2 - nq_r^2 = \pm d_r$ where p_r/q_r is a convergent and d_r a denominator of comp.

quotient in development of $\sqrt{n}$ we have $(n/13) = (n/79) = (n/5) = 1$. Hence also $(k/5) = (k/13) = (k/79) = 1$. This gives $k \equiv 0, 1, 4 \pmod{5}$ and $k \equiv 0, 1, 3, 4, 9, 10, 12 \pmod{13}$. We set down the integers

$$\begin{array}{ccccccc}
1 & 3 & 4 & 9 & 10 & 12 & 13 \\
14 & 16 & 17 & 22 & 23 & 25 & 26 \\
27 & 29 & 30 & 35 & 36 & 38 & 39 \\
40 & 42 & 43 & 48 & 49 & 51 & 52 \\
53 & 55 & 56 & 61 & 62 & 64 & 65 \\
66 & 68 & 69 & 74 & 75 & 77 & 78 \\
79 & 81 & 82 & 87 & 88 & 90 & 91 \\
92 & 94 & 95 & 100 & 101 & 103 & 104 \\
105 & 107 & 108 & 113 & 114 & 115 & 117.
\end{array}$$

We note also that $k \not\equiv 0, 3, 4, 5 \pmod{8}$; using this with $k \not\equiv 2$ or $3 \pmod{5}$ we have remaining

$$\begin{array}{cccccc}
1 & 9 & 10 & 14 & 25 & 26 \\
30 & 39 & 49 & 55 & 65 & \\
66 & 74 & 79 & 81 & 90 & \\
94 & 95 & 105 & 114. & &
\end{array}$$

We have

$$\left(\frac{-2\cdot 5\cdot 13\cdot 79}{3}\right) = \left(\frac{-2\cdot 5\cdot 13\cdot 79}{7}\right) = -1.$$

Hence k cannot be a multiple of 3 or 7. Moreover, it is evident that k is not divisible by 5^2. This leaves

$$\begin{array}{ccccc}
1 & 10 & 26 & 55 & 65 \\
 & 74 & 79 & 94 & 95.
\end{array}$$

From $x^2 + ay^2 = 10n$ we have $5x^2 + 2\cdot 13\cdot 79y^2 = 2n$ whence $(13\cdot 79/5) = 1$ which is incorrect; this excludes 10. Similarly 26 is excluded, also 65. We have $(74/79) = -1 = (94/79)$ and these are then excluded leaving 1, 55, 79 and 95 as possible values of k. Let us now examine

$$x^2 + ay^2 = n.$$

Here y ranges from 1 to 7 inclusive, and it must be even, which gives us

$$701^2 + 10270\cdot 2^2 = n.$$

For $k = 55$, we have $5x^2 + 2\cdot 13\cdot 79\, y^2 = 11n$. Using Gauss' method of exclusion we have

$$y \equiv \pm 2 \pmod 5), \qquad y \equiv 1 \pmod 2),$$
$$y \not\equiv 0 \pmod 3), \qquad y \not\equiv \pm 2 \pmod 7),$$

whence

$$5\cdot 1073^2 + 2\cdot 13\cdot 79\cdot 7^2 = 11n$$

and the two representations give $n = 823\cdot 647$.

6. *Example* 2. $n = 13179643$ (an 8-figure number selected at random, except that it is not divisible by small integers).

Denominator of complete quotient	Term of cont. fraction
1	3630
$2743 = 13\cdot 211$	2
$13^2\cdot 3\cdot 7 = 3549$	1
2906	1
$3\cdot 17\cdot 79 = 4029$	1
1303	4
$3\cdot 11\cdot 173 = 5709$	1
$2\cdot 193 = 386$	

Hence we may take $a = 2\cdot 3\cdot 7\cdot 193 = 8106$,

$$x^2 + ay^2 = nk, \tag{1}$$

$k < 2\sqrt{a/3}$ or $k < 104$. Also $(n/3) = (n/7) = (n/193) = 1$ and $(k/3) = (k/7) = (k/193) = 1$. $k \equiv 0$ or $1 \pmod 3$, $k \equiv 0, 1, 2, 4 \pmod 7$; hence, modulo 21, the residues of k are 1, 4, 7, 9, 15, 16, 18, 21, so the values of k may be written

1	4	7	9	15	16	18	21
22	25	28	30	36	37	39	42
43	46	49	51	57	58	60	63
64	67	70	72	78	79	81	84
85	88	91	93	99	100	102.	

Also $k \not\equiv 0, 4, 5, 7 \pmod 8)$. Using these conditions together with the methods employed in the last case gives

$$k = 1, 25, 42, 43, 46, 67.$$

Hence the factors of n may be found by using these values of k in (2).

For the same value of n,

$$n = 3630^2 + 13\cdot 211,$$

we may take $a = 13\cdot 211$. This gives

$$k = 1,\ 13,\ 16,\ 43,\ 49,\ 53,\ 56.$$

Let $n = 11432767$. Here $n = 3381^2 + 1606$; $1606 = 2\cdot 11\cdot 73$ Let us now examine

$$x^2 + 1606\,y^2 = kn.$$

We find $k = 1$, 25, 23 or 38.

Let $n = 236364091$, the numerator of the 12th Bernoulli number; then

$$n = 15374^2 + 4215,$$
$$kn = x^2 + 3\cdot 5\cdot 281\,y^2,$$

whence, using previous methods, $k = 1$, 16, 31, 39 or 64.

7. *Case of Periodic Continued Fractions.* In applying the schemes outlined in the second method we may find at the beginning that, after a few operations, we have developed a period of the continued fraction for $\sqrt{n}$. Suppose* that the period has $2r$ terms, say

$$\mu_1, \mu_2, \ldots, \mu_{r-1}, b, \mu_{r-1}, \mu_{r-2}, \ldots, \mu_1, c,$$

then d_r, the denominator of the complete quotient corresponding to b is a divisor of $2n$. Since b is not the last quotient in a period then $(-1)^r d_r \neq 1$, and therefore $(-1)^r d_r = -1$, ± 2, or some factor of n not unity. Also, if the period has an odd number of terms then $x^2 - ny^2 = -1$ has solutions. It then follows that *either a factor of n has been found, or else n is expressible in one of the forms $x^2 + y^2$, $x^2 + 2y^2$ or $2x^2 - y^2$, x and y each $< \sqrt{p}$.* In cases where the factor is not derived as a divisor of d_r, however, it is more expeditious to expand the square root of some multiple of n as a continued fraction to find suitable values for a.

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* Märcker, CRELLE, vol. 20 (1840), pp. 355–59.

8. *Example.* Let us factor $n = 36343817$. A period in the development of $\sqrt{n}$ is 1, 1, 2, 1, 1, 12056, and $d_3 = 4019$, whence $n = 4019 \cdot 9043$.

9. *Third Method of Factorization.* The third method depends on representing the number as an indefinite quadratic form. In a paper already cited I proved that there exists at least one set of positive integers (x, y) such that $ay \equiv \pm x \pmod{m}$ where a is any integer prime to the integer m, x and y each $< \sqrt{m}$. Hence if there exist roots of the quadratic congruence $\mu^2 \equiv a \pmod{n}$ then corresponding to each root μ_1, we have a set (x, y) such that $\mu_1 y \equiv \pm x \pmod{n}$, x and $y < \sqrt{n}$, and we have $ay^2 - x^2 \equiv 0 \pmod{n}$ or $ay^2 - x^2 = kn$ where $0 < k < a$.

10. *Example.* Let us consider $n = 13179643$. The development of $\sqrt{n}$ as a continued fraction (given above) shows that 21 is a quadratic residue of n, and therefore

$$21x^2 - y^2 = kn$$

where $0 < k < 21$, also x and $y < \sqrt{n}$. The continued fraction development gives $(n/7) = 1$, $(n/3) = 1$, hence $(k/7) = -1$, $(k/3) = -1$. Using $(k/3) = -1$ we have as possible values of k

$$\begin{array}{cccccc} 2 & 3 & 5 & 6 & 8 & 9 \\ 11 & 12 & 14 & 15 & 17 & 18 \end{array}$$

We have $k \not\equiv 1, 2, 4 \pmod{7}$ and $k \not\equiv 0, 2, 6 \pmod{8}$, which leaves 3, 5, 12 and 17 as possible values. Note that $\sqrt{(k+1)n/21} > x > \sqrt{kn/21}$.

Since writing what precedes, I have examined Kraïtchek's *Théorie des Nombres* (Paris, 1922). This book contains tables that are admirably adapted for use in connection with any of the three methods described in this paper. Kraïtchek tabulates the incongruent values of x in $x^2 + Dy^2 \equiv N \pmod{\varrho}$, where ϱ assumes various small values.

Cornell University

THE THIRD EDITION OF PICARD'S TRAITÉ

Traité d'Analyse. By Émile Picard. Troisième édition. Revue et augmentée avec la collaboration de Gaston Julia. Volume I. Paris, Gauthier-Villars, 1922. xvii + 593 pp.

Professor Picard's *Traité d'Analyse* has for many years past been regarded as one of the classics of modern mathematical literature. It is therefore to be presumed that most of the readers of the BULLETIN are familiar with the second edition of the present volume, which appeared more than twenty years ago and was reviewed* at that time by the late Professor Bôcher. Hence the present review will be concerned only with the changes and additions that have been made in the third edition.

The additional material comprised in the present volume amounts to about one hundred and ten pages, nearly one-fourth the content of the first volume of the second edition. With the exception of a few pages on non-euclidean geometry added to part three, the additions are scattered through parts one and two, the great bulk of them being found in part two, and particularly in that portion of it which deals with trigonometric series.

In part one the additional material is found mainly in more complete discussions of certain standard topics already treated in earlier editions, such as the first and second laws of the mean, functions of bounded variation, and change of variables in double integrals. Aside from the introduction of this supplementary material, a number of changes in the exposition have been introduced in certain portions of the text.

Part two is devoted mainly to a discussion of potential theory and trigonometric series. As mentioned before, the additions are much more extensive than in part one, covering in all some eighty-two pages. Moreover, the changes in that portion of the text carried over from the second edition are much greater than in the case of part one. In those sections dealing with potential theory, the order of presentation has been considerably changed and much rewriting has been done. But it is the chapter dealing with trigonometric series (now Chapter X) that has undergone the most extensive alteration. This was to be expected, of course, since this is the particular topic of volume one whose theory has been most enriched by the researches of the past twenty years. The major portion of the new material introduced is related to certain of these researches, and as only a few could be utilized in this fashion, it is interesting to note what choice Professor Picard has made. We discover that the inspiration for most of the additions is to be found in the writings of three men, Lebesgue, Bôcher and Fejér.

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* Cf. this BULLETIN (2), vol. 8 (1901–02), p. 124.

In discussing the Jordan criterion for the convergence and uniform convergence of Fourier's series and related topics connected with the representation of arbitrary functions, the author has adopted the point of view introduced by Lebesgue in his fundamental memoir of 1910, *Sur les intégrales singulières*. The essence of this method consists in proving first a number of general theorems regarding the behavior of integrals of the type $\int_a^b f(\alpha)\varphi(\alpha, n)\,d\alpha$ when the parameter n of the so-called kernel, $\varphi(\alpha, n)$, becomes infinite. It is then found that many highly important results, such as the classical results regarding the convergence of Fourier's series, Fejér's theorem regarding their summability, and the behavior of Poisson's integral, appear as special cases of one or the other of these general theorems. The advantages of this form of treatment are considerable. The reader is spared needless repetition of proofs that are essentially similar, and he is shown the underlying unity of a certain group of fundamental theorems.

One of the interesting properties of Fourier's series that for a long time remained unnoticed, is the fact that in the neighborhood of a finite jump of the function developed, the oscillation of the approximation curves (the curves $y = S_n(x)$ where $S_n(x)$ is the sum of the first n terms of the series) does not approach as a limit the value of the finite jump, but a quantity exceeding it by about eighteen per cent of its magnitude. This property was first pointed out by Gibbs in 1899, in a letter to NATURE. Gibbs's statement was extremely brief and apparently attracted no general attention. In 1906 Bôcher included in his well known monograph, *Introduction to the theory of Fourier's series*, a discussion of the property in question (called by him Gibbs's phenomenon), which exhibited in a very illuminating manner its most important features. In the volume under review, section 16 of Chapter X is devoted to an exposition of Gibbs's phenomenon which follows quite closely the treatment in Bôcher's monograph. In the title heading of the section, however, the property in question is referred to as "Phénomène de Du Bois-Reymond et Gibbs", and at the end of the section reference is made to an article by Du Bois-Reymond in volume 7 of the MATHEMATISCHE ANNALEN. An examination of that article has failed to convince the reviewer that there is any justification for associating the name of Du Bois-Reymond with the phenomenon under discussion. It is true that there are certain formulas in the article from which Gibbs's phenomenon might have been deduced, but the deduction was not made. It is a little difficult to interpret the meaning of such discussion of the formulas as is given,* inasmuch as this discussion contains several erroneous statements. The most natural interpretation, however, would lead one to conclude that Du Bois-Reymond was under the impression that in the neighborhood of a finite jump the approximation curve approaches a limiting curve in which the vertical portion is the line joining the loose ends of the curve representing the function, and that he was entirely unaware of the fact that this vertical part extends beyond the loose ends in question by an amount that bears a definite ratio to the magnitude of the finite jump. Therein lies the whole point of Gibbs's noteworthy discovery.

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* Loc. cit., p. 254.

In the case of so keen an analyst as Du Bois-Reymond, who had, moreover, penetrated deeply into a number of questions concerning the convergence and divergence of Fourier's series, one is at liberty to suppose that he may actually have noted Gibbs's phenomenon and that this fact is concealed by the errors above mentioned. But at best this is mere conjecture, and is hardly a sufficient reason for attempting to give him priority over Gibbs in the matter.

One of the most important of the well established contributions of Du Bois-Reymond to the theory of Fourier's series is his discovery of continuous functions whose developments in Fourier's series diverge at certain points. The examples he gave of such functions are quite complicated, and as the property is such a fundamental one it is highly desirable to have simpler examples, particularly for use in instruction. This need has been admirably met by Fejér, who, starting from certain principles of construction introduced by Lebesgue and Haar, succeeded in forming very simple examples of continuous functions that exhibit the so-called singularity of Du Bois-Reymond. One of these examples is discussed in detail by Professor Picard in section 17 of Chapter X. This example leads naturally to the consideration of methods of summing Fourier's series in the cases where it is divergent. In sections 19-22, Cesàro summability of order one is discussed, and Fejér's theorem regarding its application to Fourier's series is established.

Most of the other new material in Chapter X does not deal with recent researches, but is based on results obtained before the publication of the second edition. Among the topics discussed may be mentioned Dini's condition for the convergence of Fourier's series, and de la Vallée Poussin's theorem regarding the series formed from the squares of the Fourier coefficients of a given function.

With the exception of the introduction of a brief discussion of non-euclidean geometry, mentioned above, the remainder of the book exhibits no material change from the second edition.

C. N. Moore

PEET'S TRANSLATION OF THE RHIND PAPYRUS

The Rhind Mathematical Papyrus. By T. Eric Peet. Liverpool, The University Press, and London, Hodder and Stoughton, 1923. Pp. (4) + 136 + 24 plates.

In the history of mathematics the one outstanding original document of the ancient world is the so-called Ahmes Papyrus. Its importance is not due to the value of the work as a contribution to the science of mathematics, but to its value as a historical document, it being the only extensive treatise upon the subject existing in the form of a manuscript written before the beginning of the Christian era. It dates from about the middle of the second millenium before Christ and is now, with the exception of a small fragment, in the British Museum Curiously enough, this small fragment has recently been found in the collection of the New York Historical Society, the original discoverer having apparently sold this piece separately. We have many older documents relating to mathematics, chiefly in the form of Babylonian tablets, but they are not treatises of any extent, valuable though they are as showing the symbols and tables of the Mesopotamian civilization.

The general nature of the work is well known. Eisenlohr published a German translation of it more than fifty years ago, and Professor Archibald has listed a large number of articles upon the subject. written by some seventy-five different scholars. For the present purposes, therefore, and in view of the limited space at the disposal of the reviewer, it is not proposed to speak further of the significance and the special features of the manuscript itself, but to call attention to the nature of the work done by Professor Peet.

It has long been patent, even to those who have no special training in Egyptology, that both the translation made by Professor Eisenlohr and his commentary upon it were in need of revision. Our knowledge of the history of the Egyptian people and their language and rulers has made much progress in half a century. There was therefore a need for a first-class student of history, of ancient epigraphy, of language, and of metrology who should undertake the task of translating the papyrus and of explaining its contents and its general significance. To undertake such a piece of work Professor Peet, Brunner professor of Egyptology in the University of Liverpool, an Oxford man of high standing, was unusually well qualified and he seems to have executed his mission with the thoroughness of a British scholar.

As one examines the impressive work, mechanically a product of the University Press of Liverpool, three questions of moment arise: (1) What is the distinguishing feature of Professor Peet's work? (2) What does the work tell us that is new concerning the man Ahmes (whose name Eisenlohr transliterates as Aah-mesu, and Professor Peet as Aḥmōse) and his time? (3) Is the work a scholarly production that should stand forth as an authority and have a place in our libraries of reference?

As to the first question, the author has given us a new translation, made from the facsimile published by the British Museum in 1898 and from the New York fragment. Moreover, he has greatly assisted the student by transliterating from the difficult hieratic of the original to the relatively simple hieroglyphic form of which a considerable part can be understood after only a little study. The translation and discussion show evidence of the scholarly way in which the whole undertaking has been carried through. If the work contained nothing else, it would rank high as a source book in the study of an interesting and important phase of the ancient Egyptian development of "the science venerable." No doubt the layman would have welcomed an explanation of the diacritical marks and a brief statement as to the probable pronunciations used by the ancients, but these are matters that can be found in linguistic treatises and which, after all, have no special bearing upon the mathematical problem involved.

As to the second question, the work is helpful as locating a little more definitely, for the time being at least, the probable period in which the manuscript was written. This has long been a matter of dispute. Eisenlohr was of the opinion that it dates from c. 1700 B. C. and was a copy of an earlier work made in the time of Amenemhat III. The manuscript itself, as now translated, contains this statement:

"Behold this roll was written in Year 33, month 4 of the inundation season, [. . .] and Lower Egypt Aauserrēʿ, endowed with life, in the likeness of a writing of antiquity made in the time of the King of Upper and Lower Egypt Nemarēʿ. It was the scribe Aḥmōse who wrote this copy." This Nemarēʿ seems to have been the Amenemmes (Amenemhat) III of the twelfth dynasty, who reigned from c. 1849 to 1801 B. C., so that the original work of which we have the copy made by "the scribe Aḥmōse" seems to date from c. 1850 B. C., or the half century following.

As to Ahmes himself, Professor Peet places the date of Aauserrēʿ (Apophis, Apepa I) as "some time between 1788 and 1580 B. C." There is an indorsement on the roll, seemingly of about the same period, which has given rise to much speculation, some authorities having placed it as late as 1321 B. C. It can hardly be said, therefore, that the date is yet fixed with certainty, and perhaps we are fairly safe in saying, at least until more precise data are found to change the opinion, that it was written c. 1600 or c. 1550 B. C. In a work on the Egyptian Collection in the British Museum, written by E. A. Wallis Budge and published in 1908, no precise dates were given for Āa-user-Rā (Aausserrēʿ) and he was not included in the Museum list of prominent rulers. We have, however, in the work under review, the latest chronological information at hand. A more precise date is of little moment in this connection.

As to the third question, the scholarly nature of the work, the answer would seem to be definite. The translation doubtless represents a degree of excellence not heretofore reached with respect to this important document, and perhaps not to be surpassed. It is futile to say that research in the field of Egyptology will not permit of the removal of certain doubts which Professor Peet himself has mentioned, and for a more satisfactory interpretation of certain problems, but for the present we have reason to be thankful to him, to the University of Liverpool, and to Messrs. Hodder and Stoughton, for what has been done in the preparation and the printing of this noteworthy work.

As to the other side of the case it may be said in general that after a treatise like this is in print there is little or no use in wishing for changes. Fault finding for its own sake is an easy task, but it is an ungracious act unless it results in corrections that would otherwise escape attention or warns the reader against a book which is poorly written or is generally unreliable.

The chief adverse criticism of Professor Peet's work will probably be that it is so good that the reader wishes it had been carried to greater length,—say with the hieratic along with the hieroglyphic, with perhaps a transliteration in Latin letters and a translation directly connected with it, and possibly with a table of pronunciations,—a criticism of no real significance. The only complaint of the present reviewer is that the descriptive statement preceding the translation is not arranged as he would have planned it, which only shows that we all have our idiosyncracies.

The book has a list of the technical words discussed in the text, and an index which has not proved as helpful in actual use as the reviewer had hoped.

Professor Peet's work should be in all our university libraries, and in public libraries of any pretense to completeness in the matter of standard books of reference. Those who are interested in the history of science will naturally wish, in spite of the high post-war price, to add it to their own collections.

David Eugene Smith

SHORTER NOTICES

A Manual of Field Astronomy. By Andrew H. Holt. New York, Wiley, 1917. 10 + 128 pp.

This book is designed to meet the demand for something "less extensive than most texts on the subject and yet more complete than the usual chapters on the subject in books on surveying." It is "planned especially for engineers, rather than astronomers, the methods described being those for use with field instruments under ordinary field conditions."

After an introductory chapter, the various systems of coordinates and the astronomical triangle are explained. Then follow chapters on Measurement of Time, The American Ephemeris and the Nautical Almanac, Problems in Conversion of Time, Corrections to Observations, Observations for Latitude, Azimuth, Time and Longitude. The appendix contains the derivations of the necessary formulas in spherical trigonometry, a description of the solar attachment for transits, a set of tables, and a number of typical problems completely solved.

The treatment is eminently practical throughout. Under each topic there is given: first, the relations and theory on which the method depends, then the procedure is outlined, step by step, under the general headings *Computation preceding field work, Field work,* and *Computation following field work.* This outline is supplemented by reference to the solution of a typical problem. Thus the student learns exactly what things he is to do, and exactly how, when, and in what order, he is to do them. The diagrams are excellent, the book is attractive in make-up, and on the whole this is the best brief treatment of the subject that the writer has seen.

C. H. CURRIER

The Elements of Theoretical and Descriptive Astronomy. By the late Charles J. White. 8th edition, revised by Paul P. Blackburn New York, Wiley, 1920. 11 + 309 pp.

Originally published in 1869 to meet the requirements of the U. S. Naval Academy, successive editions "have simply brought the treatment of the subject up to date without making material changes in its form." The book is largely a discussion of theoretical astronomy, descriptive astronomy being treated very briefly. Thus in the chapter on the planets, Mars is dismissed with a page, while two pages are devoted to a discussion of how to find the heliocentric longitude of the node of a planet. No exercises to be solved or questions to be answered by the student are inserted.

Some important topics are either left out altogether or treated briefly. Thus there is no mention whatever of standard time in the chapter on time; the discussion of spectral types is incomplete, as is that of the form of the earth, while the problem of the source of the sun's heat is not considered at all. The discussion of longitude would be clearer if the chapter on time were placed earlier in the book.

For the most part the explanations are clearly given, and few errors have been noted. An annular eclipse is not limited to the point where the axis of the cone, prolonged, meets the earth (p. 154). A planet east of the sun does not always set after the sun has set (p. 176). Saturn is not usually credited with ten satellites (p. 196). The light of the planets, we are told (p. 204) produces only the ordinary spectrum of reflected solar light. The explanation of the method of numbering comets (p. 208, note) is not correct. We learn (p. 243) of a first magnitude star in Ursa Major.

The most valuable feature of the book is the series of diagrams, 84 in number, illustrating the more important terms and relations of theoretical astronomy. The diagrams are excellent, and the derivations of the formulas are clearly explained. The appendix contains a summary of the necessary mathematical definitions, theorems, and formulas from plane and spherical trigonometry, analytic geometry, and mechanics, as well as a brief chronological history of astronomy and navigation, tables summarizing the more important facts, and some excellent reproductions of Yerkes Observatory photographs of celestial objects.

C. H. Currier

Mathematische Bevölkerungstheorie, auf Grund von G. H. Knibbs' "The Mathematical Theory of Population", dargestellt von E. Czuber. Leipzig, B. G. Teubner, 1923. xvi + 357 pp.

Knibbs' *Theory of Population*, printed in 1917 in Australia, is one of those remarkable original works that we have from time to time from Englishmen. There are probably many things in it that will not stand the test of time, but the conception of the work is so large that its influence will remain for a long while. Czuber saw a notice by Knibbs in the first number of Metron, obtained a copy of the work through the kindness of the author, recognized at once, as Czuber would, the great importance of the work, and has given us a German reworking of it in somewhat condensed form. The reduction in total length is about 25%; the number of tables has been cut by 15%, and the list of serially numbered formulas by 40%. Yet a comparison of the German with the original version does not reveal any cuts or modifications so serious that the original author could well feel that he has not been adequately rendered, or that the reader to whom the German may be more accessible need feel the necessity of consulting the original text.

E. B. Wilson

Cours d'Astronomie. By H. Andoyer. Paris, Hermann.
Première partie: *Astronomie théorique*, Third edition, 1923. 455 pp.
Seconde partie: *Astronomie pratique*, Second edition, 1924. 316 pp.

The third edition of the first part of Andoyer's comprehensive text on spherical astronomy (*Astronomie théorique*) has undergone considerable changes when compared with the two previous editions. The changes affect the theory of precession and that of the eclipses of the Sun and the Moon. There is besides in the new edition an additional chapter which deals with the determination of a Keplerian orbit, when three approximate observations are given. The material of the text is subdivided into four larger sections (livres); each section contains a number of chapters. There are altogether twenty chapters. The author's acknowledged mastership has given to science a text which ranks well with the four volumes of Tisserand's *Mécanique Céleste.* Very few and rather insignificant misprints have been noticed. It is observed that very few references are given, and an index would have added considerably to the usefulness of the text. The reader familiar with the English or German notations in spherical astronomy will probably wish that terms like *hour angle* (τ) and *sidereal time* (θ) had undergone no changes in notation. Likewise it would seem that to represent a star by letter M is not very fitting, particularly when the parallactic angle is represented by S. We notice on page 120 the notation t for sidereal, τ for mean solar, and H for apparent solar time, which seems not very good usage.

In the introductory section one chapter is devoted to spherical trigonometry. No restrictions are based on the magnitude of sides and angles of the "general" spherical triangle. The formula $\cos V = \cos\alpha\cdot\cos\alpha' + \cos\beta\cdot\cos\beta' + \cos\gamma\cdot\cos\gamma'$ is made the basis of the general study and by purely algebraic transformations all of the systems of formulas used in spherical trigonometry are derived. The consideration of adjoined spherical triangles and polar triangles employed for elementary spherical trigonometry are here replaced by transformations "S" and "T". The invariants

$$D = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c$$
$$= 4\sin s\sin(s-a)\sin(s-b)\sin(s-c)$$

and
$$D' = 1 - \cos^2 A - \cos^2 B - \cos^2 C + 2\cos A\cos B\cos C$$
$$= 4\sin\sigma\sin\left(A-\frac{\sigma}{2}\right)\sin\left(B-\frac{\sigma}{2}\right)\sin\left(C-\frac{\sigma}{2}\right)$$

where $2s = a+b+c$, $2\sigma = A+B+C-\pi$, play in this theory an important role.

In the second and third sections the theory of the corrections is given which are to be applied to given observations. The reader will find among these the theory of refraction very attractive since very short and quite original. Designating the refraction on the path C

by R, the radius vector by r, the index of refraction by n, the zenith distance of observed ray of light by z_0, the author employs for R a power series development in tan z_0 as follows:

$$R = \int_C \frac{\frac{r_0}{r}\, d\left(\frac{n_0}{n}\right)}{\sqrt{\operatorname{ctn}^2 z_0 + 1 - \left(\frac{n_0}{n}\right)^2 \cdot \left(\frac{r_0}{r}\right)^2}} = A \tan z_0 - B \tan^3 z_0 + C \tan^5 z_0 - D \tan^7 z_0 \pm \cdots .$$

He shows that the coefficients A, B, C, D can be determined approximately for $z_0 < 75°$ with an error not exceeding $0\overset{''}{.}1$ without establishing first the connections between the variables r and n by means of introducing three new functions in the customary way. These functions involve the density of the air, the atmospheric pressure and the temperature. The horizontal refraction cannot be dealt with so easily. Here an assumption of a relation between the temperature and density of the air is of course necessary. The auxiliary function

$$\varphi(u) - \frac{r_0}{r} d\left(\frac{n_0}{n}\right) = \varphi(u)\, du$$

is put in the form $\varphi(u) = a \cdot e^{-ku}$ and Andoyer shows that the computation of R will then be in such a form that Radau's tables for $\Psi(x)$ where $\Psi(x) = e^{x^2}\int_x^\infty e^{-x^2} dx$ become readily available. A comparison of Andoyer's value of R with that of Radau for a mean condition of the atmosphere shows that his value differs only by 20″ from that of Radau, obtained in a much more laborious manner. Such an agreement for horizontal refraction is all one may ask for.

For the presentation of the theory of precession and nutation the author has written an introductory chapter (X) on Keplerian motion and a general outline of planetary perturbations. It would have been highly desirable to extend the scope of this chapter so as to include the fundamentals in the theory of a rotating body. Chapter XI requires a facile knowledge of both and presents probably more difficulties to the reader than any other chapter.

Of the second part written in collaboration with A. Lambert it will suffice to say that W. W. Campbells *Practical Astronomy* is probably the nearest approach in the English language to the presentation of the subject matter in this book. While, in the study of the equatorial, reference is given to the photographic instrument by which the "carte du ciel" is obtained, there is no chapter devoted to the reflecting telescope. The text contains nine photographic cuts some of which illustrate new instruments or instrumental accessories. Since only French makes are selected, it can hardly be said that the great advance in instrumental equipment which the last quarter of century has witnessed in this country and elsewhere is given sufficient recognition.

KURT LAVES

Leçons de Mécanique Rationnelle. By François Bouny. Volume I. Paris, A. Blanchard, and Mons, Librairie Leich. viii + 600 pp.

This is the first volume of a course on theoretical mechanics given to students at the École des Mines et de Métallurgie du Hainaut, and extending throughout the first two years. As indicated in the title, it is a course in theoretical mechanics. As such it can be read by the non-technical student quite as well as by the prospective engineer. The subjects considered are explained very fully, and as a result the author has produced a book which is easy to read. A particularly valuable feature is the large number of exercises which are given at the end of each chapter. It is stated in the preface that the solution of a large number of exercises is an important part of the course, and an essential for the clear understanding of the theory. This opinion undoubtedly will be shared by all who have had experience in teaching the subject.

The opening chapter takes up the parts of vector analysis that are needed for the chapters on kinematics and statics. Plückers line coordinates are introduced very neatly by writing the condition that a certain vector product should be zero. The properties of a system of localized vectors or strokes are also developed. This is a natural thing to do because both forces and angular velocities are represented by localized vectors, and any theorem regarding a system of localized vectors leads at once to a corresponding theorem in kinematics and statics. The four following chapters are devoted to kinematics. In Chapter III, page 125, it seems that it would have been better to have used $\overline{\omega}_1$, and $-\overline{\omega}_1$, for the two components of a couple of rotations. Although the article is headed *Couples de rotations*, when it is stated that a body S has a constant angular velocity $\overline{\omega}_1$, with respect to a system S_1, and S_1 has a constant angular velocity $\overline{\omega}_2$ with respect to a fixed reference system, we can infer only from what follows that $\overline{\omega}_1$ and $\overline{\omega}_2$ are to form a couple.

Part two contains four chapters on statics which include the graphical and analytical determination of the stresses in a framework, and also the equilibrium of strings. In § 218 the centroid of a homogeneous volume bounded by the three planes of reference and the surface of a sphere whose center is at the origin is determined. It may be noted that this result is already known from the last equation in § 209, which gives the position of the centroid of a spherical segment of one base. The fifth chapter is a continuation of the study of vector analysis. It takes up such concepts as *div*, *grad*, *curl*, and the linear vector function. The div, grad, and curl are each defined with reference to a set of axes, and it is shown later that they are independent of the reference system. Although some writers on vector analysis object to this method, it is only a question of individual taste. The final chapters treat of work, the theory of the potential, the method of virtual velocities, simple machines, theory of friction, homogeneity and similitude.

There are a number of misprints, but practically all of them are of the kind that do not cause any difficulty. Although the figures are good, some of them have been reduced so much that it is an effort to read the lettering. We should congratulate engineering students who get so thorough a training in theoretical mechanics.

PETER FIELD.

Projective Geometry with Applications to Engineering. By Peter Field. New York, D. Van Nostrand Company, 1923. viii + 98 pp.

The purpose of this text book is concisely stated in the preface:

"In most American universities the course in descriptive geometry is purely a course in drawing. On the other hand, writers on technical mechanics frequently assume their readers have a knowledge of the fundamental theorems of projective geometry. It therefore seems there is a good opportunity for the mathematical departments in the technical colleges to offer an elective course in projective geometry which emphasises the technical applications."

Professor Field's projective geometry covers the subject matter of such a course given by him at the University of Michigan.

"No attempt has been made to give references. The authors most frequently consulted in working up the course were Cremona, Veblen and Young, Emch, Foppl, Mohr, Ritter, Culmann, and Koenigs."

The reader who is familiar with the works of these authors may therefore approximately surmise what a treatise in which the applications are emphasised must contain. There are altogether ten chapters, comprising definitions and fundamental forms, plane homology, linkages, the complete quadrangle and quadrilateral, the cross ratio, ranges and pencils, the hexagon, involution, pole and polar, and the null system. A course of this sort in technical colleges and universities with colleges of engineering is certainly very desirable. One of the most effective methods of reinforced concrete arch-construction, for example, depends largely upon projective geometry and projective properties of conics. The only reason why these methods are not taught is the lack of preparation in projective geometry of most of the teachers as well as the students of this branch of engineering.

Professor Field's attempt to remedy such defects is therefore very commendable and his little book will be welcomed by all, teachers and students, who seek information along the line of technical applications of projective geometry. The null system is of such importance that the reviewer should have liked to see a much more extended treatment and range of applications of this chapter.

ARNOLD EMCH

NOTES

The third Pan-American Scientific Congress will be held at Lima Peru, beginning December 20, 1924. The officers of Section II (Mathematical and Physical Sciences) are Vice-Admiral M. M. Carbajal, chairman, and Professor J. R. de la Puente, secretary. The chairman of Subsection 1 (Pure Mathematics, Rational Mechanics, Mathematical Physics) is Professor E. Villarán. Professors E. V. Huntington, of Harvard University, and D. N. Lehmer, of the University of California, have been appointed as the official delegates of the American Mathematical Society to the Congress, and will be in attendance.

The concluding number of volume 25 of the TRANSACTIONS OF THIS SOCIETY (October, 1923) contains the following papers: *Generalized limits in general analysis. Second paper*, by C. N. Moore; *The equilong transformations of euclidean space,* by B. H. Brown; *Invariant sets of equations in Riemann space,* by P. Franklin; *Some properties of spherical curves, with applications to the gyroscope*, by O. D. Kellogg; *The greatest and the least variate under general laws of error*, by E. L. Dodd; *The intersection numbers,* by O. Veblen; *The geometry of paths*, by O. Veblen and T. Y. Thomas. The opening number of volume 26 (January, 1924) contains: *An existence theorem for the characteristic numbers of a certain boundary value problem,* by H. T. Davis; *A theorem on the factorization of polynomials of a certain type,* by L. L. Dines; *A fundamental class of geodesics on any closed surface of genus greater than one,* by H. M. Morse; *The Hilbert integral and Mayer fields for the problem of Mayer in the calculus of variations*, by G. A. Larew; *Normal congruences and quadruply infinite systems of curves in space,* by J. Douglas; *The equivalence of certain regular transformations,* by L. L. Silverman; *Maclaurin expansion of the interpolation polynomial determined by $2n+1$ evenly spaced points,* by G. Rutledge; *On covariants of linear algebras,* by C. C. MacDuffee; *A generalized problem in weighted approximation,* by D. Jackson.

The second number of volume 46 of the AMERICAN JOURNAL OF MATHEMATICS (April, 1924) contains: *Two-dimensional tensor analysis without coordinates,* by G. Y. Rainich; *Fractional operations as applied to a class of Volterra integral equations,* by H. T. Davis; *Representation of three-element algebras,* by B. A. Bernstein; *The Riemann adjoints of completely integrable systems of partial differential equations,* by C. A. Nelson; *Further types of involutorial transformations which leave each cubic surface of a web invariant,* by V. Snyder.

The opening number of volume 25, series 2, of the ANNALS OF MATHEMATICS contains: *The history of the notations of the calculus*, by F. Cajori; *New applications of a fundamental theorem of substitution groups*, by G. A. Miller; *Geodesic representation between Riemann spaces*, by H. Levy and A. Bramley; *On the residues of figurate numbers*, by O. E. Glenn; *On symmetric forms in n variables*, II, by A. Dresden, *On an infinite system of non-abelian groups of order* nm^{n-1}, by W. E. Edington.

A complete edition, in five volumes, of the works of N. J. Lobatchefsky is being prepared under the auspices of the Mathematical Institute of Moscow and with the support of the Russian government. It is expected that the first volume, which will contain a biography by A. Vassilieff, will appear in 1926.

The Class of Sciences of the Royal Academy of Belgium announces the following subjects for prizes to be awarded in 1925: (1) a contribution to infinitesimal geometry; (2) a contribution to the problem of n bodies in Einstein's theory. Competing memoirs should be presented by August 1, 1925.

The Franklin Institute of Philadelphia has awarded a Franklin medal and an honorary certificate of membership to Sir Ernest Rutherford, Cambridge University.

The National Academy of Sciences has awarded its Watson medal to Professor C. V. L. Charlier, of the University of Lund, and its Henry Draper medal to Professor A. S. Eddington, of Cambridge University. Professor Eddington will lecture on general relativity at the University of California during the first semester of 1924-25, and will conduct a seminar on sidereal astronomy.

Professor Ludwig Bieberbach, of the University of Berlin, has been elected a member of the Prussian Academy of Sciences.

The University of Manchester has conferred its honorary doctorate of science on Professor Niels Bohr.

Professor G. H. Hardy has received an honorary doctorate from the University of Birmingham.

The London Mathematical Society has recently elected the following honorary members: L. Bianchi, A. Einstein, J. Hadamard, E. Landau, H. Lebesgue, T. Levi-Civita, L. Prandtl, and A. Sommerfeld.

At the meeting of the International Mathematical Congress held during August in Toronto, Professor S. Pincherle, of the University of Bologna, was elected president of the International Mathematical Union, and Professor J. C. Fields, of the University of Toronto, was elected president of the Congress. A full account of the Congress will appear in an early issue of this BULLETIN.

New York University has conferred the honorary degree of doctor of laws on Professor M. I. Pupin, of Columbia University.

Professor A. B. Coble has been elected a member of the National Academy of Sciences.

Professor O. Chisini has been appointed professor of projective geometry at the University of Cagliari.

At the University of London, Professor G. B. Jeffery, of King's College, has been appointed to the Astor chair of mathematics at University College, and Professor A. E. Joliffe, of Royal Holloway College, has been appointed to the university chair of mathematics tenable at King's College.

Professor O. W. Richardson, of King's College, London, has been appointed third Yarrow research professor of the Royal Society.

Mr. J. C. Burkill, director of studies at Trinity Hall and Fitzwilliam Hall, Cambridge, has been appointed professor of pure mathematics at the University of Leeds.

The following persons have been admitted as privat docents: Dr. P. Finsler, at the University of Cologne; Dr. A. Ostrowski, at the University of Gottingen; Dr. H. Prüfer, at the University of Jena; Dr. E. Schonhardt, at the University of Tubingen; Dr. N. Spampinato, at the University of Catania.

Adjunct Professor P. M. Batchelder, of the University of Texas, has been appointed acting assistant professor of mathematics at Brown University for the academic year 1924–25.

At the University of California, Assistant Professors B. A. Bernstein and Frank Irwin have been promoted to associate professorships, and Dr. Sophia H. Levy has been promoted to an assistant professorship of mathematics. At the Southern Branch, at Los Angeles, Associate Professor G. E. F. Sherwood has been promoted to a full professorship, and Dr. P. H. Daus to an assistant professorship.

At the University of Illinois, Professor H. Blumberg has been granted leave of absence for the academic year 1924–25, and Dr. C. C. Camp has been promoted to the rank of associate.

Dr. R. F. Borden, of Brown University, has been appointed assistant professor of mathematics at George Washington University, Washington, D. C.

At the University of Cincinnati, Associate Professor C. N. Moore has been promoted to a full professorship and Dr. I. A. Barnett to an assistant professorship of mathematics.

At Cornell University, Assistant Professors W. A. Hurwitz and Arthur Ranum have been promoted to full professorships. Dr. D. S. Morse and Assistant Professor W. L. G. Williams have resigned, to accept assistant professorships at Union College and McGill University, respectively.

Associate Professors J. W. Bradshaw and T. H. Hildebrandt have been promoted to full professorships at the University of Michigan.

At the University of Missouri, Professor G. E. Wahlin, of the University of Illinois, has been appointed professsor, Associate Professor Louis Ingold has been promoted to a full professorship and has been granted leave of absence for the year 1924-25, and Dr. Herman Betz, of Yale University, has been appointed assistant professor of mathematics. Dr. E. F. Allen has been promoted to an assistant professorship.

Associate Professor C. L. Arnold has been promoted to a full professorship of mathematics at Ohio State University.

Dr. B. H. Brown and Dr. R. E. Langer have been promoted to assistant professorships of mathematics at Dartmouth College.

Assistant Professor R. W. Burgess, of Brown University, has been granted leave of absence for the academic year 1924-25, and has accepted a position in the office of the Chief Statistician of the Western Electric Company in New York City.

At the University of Arizona, Associate Professor G. H. Cresse has been promoted to a full professorship of mathematics.

Dr. L. S. Dederick of the United States Naval Academy has been appointed professor of mathematics at the University of British Columbia.

Assistant Professor L. C. Emmons, of Michigan Agricultural College, has been promoted to a full professorship of mathematics.

Associate Professor H. S. Everett has been promoted to a full professorship of mathematics and astronomy at Bucknell University.

Assistant Professor Ernest Flammer has been promoted to an associate professorship of mathematics at Queen's University, Kingston, Ontario.

Dr. C. A. Garabedian, of Harvard University, has been appointed assistant professor of mathematics at Northwestern University.

President W. A. Granville, of Gettysburg College, has resigned to become educational director of the United States National Life and Casualty Company, Chicago.

Professor W. A. Hamilton, formerly of Beloit College, who has been a special lecturer at the University of Wisconsin during the past year, has been appointed head of the department of mathematics at Antioch College, Yellow Springs, Ohio.

Dr. Harold Hotelling, of Princeton University, has been appointed junior assistant at the Food Research Institute, Stanford University.

Dr. H. M. Jeffers, of Iowa State University, has been appointed assistant astronomer at the Lick Observatory.

Professor S. Lefschetz, who is on leave of absence from the University of Kansas, has been appointed visiting professor of mathematics at Princeton University for the academic year 1924–25.

Associate Professor J. J. Luck has been promoted to a full professorship of mathematics at the University of Virginia.

Assistant Professor C. C. MacDuffee, of Princeton University, has been appointed assistant professor of mathematics at Ohio State University.

Associate Professor W. D. MacMillan has been promoted to a full professorship of astronomy at the University of Chicago.

Associate Professor I. L. Miller, of South Dakota State College, has been promoted to a full professorship of mathematics.

At the Case School of Applied Science, Assistant Professor J. J. Nassau has been promoted to an associate professorship of mathematics and astronomy.

Dr. J. A. Nyswander has been appointed assistant professor of mathematics at Swarthmore College.

Assistant Professor E. J. Oglesby, of New York University, has been promoted to an associate professorship of mathematics.

Assistant Professor F. W. Reed, of Ohio University, Athens, Ohio, has been promoted to an associate professorship.

At Worcester Polytechnic Institute, Assistant Professor Harris Rice has been promoted to a full professorship.

Mr. L. V. Robinson, of the University of Virginia, has been appointed assistant professor of mathematics and astronomy at Oklahoma City College.

Assistant Professor R. A. Sheets, of Denison University, has been appointed assistant professor of mathematics at Miami University.

Mr. J. A. Shohat (J. Chokhate), who has held a reading assistantship at the University of Chicago during the past year since he came to this country from Russia, has been appointed instructor in mathematics at the University of Michigan.

Assistant Professor W. G. Simon has been promoted to an associate professorship of mathematics at Western Reserve University.

Dr. L. T. E. Thompson has been appointed physicist at the Naval Proving Ground, Dahlgren, Va.

Dr. F. M. Weida, of Iowa State University, has been appointed assistant professor of mathematics at Montana State College.

At the Massachusetts Institute of Technology, Dr. Norbert Wiener has been promoted to an assistant professorship of mathematics.

Associate Professor K. P. Williams has been promoted to a full professorship of mathematics at Indiana University.

Assistant Professor C. O. Williamson has been promoted to a full professorship of applied mathematics at the College of Wooster.

Dr. Frederick Wood, of the University of Wisconsin, has been appointed professor of mathematics at Lake Forest College.

The following appointments to instructorships are announced:
University of Florida, Mr. C. G. Phipps;
University of Missouri, Dr. L. H. MacFarlan;
New York University, Dr. Constance R. Ballantine;
Smith College, Miss Bess M. Eversull;
Wellesley College, Miss Ethel L. Anderton.

Professor Martin Disteli, of the University of Zurich, died October 25, 1923, at the age of sixty-three years.

Professor Claude Guichard, of the Sorbonne, died May 12, 1924, at the age of sixty-two years.

Professor Giacomo Bellacchi, of the Technical School of Florence, died February 25, 1924, at the age of eighty-four years.

Professor Corrado Segre, of the University of Turin, distinguished for his work in geometry, died May 18, 1924, at the age of sixty years.

Professor Charles Godfrey, late headmaster of the Royal Naval College, Osborne, died April 4, 1924.

Sir Asutosh Mookerjee (Mukhopâdhyây) formerly vice-chancellor of the University of Calcutta and founder-president of the Calcutta Mathematical Society, died May 25, 1924, at the age of fifty-nine years. He had been a member of the American Mathematical Society since 1900.

Professor J. E. Hodgson, of West Virginia University, died April 11, 1924.

Dr. R. S. Woodward, retired president of the Carnegie Institution, died June 29, 1924, at the age of seventy-four years. He had been a member of the American Mathematical Society since its foundation, and served as president from 1898 to 1900.

NEW PUBLICATIONS

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Granville (W. A.) et Smith (P. F.). Eléments de calcul différentiel et intégral. Traduit de l'anglais par A. M. M. Sallin. Paris, Vuibert, 1924. 548 pp.

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——. Lezioni di calcolo infinitesimale. Parte 2: Calcolo integrale. 5a edizione, riveduta. Milano, Hoepli, 1924. 8 + 330 pp.

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Picone (M.). Lezioni di analisi infinitesimale. Volume 1: La derivazione e l'integrazione. Parte 2. Catania, pubblicazione del "Circolo Matematico," 1924. 391 pp.

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THIRTY-THIRD ANNUAL LIST OF PAPERS

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